
Homework 12

Exercise 1 Compute the integral of the function $f : \mathbb{R}^2 \rightarrow \mathbb{R}$, defined by $f(x, y) = x$, on the domain Ω with

$$\Omega := \{(r \cos(\theta), r \sin(\theta)) \in \mathbb{R}^2 \mid 0 \leq \theta \leq \pi/2 \text{ and } 0 \leq r \leq 2 \cos(\theta)\}.$$

Exercise 2 Find the integral of the function $f : \mathbb{R}^2 \rightarrow \mathbb{R}$ defined by $f(x, y) = \frac{1}{(x^2+y^2+1)^{3/2}}$ on the disc of radius R and centered at the origin of \mathbb{R}^2 .

Exercise 3 Compute the area enclosed by the curve given in polar coordinate by $r^2 = \cos(\theta)$. Sketch first this area.

Exercise 4 Find the mass of a spherical ball of radius R if the density of the ball at any point is equal to a constant k times the distance of that point to the center of the ball.

Exercise 5 Find the integral of the function $f : \mathbb{R}^3 \rightarrow \mathbb{R}$ given by $f(x, y, z) = x^2$ over the portion of the cylinder defined by $x^2 + y^2 = a^2$ and lying between the planes defined by $z = 0$ and $z = b$, with $a > 0$ and $b > 0$.

Let Ω be a body in \mathbb{R}^n and let $\varrho : \Omega \rightarrow \mathbb{R}_+$ denote its density function (for $X \in \Omega$ the value $\varrho(X)$ denotes the density of Ω at X). Let M denote the total mass of the body, and let \bar{X} denote the coordinates of its center of mass. These quantities are defined by

$$M = \int_{\Omega} \varrho(X) dX,$$
$$\bar{X} = \frac{1}{M} \int_{\Omega} X \varrho(X) dX.$$

Note that the second line represents in fact n equalities.

Exercise 6 (i) Find the center of mass of the quarter of a unit disc Ω defined in polar coordinates by

$$\Omega = \{(r \cos(\theta), r \sin(\theta)) \in \mathbb{R}^2 \mid 0 \leq r \leq 1 \text{ and } 0 \leq \theta \leq \pi/2\},$$

(ii) Find the z -coordinate of the center of mass of the upper half of a unit ball centered at $0 \in \mathbb{R}^3$.

Exercise 7 Let X, Y be two vectors in \mathbb{R}^2 . Check that the area of the parallelogram spanned by X and Y is equal to the absolute value of the determinant of the matrix $(X \ Y) \in M_2(\mathbb{R})$. More generally, if X_1, \dots, X_n are n vectors of \mathbb{R}^n , one writes $\text{Vol}(X_1, \dots, X_n)$ for the volume of the n -dimensional box spanned by X_1, \dots, X_n . Why is it natural to have

$$\text{Vol}(X_1, \dots, X_n) = |\text{Det}(X_1 \dots X_n)| ?$$