Homework 13

Exercise 1 *Use Green's theorem to compute the integral* $\int_c f$ *with* $f(x,y) = (y^2, x)$ *when* c *corresponds to the following curves, taken counterclockwise:*

(i) The square of vertices $(0,0)$, $(2,0)$, $(2,2)$, $(0,2)$,

(ii) The square of vertices $(\pm 1, \pm 1)$,

(iii) The circle of radius 1 *and centered at* $(0,0)$,

(iv) The ellipse of equation $(x/a)^2 + (y/b)^2 = 1$ *for some* $a, b > 0$.

Exercise 2 *Check the validity of Green's theorem for the function* $f : \mathbb{R}^2 \to \mathbb{R}^2$ *defined for* $(x,y) \in \mathbb{R}^2$ *by* $f(x,y) = (2xy, x^2)$ *on the domain* $\Omega = [-1,2] \times [-1,3] \subset \mathbb{R}^2$.

Exercise 3 *Consider the function* $f : \mathbb{R}^2 \setminus \{0,0\} \to \mathbb{R}^2$ *defined for* $(x,y) \in \mathbb{R}^2$ *by*

$$f(x,y) = \Big(\frac{-y}{x^2+y^2}, \frac{x}{x^2+y^2}\Big).$$

Let $c : [a,b] \to \mathbb{R}^2 \setminus \{(0,0)\}$ *be a parametric curve of class* C^1 *and non-intersecting such that its interior* Ω *is located on the left of the curve. We also assume that* $(0,0) \in \Omega$. *Compute* $\int_c f$, *and explain your result.*

Exercise 4 *Let* $f : \mathbb{R}^2 \to \mathbb{R}$ *be sufficiently many times differentiable and satisfying the Laplace equation*

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = 0.$$

(i) Let c *be a closed parametric curve oriented counterclockwise and non-intersecting. Show that*

$$\int_c \begin{pmatrix} \partial_y f \\ -\partial_x f \end{pmatrix} = 0,$$

(ii) Show that $f(0,0) = \frac{1}{2\pi}\int_0^{2\pi} f\big(r\cos(\theta), r\sin(\theta)\big)\mathrm{d}\theta$ *for any* $r > 0$.

Exercise 5 *Compute the area of the domain defined in* $\mathbb{R}_+ \times \mathbb{R}_+$ *by the four curves of equation*

$$y = ax, \quad y = x/a, \quad y = b/x, \quad y = 1/bx \qquad \text{for } a > 1, b > 1.$$