NONCOMMUTATIVE RESIDUE

Chapter I. Fundamentals

by

Mariusz Wodzicki*

Mathematical Institute, University of Oxford, 24-29 St. Giles, Oxford OX1 3LB, England**.

Mathematical Institute, Polish Academy of Sciences, ul. Śniadeckich 8, P.O. Box 137, 00-950 Warsaw, Poland.

* Supported by the Royal Society Grant.

** Current address: The Institute for Advanced Study, Princeton, NJ 08540, U.S.A.
Introduction

Noncommutative residue was discovered originally about 1978 in the special case of one-dimensional symbols by (working independently) M. Adler (cf. [2]) and Yu. I. Manin (cf. [10]). They considered the half of the algebra of one-dimensional symbols whose elements are formal Laurent series:

$$ a = \sum_{i=-\infty}^{m} a_i \partial^i $$

$$ (m = m(a)) $$

($$ a_i $$ belong to a K-algebra A with differentiation $$ \partial : A \to A; $$ K is a field of characteristic zero). Two such series are composed as:

$$ a \circ b = \sum_{i,j+k=i}^{\infty} \partial^i \sum_{l=0}^{j} \frac{(j+l) \cdots (j+1)}{l!} a_j \partial^l b_k $$

(2)

$$ (b^{(l)} = \partial^l b). $$

Let $$ \overline{A} $$ denote the K-vector space $$ A/(\langle A, A \rangle \cup \text{Im}\partial). $$ It was observed that assigning to a series (1) the class of its coefficient $$ a_{-1} $$ in $$ \overline{A} $$ defines a trace functional on the algebra of series (1) with the composition law (2). This functional was given a name of noncommutative residue in analogy to Cauchy residue on ordinary (commutative) Laurent series. Noncommutative residue was discovered in the context of completely integrable systems and since then it was recognized as one of standard tools in the theory of such systems. It allows, e.g. to view linear functionals on the space of differential operators $$ A[\partial] $$ as being represented by symbols of negative order (in terminology of Yu.I. Manin: formal Volterra operators) by means of the correspondence

$$ A[[\partial^{-1}]] \partial^{-1} \ni q \mapsto \phi_q \in (A[\partial])', \quad \phi_q(a) = \chi(\text{res}(q \circ a)) $$

$$ (\chi \in \overline{A}'; \overline{A} $$ is usually a one-dimensional space which is canonically identified with K so that there is no need in choosing $$ \chi). $$
In applications $A$ is generally an algebra of functions on the real line (e.g. fast decaying, periodic etc.) with $\mathfrak{a} = d/dx$, and such that the projection $A \rightarrow \overline{A}$ has the form of integration:

$$A \ni f \mapsto \mathfrak{f} = \int f \, dx .$$

Noncommutative residue has in this situation the extra feature: the form $a_{-1} dx$ is functorial with respect to orientation preserving automorphisms of $\mathbb{R}^1$ which also preserve $A$.

All mentioned facts seemed for a long time to be fairly peculiar to dimension one, probably because of their relation to the invariant decomposition of the algebra of one-dimensional symbols into the direct sum of its positive and negative parts (i.e. differential and formal Volterra operators). No such decomposition exists in higher dimensions indeed. So it was rather surprising when the author, when thinking about the residues at negative integers of the zeta function of an elliptic pseudodifferential operator ($\psi$DO), discovered in Spring 1983 that $\text{Res}_{s=-1} \zeta(s;A)$ behaved much like a trace functional. This suggested to put for an arbitrary $\psi$DO $Q$:

$$\text{res } Q := \frac{d}{dt} \left( \text{ord } A \text{ Res}_{s=-1} \zeta(s;A+tQ) \right) \bigg|_{t=0}$$

(3)

where $A$ was an elliptic $\psi$DO admitting complex powers and of the order strictly bigger than that of $Q$. Since, as can be shown, (3) does not depend on $A$, it is not difficult to deduce that (3), in fact, defines a trace functional. In early Summer 1983 another discovery was made - that this trace on the algebra of symbols is not only exotic - it is also unique. Since then it was established that noncommutative residue was inherent in all known local invariants of $\psi$DOs (e.g. heat kernel expansion coefficients, local formulae for the variation of $\log \det A$). In dimension one this new trace, defined in the framework of spectral geometry, reduces to the one introduced by M. Adler and Yu.I. Manin.
The progress which followed the discovery of higher-dimensional noncommutative residue allowed to change a set up from rather subtle analytical methods of the zeta-function theory to more general and more flexible methods of homogeneous symplectic geometry and homological algebra. There is still no satisfactory complete theory however. Despite this, the author prompted by richness of the material available decided to publish first few "chapters" of the theory of noncommutative residue to come. These will include:

Chapter I. Fundamentals.
Chapter II. The commutator structure of pseudodifferential operators [18].
Chapter III. Bi-multiplicative index formula and exotic log det [19].
Chapter IV. Homology of the algebra of differential operators and the algebra of symbols [20].

The contents of Chapter I is, very briefly, as follows. In Section 1 we introduce an elementary formalism of the symplectic residue in a fairly general set up. Basic for all further applications and constructions is the morphism of chain complexes

\[ \text{Res}: C_\ast(P^\ast(Y); \text{ad}) \to \Omega_\ast(Z) \]

where \( P^\ast(Y) = \bigoplus_{\ell \in \mathbb{Z}} P_{\ell}^\ast(Y) \) denotes the graded Poisson algebra of a symplectic cone \( Y \times \mathbb{Z} \) with a base \( z^{2n-1} \), and \( \Omega_q(Z) \equiv \Omega_{2n-1-q}(Z) \) (see 1.17 and 1.28 below).

The core of Chapter I is the next section. There we further develop the technique of Section 1 in the special case: \( Y = T^\ast_U, Z = S^\ast U \) (\( U \) is a domain).

This allows us to define for an arbitrary \( \Psi DO A \) in a domain a certain functorial matric density \( \text{res}_u A \) called its noncommutative residue (cf. 2.13-20). Then we prove (cf. Proposition 2.28) that
$$\text{tr res}_u AB - \text{tr res}_u BA = d\rho_u(A,B) \quad (4)$$

holds for a certain explicit form $\rho_u(A,B) \equiv -\rho_u(B,A)$. The form $\rho_u$ is functorial with respect to gauge transformations (Proposition 2.31) but fails in general to be functorial with respect to open embeddings (cf. Example 2.44). In Proposition 2.38 we establish that its "non-functoriality" is measured by another explicit exact form $d\sigma_u$. One should dwell, at this point, on a special feature of the considered situation: whenever some equality holds only up to certain exact terms (and this happens more often than the point-wise equality does) it appears always possible to write down a suitable local primitive form.

In Section 3 we consider global $\psi$DOs acting on sections of general vector bundles on arbitrary manifolds. We introduce a canonical vector-valued density $\text{res}_x A$ (see 3.1) and deduce its basic properties from results of previous section (see Proposition 3.2). We prove that (4) still holds but now the analogue of form $\rho_u$ is canonical only when defined up to an exact form (Proposition 3.7). The corresponding class modulo exact forms turns out to be a cyclic 1-cocycle on the algebra of $\psi$DOs (Proposition 3.9).

Still another invariant quantity related to the noncommutative residue can be associated with an arbitrary $\psi$DO. It is called a subresidue, and is a higher-dimensional analogue of a quadratic differential (see 3.14-17).

In 3.18-21 we rewrite R.T. Seeley's formulae for the residues and values of the zeta-function of an elliptic $\psi$DO purely in terms of the noncommutative residue formalism. This includes a new proof of the invariance under gauge transformations and changes of local coordinates. Our proof is intrinsic and purely algebraic. We believe that our reformulation of Seeley's results clarifies many issues. For example, it turns out that the zeta-function of a $\psi$DO
is not so much different from that of an endomorphism of a finite-dimensional space. The difference is comparable, using a free parallel, to that between rational elliptic curves of rank one and rank zero (if Birch-Swinnerton-Dyer conjecture has to hold). For both \( \psi \)DOs and matrices there are similar formulae for coefficients at leading terms in power expansions of zeta-function at certain points. In the case of \( \psi \)DOs the residues of \( \zeta(s;A) \) are given by

\[
\text{Res}_{s=0} \zeta(s;A) = \frac{\text{res}_{A} A^{-s}}{\text{ord } A}
\]

(see (8) of Section 3; one is reminded that \( \text{res} \) is the only trace available in this situation), whereas for matrices the values of \( \zeta(s;A) \) are given by

\[
\zeta(0;A) = \text{tr } A^{-s}.
\]

Both formulae are virtually the same if to neglect the standard in \( \psi \)DO theory appearance of the order of an operator.

Next turn to the behaviour of \( \zeta(s) \) at the origin. In the former case \( \zeta(s;A) \) has \textit{a priori} residue at that point but this turns out to be equal to zero. For a matrix the value \( \zeta(0;A) \) also does not depend on \( A \): it is equal to the dimension of the linear space (in both cases we assume for brevity that \( A \) is invertible). Thus the leading terms in both cases can be fairly ignored, and one may examine the next terms. For matrices this turns out to be the derivative

\[
\zeta'(0;A) = -\log \det A \equiv -\text{tr log } A,
\]

for \( \psi \)DOs - the value:

\[
\zeta(0;A) = \frac{Z(A)}{\text{ord } A}
\]

where \( Z(A) \equiv \int_{X} \text{tr } Z_{\psi,x}(A) \), and \( Z_{\psi,x}(A) \) is defined in Lemma 3.19.
In [17] we proved that $\text{tr} Z_\theta \cdot \chi(A)$ does not depend on the choice of a cut in the spectral plane, so, we omit the subscript $\theta$. We shall prove, indeed, in Chapter III [19] that the functional $Z(A)$ extends to a certain exotic "log det" on the group of invertible elliptic PDOs and that it is even a homomorphism from this group into the additive group of complex numbers.

The noncommutative residue approach yields equally simple formulas for more general zeta-functions (see 3.22). It gives also a fresh insight into the standard high-temperature (or small-time) expansion

$$\text{tr} e^{-t\Box} \chi(x,x) - \text{tr} e^{-t\Box_F} \chi(x,x) = \sum_{j=0}^{\infty} \hat{\alpha}_j(x) t^{j-n}/2^m + \sum_{q=1}^{\infty} \beta_q(x) t^q \log t$$

$(t \to 0)$; for notation cf. 3.23 and 3.28). As is well known, all coefficients of this expansion except $\hat{\alpha}_n(x)$ integrate to zero. We deduce from the noncommutative residue formalism (and its further extension called the $\Lambda$-twisted noncommutative residue which is sketched in 3.26-27) that the coefficients mentioned can be expressed as differentials of explicit forms which depend locally on the symbol of an operator and metrics involved (see Prop. 3.24 and the further discussion). From an entirely different point of view, an analogous statement was proved (by P.B. Gilkey and, perhaps, also by others) for certain differential operators arising in Riemannian geometry (cf. e.g. [4] and [7]). The two approaches seem to provide different answers, the difference being measured by certain cohomology classes (see 3.31). Finally, local cancellation phenomena which are widely known in physics and Riemannian geometry are responsible in a number of cases for vanishing of $\hat{\alpha}_j(x)$'s $(j < n \equiv \dim X)$ point-wise. Whenever such a case occurs, our approach produces a certain secondary cohomology class which lives in $H^{n-1}(X,\xi_X)$ where $\xi_X$ denotes the orientation local system (see 3.28). These secondary classes call for further investigation (cf., however, the related example in 3.29).

Chapter I was designated to contain basic constructions so that it could serve (for a while) as a standard reference source for the
subsequent chapters' and other works. (When the higher-dimensional residue was originally discovered and some of its fundamental properties announced in the final section of [17] the complete exposition was intended to appear in Functional Analysis and its Applications. This soon turned out to be unrealistic since the bulk of the available material overgrew limits of an article permissible for that journal.) The first draft of the first two chapters was published (in September 1984) as a part of the author's Steklov Habilitation Thesis [21]. In the different context some results were reported in [16]. Even in the present long overdue publication one fundamental topic is still absent: the formalism of "super-residue". As in a purely even case its original discovery was made in the different context of super-integrable systems, the underlying space having dimension 1|1 (actually, the "superisation" used there is fairly nontrivial, e.g. the corresponding time-direction is odd, for details the reader is referred to the article by Yu.I. Manin and A.O. Radul [12]). The author introduced general n|m-dimensional super-residue and proved it to be a super-trace as well as being essentially unique. The notion of super-residue might also deserve more attention because it seems to be well suited to heat kernel methods which are current in physics and Riemannian geometry.

One final remark must be made. Although the name of Alain Connes is not mentioned in the present paper it is needless to say that noncommutative residue is well suited to his general picture of noncommutative integration. The careful reader will find in 1.25 an indication of a more direct link existing between noncommutative residue and his approach based on relations derived from the failure of trace property (of the ordinary trace). We hope to trace this link in subsequent papers. It should be also clear from the exposition of noncommutative residue which is presented below that all ways lead us to homological algebra. Despite this we consciously decided to remain in the introductory Chapter I on an elementary level of exposition.
During my long stay in Moscow I was fortunate to be surrounded by a unique community consisting of outstanding mathematicians as well as remarkable people. To all of them my warmest thanks. I owe special gratitude to Yuri Ivanovich Manin (among other things for his invaluable support) and to Sasha Beilinson. I wish also to thank Mikhail Alexandrovich Shubin for numerous discussions on the subject of the present paper.

These notes were finally prepared during my appointment at Oxford with Sir Michal F. Atiyah, whose support I greatly acknowledge.

0. Notation

0.1. Manifolds are usually denoted by $X$, $Y$, $Z$ etc. Domains in $\mathbb{R}^n$ are denoted $U$, $V$ etc. The space of vector fields on $X$ is denoted by $TX$, the cosphere bundle by $S^*X$.

0.2. Vector bundles are denoted by $E$, $F$, $G$, $H$ etc. The canonical trivial bundle of rank $k$ is denoted $\theta^k$, sometimes $\theta^k_X$, to indicate its base space.

0.3. Triples $(X;E,F)$ consisting of a manifold and two vector bundles on it form the category whose morphisms are triples $\phi = (f;r,s) : (X';E',F') \rightarrow (X;E,F)$ such that

1) $f : X' \rightarrow X$ is an open embedding,

2) $r \in \text{Hom}(E',f^*E)$ and $s \in \text{Hom}(f^*F,F')$

0.4. Pairs $(X;E)$ form a subcategory whose morphisms $\psi = (f;s)$ correspond to triples $(f;s^{-1},s)$. In particular $s$ should be an isomorphism.

0.5. For every linear map $A : C^\infty_{\text{comp}}(X,E) \rightarrow C^\infty(X,F)$ a morphism $\phi : (X';E',F) \rightarrow (X;E,F)$ defines the induced map
The diagram
\[ \begin{array}{ccc}
\mathcal{C}^\infty_{\text{comp}}(X,E) & \xrightarrow{\phi} & \mathcal{C}^\infty(X,F) \\
\phi_1 \downarrow & & \downarrow \phi^* \\
\mathcal{C}^\infty_{\text{comp}}(X',E') & \xrightarrow{\phi} & \mathcal{C}^\infty(X',F')
\end{array} \]
commute. If \( E' = f^*E, F' = f^*F \) and \( r, s \) are identical maps we shall use also notation \( f^\#A \). If \( Y \subset X \) and \( f \) is the natural embedding we shall write also \( A|_Y \) instead of \( f^\#A \).

0.6. By a local coordinate patch we mean a morphism \( \phi = (j;r,s) : (U;k_1,k_2) \to (X;E,F) \) where \( k = \text{rank } E, \xi = \text{rank } F \) and both \( r \) and \( s \) are isomorphisms.

0.7. All pseudodifferential operators (\( \psi \)DOs) considered are supposed to be of classical type. Recall that \( A : \mathcal{C}^\infty_{\text{comp}}(X,E) \to \mathcal{C}^\infty(X,F) \) is a \( \psi \)DO of classical type if for every local coordinate patch \( \phi \) the induced operator \( B := \phi^\#A \) can be expressed as
\[
Bf(u) = \int_{\mathbb{R}^n_x} \int_{\mathbb{R}^n_v} e^{i(u-v) \cdot \xi} \alpha(u,\xi)f(v)\xi^\alpha dv + Tf(u)
\]
where \( \xi^\alpha \equiv (2\pi)^{-n}d\xi \), \( n = \text{dim } X \) and \( T \) is an operator with smooth kernel, and \( \alpha \in \mathcal{C}^\infty(T^*U) \otimes \text{Mat}_{\xi,k}(\mathbb{C}) \) admits the asymptotic expansion
\[
\alpha \sim \sum_{j=0}^{\infty} a_{m-j} \text{ as } |\xi| \to \infty
\]
\[
(a_{m-j}(u,t\xi) \equiv t^{m-j}a_{m-j}(u,\xi), \quad j = 0,1,2,...)
\]

0.8. As opposed to the hitherto existing practice we shall mean by the symbol of \( B \) the formal sum on the right-hand side of (1). When we refer to \( \alpha \) itself we shall call it the smoothed symbol. Usually it is not uniquely defined unlike the formal sum in (1).

0.9. The number \( m \) in (1) is called the order of \( B \) (it can be any complex number). An operator \( A \) (as in 0.7) is called of order \( m \)
(or 'less') if $\phi \# A$ is such for every local coordinate patch. The space of such $\psi$DOs is denoted by $CL^m(X;E,F)$. If $E = F$ we shall also write $CL^m(X;E)$.

Usually $m$ will be an integer.

0.10. A $\psi$DO is called proper if the support of its Schwartz kernel is a proper neighbourhood of the diagonal $\Delta \subset X \times X$. A proper $\psi$DO can be composed with other $\psi$DOs. In particular, the linear space $CL^m_{\text{prop}}(X,E) \cup CL^m_{\text{prop}}(X,E)$ forms an associative algebra.

0.11. By $L^\infty(X;E,F)$ we denote the space of operators with smooth kernels. It is a subspace in $CL^m(X;E,F)$ and the factor-space $CL^m(X;E,F)/L^\infty(X;E,F)$ is denoted by $CS^m(X;E,F)$ and called the space of (global) symbols of order $m$. It is isomorphic locally to the space of formal series as in (1).

0.12. Composition formula. The composition of $\psi$DOs defines the composition law on symbols:

$$a \circ b = \sum_{\alpha \in \mathbb{Z}} a^\alpha \partial_\xi b^\alpha,$$

(the sum over all multi-indices $\alpha$; $D_u = -\sqrt{-1}\partial_u$).

0.13. Base change formula. Let $f : V \to U$ be an open embedding, and $(u,\xi)$ and $(v,\eta)$ be canonical coordinates in $T^*U$ and $T^*V$ respectively. We shall denote by $T^*f$ the map

$$T^*V \to T^*U, \quad (v,\eta) \to (f(v),(f_\eta)^{-1}\eta).$$

Let $a(u,\xi)$ be the symbol of $A \in CL^m(U;\theta^k,\theta^l)$ and $a^f(v,\eta)$ be the symbol of $f^\# A \in CL^m(V;\theta^k,\theta^l)$. The base change formula expresses $a^f$ in terms of $a$ and $f$:

$$a^f = (T^*f)^*\left( \sum_{\alpha} a^{(\alpha)}(\psi_{\alpha}) \right) \quad (2)$$

where $a^{(\alpha)}(u,\xi) = \partial^\alpha_\xi a(u,\xi)$, and $\psi_{\alpha} \equiv \psi_{\alpha}(u,\xi)$ are universal functions.
depending only on $f$, and polynomial with respect to $\xi$'s of degree not exceeding $|a|/2$.

Description of $\psi^i\alpha$. Let $g: f(V) \rightarrow V$ denote the inverse mapping, i.e. $gf = \text{id}_V$, considered as a function $f(V) \rightarrow \mathbb{R}^n$. Its linear part at a point $u \in f(V)$ will be denoted by $j^1_u(g)$. Then $\bar{g}_u := g - j^1_u(g)$ vanishes at $u$ up to second order. We put

$$
\psi^i_u(\xi) = \frac{1}{a!} \sum_{\alpha} a^\alpha \sqrt{-1} \langle \bar{g}_u(z), f \xi \rangle \quad (3)
$$

($\langle \cdot, \cdot \rangle$ denotes the ordinary scalar product in $\mathbb{R}^n$).

Formula (2) can be found in a slightly different form in two standard reference works on $\Psi$DOs: [9] (cf. 2.1.14-16), and [15] (cf. Thm. 4.2).

We shall need, however, still another form of the base change formula.

0.14. Base change formula (Second form).

$$
a^\xi = (T^*f)^*\left\{ (1 + \sum_{j=1}^{n} \partial_{\xi_j} \circ \psi_j) a \right\} \quad (4)
$$

where $\psi_j \equiv \psi_j^f(u, \xi, \partial_{\xi_j})$ are certain differential operators on $T^*U$ of infinite order (such that $\psi_j a$ has sense). Their explicit form can be deduced easily from (2) and (3).

0.15. The complex of integral forms. Let $\alpha: \tilde{X} \rightarrow X$ be the canonical orientation double cover of $X$. Then the complex $\alpha_* \Omega^\alpha \tilde{X}$ is acted by the deck involution. The anti-invariants of this action form the complex of integral forms denoted by $\mathcal{A}'(X)$ (N. Bourbaki uses different terminology: $T(X)$-twisted differential forms (formes différentielles $T(X)$-tordues), cf. [3], 10.4.1). It coincides, of course, with the complex of smooth currents. When choosing its name the author was guided by two reasons: integral volume forms (known
also under the name of l-densities) can be canonically integrated and, hence, the corresponding sheaf is a dualizing sheaf in $C^\infty$-category (and not $\mathbb{V}^1_{\text{vol}}$). An analogous situation is well-known in super-geometry where the corresponding dualizing sheaf is called Berezinian and the complex generated by it - the complex of integral forms (cf. e.g. [11], 4.5.4. or [13]). In the classical (even) complex geometry the complex of integral forms is canonically isomorphic to de Rham complex.

In this connection we want to draw attention to the fact that local invariants of \(\Psi DOs\) (as e.g. local residues and values of the zeta function) are usually canonically integrable. When they happen to be 'exact' or when one looks on higher relations existing between them one immediately evokes the whole complex of integral forms.

0.16. The properties of integral forms are fairly dual to those of differential forms. In particular, an integral form of codimension \(q\) (the space of such forms will be denoted by $A_q$) can be integrated over co-oriented chains of the same codimension (co-orientation = orientation of the normal bundle).

0.17. For any submersion $\tau: Z \rightarrow X$ there is defined a push-out morphism $A_\tau(Z) \rightarrow A_\tau(X)$ dual to the pull-back of differential forms. This is the composition of the canonical projection $A_\tau(Z) \rightarrow A_\tau(X, \tau_* A_Z/X, 0)$ with the integral

$$\int : \tau_* A_Z/X, 0 \rightarrow \mathcal{O}_X^\infty (A_Z/X, 0)$$

\(\mathcal{O}_X^\infty\) denotes the sheaf of relative densities while \(\mathcal{O}_X^\infty\) is the sheaf of (complex valued) smooth functions on \(X\).

0.18. Integral forms can be pulled back under open embeddings. Pull-backs commute with push-outs:

$$\tau^* g^* = f^* \tau_*$$

if in the corresponding commutative diagram
$g$ induces diffeomorphisms between fibres of projections $\tau$ and $\tau'$

(5) is nothing but change of variables under the sign of integral).

0.19. $\lVert \cdot \rVert : \Omega^\prime(X) \rightarrow A^\prime(X)$ denotes the identification induced by an orientation (if $X$ is oriented).

0.20. For a general chain complex $C[k]$ is the standard notation for its $k$-shift:

$$(C[k])_q \equiv C_{q-k}$$

(boundary maps are those in $C_*$ but multiplied by $(-1)^k$).

§1. Symplectic residue

Our aim in this section will be to introduce and briefly discuss in a general setting of homogeneous symplectic geometry an elementary formalism of symplectic residue which we expect to become one of 'standards' in the global $\Psi DO$-theory.

1.1. Let $Y^{2n}$ be a symplectic manifold and $\omega$ be the corresponding non-degenerated 2-form. Recall that any function $f \in C^\infty(Y)$ determines the associated Hamiltonian vector field $H_f$ such that

$$i_{H_f} \omega = -df.$$ 

(For brevity reasons $i_{H_f}$ will be later denoted as $i_f$ and similarly Lie derivative $L_{H_f}$ as $L_f$).

The Poisson bracket
can be alternatively defined in terms of volume forms:

\[ \{f,g\} = \frac{ndf \wedge dg \wedge \omega^{n-1}}{\omega^n} = \frac{d(gi_f \omega^n)}{\omega^n}. \]

Indeed, we have:

\[
0 = i_f g^{n+1} = -(n+1) i_f (dg \wedge \omega^n) = -(n+1) [dg(H_f) \omega^n + ndg \wedge df \wedge \omega^{n-1}]
= (n+1) [ndf \wedge dg \wedge \omega^{n-1} - \{f,g\} \omega^n].
\]

1.3. From now on \( Y \) is assumed to be acted by the multiplicative group of positive real numbers and the action (denote it by \( \chi : \mathbb{R}_+^X \to \text{Diff}(Y) \)) is required to be a conformal one:

\[ \chi^*_t \omega = t \omega \quad (t \in \mathbb{R}_+^X). \]  

The infinitesimal action \( \chi_* : \text{Lie}(\mathbb{R}_+^X) \to TY \) defines the Euler vector field \( \Omega = \chi_*(r \frac{dr}{dF}) \) on \( Y \). Put:

\[ \alpha = i_\Omega \omega \quad \text{and} \quad \mu = \alpha \wedge (d\alpha)^{n-1} \]

(\( \alpha \) is the standard "canonical" 1-form of classical mechanics).

Then the following identities:

1.4. \( \chi^*_t \alpha = t \alpha, \quad \chi^*_t \mu = t^n \mu; \)

1.5. \( da = L^a \omega = \omega, \quad L^a \alpha = \alpha, \quad d\mu = \omega^n, \quad L^a \mu = n \mu. \)

can be easily deduced from (1) using the fact that \( \Omega \) generates the flow \( \phi_t = \chi_{\text{expt}}. \) The next identity:

1.6. \( \{f,g\} \mu = d(gi_f \mu) - (n-1) gdf \wedge \omega^{n-1} - L^g(gdf) \wedge \omega^{n-1} \)

follows from:
\[
\{f, g\} = \frac{1}{n} \int_\Sigma (f \omega g - g \omega f) = \frac{1}{n} \int_\Sigma \left( \partial_i f \omega^j - \partial_j f \omega^i \right) = \frac{1}{n} \int_\Sigma \left( L^\Sigma (g \partial_i f) - \partial_i (g f) \right)
\]

Starting from this point we shall deal exclusively with functions on \(Y\) which are homogeneous. For a function \(f\) its homogeneity order will be denoted by \(\ell_f\).

The following identities will be used frequently:

1.7. \([E, f]\) = \((\ell_f - 1)F_f\).

1.8. \(L^\Sigma_f (g \mu) = (n + \ell_f)fg \mu\).

1.9. \(i_{f \omega} \mu = \ell_f f \omega, \quad \mu = \ell_f f \omega^{-1} + (n - 1) \alpha \wedge df \wedge \omega^{-1}\).

1.10. \(L^\Sigma_f \alpha = (\ell_f - 1)df, \quad L^\Sigma_f \mu = (\ell_f - 1)df \wedge \omega^{-1}\).

1.11. \(\{f, g\} = \partial (g_i f) - (\ell_f + \ell_g - 1 + n) g \omega f \wedge \omega^{-1}\).

Identity 1.7 follows from:

\[
d[E, f] = [L^\Sigma_f, i_f] = (n + \ell_f)fg \mu = (n + \ell_f)fg \mu = (\ell_f + 1)df = i(\ell_f - 1)\mu
\]

(we used the Euler identity \(L^\Sigma_f = \ell_f f\)). The last four identities follow from 1.5-6 and the Euler identity.

We finish this somewhat lengthy register with the following items:

1.12. \(L^\Sigma (f_0 i_{f_1} \cdots i_{f_q} \omega^n) = (\ell_f + \cdots + \ell_{f_q} - q + n) f_0 i_{f_1} \cdots i_{f_q} \omega^n\).

1.13. \(L^\Sigma (f_0 i_{f_1} \cdots i_{f_q} \mu) = (\ell_f + \cdots + \ell_{f_q} - q + n) f_0 i_{f_1} \cdots i_{f_q} \mu\).

1.14. \(\chi^*(f_0 i_{f_1} \cdots i_{f_q} \mu) = t f_0 i_{f_1} \cdots i_{f_q} \mu\).

The former two identities can be verified by a recurrent use of 1.7. and of the standard identity \([L^\Sigma, i_f] = i_{[\Sigma, f]}\). The last one is a global version of 1.13.
Hereafter we shall assume $Y$ to be a connected symplectic cone with a smooth base $Z^{2n-1}$, i.e. $Y$ has to be a principal $\mathbb{R}_+^\times$-bundle with $Y/\mathbb{R}_+^\times = Z$. The natural projection $Y \to Z$ will be denoted by $\rho$.

Consider the differential form $f_0 i_{f_1} \ldots i_{f_q} \mu$. Identity 1.14 says that it is $\mathbb{R}_+^\times$-invariant precisely when

$$\sum_{i=0}^q i_{f_i} = q - n.$$ 

Since it is always horizontal (i.e. is annihilated by $i_{\mathbb{Z}}$), it must exist a unique $(2n - q - 1)$-form $\mu f_0^* f_1^* \ldots f_q^*$ on $Z$ such that

$$\rho^* f_0^* f_1^* \ldots f_q^* = f_0^* f_1^* \ldots f_q^* \mu.$$ 

Its differential can be calculated by a recurrent use of Cartan identity and the mentioned identity $[L_{\eta}, i_{\xi}] = i_{[\eta, \xi]}$:

\[
\begin{align*}
\rho^* f_0^* f_1^* \ldots f_q^* & = \rho^* f_0^* f_1^* \ldots f_q^* \mu \\
& = \sum_{i=0}^q i_{f_i} (f_0^* f_1^* \ldots f_q^* \omega^n) + \sum_{i=0}^q i_{f_i} (f_0^* f_1^* \ldots f_{q-1}^* \omega^n) \\
& = \sum_{i=0}^q i_{f_i} (f_0^* f_1^* \ldots f_{q-1}^* \omega^n) \\
\end{align*}
\]

Using the identity $[L_{\eta}, i_{\xi}] = i_{[\eta, \xi]}$ and the mentioned identity:

\[
\begin{align*}
\rho^* f_0^* f_1^* \ldots f_q^* & = \rho^* f_0^* f_1^* \ldots f_q^* \mu \\
& = \sum_{i=0}^q i_{f_i} (f_0^* f_1^* \ldots f_q^* \omega^n) \\
& = \rho^* f_0^* f_1^* \ldots f_q^* \mu \\
\end{align*}
\]

This completes the proof.
Since $\mathcal{E}_q = \mathcal{E}_q + \mathcal{F}_q - 1$, each term on the right hand side of (2) must be a corresponding $(2n-q)$-form on $\mathbb{Z}$ lifted to $Y$. We obtain thus:

1.16.

\[
\begin{align*}
\delta(f_0; f_1, \ldots, f_q) &= \sum_{1 \leq h < j \leq q} (-1)^{h+j-1} f_0 \{ f_h, f_j \} \wedge f_1 \wedge \ldots \wedge \hat{f}_h \wedge \ldots \wedge \hat{f}_j \wedge \ldots \wedge f_q \\
&+ \sum_{h=1}^{q} (-1)^{h-1} \{ f_h, f_0 \} \wedge f_1 \wedge \ldots \wedge \hat{f}_h \wedge \ldots \wedge f_q .
\end{align*}
\]

1.17. **Graded Poisson algebra.** The right hand side of 1.16 is nothing but a common expression for the boundary operator in the standard chain complex of a Lie algebra with coefficients in its adjoint representation. More precisely, let us introduce the graded Lie algebra

\[ P^*(Y) = \bigoplus_{k \in \mathbb{Z}} P^k(Y) \quad (P^k(Y) = \{ f \in C^\infty(Y) | \mathcal{L}_f = k f \}) \]

with ordinary Poisson bracket as commutator. This will be called the **graded Poisson algebra** of a symplectic cone $Y$ (we shall often leave out in notation its dependence on $Y$ where it does not lead to confusion).

Recall the definition of the standard chain complex $C_q(P^*; \text{ad})$:

\[ C_q(P^*; \text{ad}) = P^* \otimes \Lambda^q P^*, \]

\[
\begin{align*}
\partial(f_0 \wedge f_1 \wedge \ldots \wedge f_q) &= \sum_{1 \leq h < j \leq q} (-1)^{h+j-1} f_0 \{ f_h, f_j \} \wedge f_1 \wedge \ldots \wedge \hat{f}_h \wedge \ldots \wedge \hat{f}_j \wedge \ldots \wedge f_q \\
&+ \sum_{h=1}^{q} (-1)^{h-1} \{ f_h, f_0 \} \wedge f_1 \wedge \ldots \wedge \hat{f}_h \wedge \ldots \wedge f_q .
\end{align*}
\]

Since $P^*$ is graded:
the complex $C(p'; \text{ad})$ splits into the direct sum of its subcomplexes $C_{\mu}(p'; \text{ad})$ which are defined by

$$C_{\mu}(p'; \text{ad}) = \bigoplus_{k \geq 0} C^k(p'; \text{ad}) .$$

Put

$$\text{Res}_q(f_0 \otimes f_1 \ldots \otimes f_q) = \begin{cases} \mu f_0 \otimes f_1 \ldots \otimes f_q & \text{if } f_0 \otimes f_1 \ldots \otimes f_q \in C_{\mu}^{(-n)} \\ 0 & \text{if otherwise.} \end{cases} \quad (3)$$

Now, 1.16 simply says that $\text{Res}_q$'s ($q \in \mathbb{Z}_+$) define the morphism of chain complexes

$$\text{Res}: C(p'; \text{ad}) \to \Omega_\mu(Z) \quad (4)$$

which, hereafter, will be called the \textit{total} symplectic residue morphism ($\Omega(Z)$ denotes de Rham chain complex of $Z$, i.e. $\Omega_q(Z) = \Omega^{2n-1-q}(Z)$).

1.18. The morphism $\text{Res}$ is surjective.

\textbf{Proof.} Recall that there is a canonical $C^\infty(Z)$-isomorphism

$$\sigma: \Omega^{\text{vol}}(Z) \otimes C^\infty(Z) \cong \Omega(Z) , \quad (5)$$

$$\nu \otimes \eta_1 \wedge \ldots \wedge \eta_q \mapsto \int_{\eta_1} \ldots \int_{\eta_q} \nu$$

which is also an isomorphism of $TZ$-modules.

On the other hand any vector field $\eta \in TZ$ clearly can be expressed as

$$\eta = \sum_{\alpha} q_{\alpha} \cdot \rho_{\alpha}(H_{f_{\alpha}}) ,$$

for some $q_{\alpha} \in C^\infty(Z)$ and $f_{\alpha} \in \mathbb{R}^1$. Moreover, any volume form $\nu$ on $Z$ lifts to $f_{0\mu}$ for some $f_0 \in \mathbb{P}^{-n}$. Therefore all forms
\text{Corollary.} The total symplectic residue induces the epimorphisms:

\[ \text{Res}_q : H_\text{q}(P'(Y); \text{ad}) \to H^{2n-1-q}(\mathbb{Z}) \]  
\[(q = 0, 1).\]  

\text{Proof: This follows from the fact that in the commuting diagram:}

\[ C_1(-n)(P'; \text{ad}) \xrightarrow{\partial} C_0(-n)(P'; \text{ad}) \]
\[ \text{Res}_1 \downarrow \quad \downarrow \text{Res}_0 \]
\[ \Omega^{2n-2}(\mathbb{Z}) \xrightarrow{\partial} \Omega^{2n-1}(\mathbb{Z}) \]

both vertical arrows are surjective whereas the right one is even bijective.

1.20. \text{Remark.} We shall determine later the commutator structure of the Lie algebra \( P'(Y) \) in the case of closed \( Z \) (see [18], Sect. 3). In particular, it will be shown that (6), for \( q = 0 \), is an isomorphism (at least when \( Z \) is closed).

1.21. The base of a symplectic cone has a canonical orientation determined by any volume form \( \mu_g \) corresponding to a positive \( g \in P^{-n} \). Hence for \( f \in P' \) with compact support the integration of \( \text{Res}_0(f) \) yields the number

\[ \text{Res}_f := \left\{ \begin{array}{ll} \text{Res}_0(f) & \text{if } f \in P' \end{array} \right. \]  
\[ (7) \]

which will be called the \textit{(symplectic) residue} of \( f \). (This definition was introduced first by V.W. Guillemin who also proved 1.22 below in the special case \( \lambda = 1 \) by a different method, cf. [8]).

Notice that (7) can be calculated by using any "section" \( Z' \subset Y \) of the projection \( \rho : Y \to Z \).
\[ \text{Res } f = \frac{\mu_f}{2}. \]

The following assertion is a corollary of identity 1.11 and (3).

1.22. Let \( f \in P^\ell \) and \( g \in P^m \) (\( \ell, m \in \mathbb{C} \) and \( \ell + m \in \mathbb{Z} \)), and \( \rho(\text{supp } f \cap \text{supp } g) \) be compact. Then \( \text{Res}(f,g) \) is defined and equal to zero.

1.23. Corollary. The map \( f \mapsto \text{Res } f \) defines a trace functional on the Lie algebra \( P'_{\text{comp}}(Y) = \{ f \in P'(Y) \mid \rho(\text{supp } f) \text{ is compact} \} \).

Later it will be demonstrated that when \( Y \) is connected and \( Z \) is compact this trace is actually unique (see [18], 3.4).

1.24. Remark. Assume that \( \dim Y = 2 \) and we are given two functions \( f_0 \) and \( f_1 \) such that \( \lambda f_0 + \lambda f_1 = 0 \) (which is the actual value of \( 1 - n \)). Then:

\[ \mu_{f_0; f_1} = f_0^1 f_1 = -f_0^1 f_1 \omega = f_0^1 d f(\Xi) = f_0^1 f_1 = \lambda f_1 f_0^1. \]

In particular, \( \text{Res}_1(f_0 \otimes f_1) \) is seen to be skew-symmetric in \( f_0 \) and \( f_1 \). Since the whole \( \text{Res} \)-morphism reduces in this case to \( \text{Res}_0 \) and \( \text{Res}_1 \) alone we infer that \( \text{Res} \) factorises through the 'anti-symmetrisation' map \( \triangledown : C(P'; \text{ad}) \to C(P')[-1] \).

\[ \begin{array}{ccc}
C(P'; \text{ad}) & \xrightarrow{\text{Res}} & \Omega(\mathbb{Z}) \\
\downarrow{\triangledown} & \downarrow{\text{R}} & \downarrow{\text{R}} \\
C(P')[-1] & & \\
\end{array} \]

where \( R_0(f_0) = \text{Res}_0(f_0) \) and \( R_1(f_0 \wedge f_1) = \text{Res}_1(f_0 \otimes f_1) \).

Although this factorisation is peculiar to dimension 2, its analogue exists in higher dimensions for symplectic cones of special type (e.g. for \( Y = T^*_0 X \) and \( Z = S^* X \), cf. 2.8).
1.25. Trace morphisms. Consider the following hierarchy of "trace" morphisms:

\[ C_*(T_X; \Omega^{\nu}(X)) \rightarrow \Omega_*(X), \quad (8a) \]
\[ C_*(T_T X; C^\infty(X)) \rightarrow \Omega_*(X), \quad (8b) \]
\[ C_*(\text{Ham}(X); C^\infty(X)) \rightarrow \Omega_*(X), \quad (8c) \]
\[ C_*(\text{Poiss}(X); \text{ad}) \rightarrow \Omega_*(X). \quad (8d) \]

Here, \( T_X \) denotes the Lie algebra of vector fields on \( X \) which preserve a fixed volume form \( \nu \). \( \text{Ham}(X) \) is the algebra of Hamiltonian vector fields and \( \text{Poiss}(X) \) is the Poisson algebra. In the last two cases \( X \) is assumed to be a symplectic manifold.

Arrow (8a) is defined as the composition of the "brute" \( C^\infty(X) \)-linearisation map

\[ \Omega^{\nu}(X) \otimes \Lambda^* T_X \rightarrow \Omega^{\nu}(X) \otimes \Lambda^* C^\infty(X) \]

with isomorphism (5). A direct computation shows that so defined map is actually a morphism of chain complexes.

The next trace (8b) is obtained from the previous one by composing it with the obvious morphism \( C_*(T_T X; C^\infty(X)) \rightarrow C_*(T_X; \Omega^{\nu}(X)) \) induced by the inclusion \( T_T X \subset T_X \) and the map \( C^\infty(X) \rightarrow \Omega^{\nu}(X), \ f \rightarrow f \nu \).

Third trace is induced by the second one and the inclusion:

\[ \text{Ham}(X) \subset T_T X \quad (\nu = \omega^n). \]

Finally, the last trace is induced by the previous one using the canonical homomorphism \( \text{Poiss}(X) \rightarrow \text{Ham}(X) \).

Assume now that the underlying manifold is a symplectic cone \( Y \) with a base \( Z \) (as in 1.15). Its graded Poisson algebra \( P' = P'(Y) \) sits inside \( \text{Poiss}(Y) \) and the restriction of (8d) to \( P'(Y) \) induces the commutative diagram
\[ C_{(\text{Poiss}(Y);\text{ad})} \to \Omega_{\cdot}(Y) \]
\[ C_{(P'(Y);\text{ad})} \to \Omega_{\cdot}(Y) \quad (9) \]

where \( \Omega_{\cdot}(Y) \) is the corresponding graded de Rham chain complex:

\[ \Omega_{\cdot}(Y) := \bigoplus_{k \in \mathbb{Z}} \Omega_{\cdot,k}(Y), \quad \Omega_{\cdot,k}(Y) := \{ \psi \in \Omega_{\cdot}(Y) | L_{\mathbf{a}}\psi = k\psi \} . \]

Identity 1.12 says that the lower arrow in (9) sends the \( k \)-th subcomplex \( C_{(k)}(P';\text{ad}) \) (cf. 1.17) to \( \Omega_{\cdot,k+n}(Y) \) \((n = \text{dim } Y/2)\). In particular, up to this shift, it is a graded morphism (called, hereafter, the \textit{graded Poisson trace}).

Replace \( Y \) by \( Y^C := Y \times \mathbb{C}^\times \). This is a \( \mathbb{C} \)-bundle over \( Z \) associated to \( \rho : Y \to Z \). Then \( \Omega_{\cdot}(Y) \) embeds canonically as a subcomplex into \( \Omega_{\cdot}^{\text{hol}}(Y^C) \) – the complex of differential forms on \( Y^C \) which are holomorphic along fibres. The ordinary fibre-wise Cauchy residue induces a canonical push-out morphism

\[ \Omega_{\cdot}^{\text{hol}}(Y) \to \Omega_{\cdot}(Z) . \quad (10) \]

Notice that its composition with the graded Poisson trace that was introduced above is exactly our symplectic residue morphism (4) (this may serve as a justification of its name).

Projection (10), however, reflects precisely one half of the homology of \( \Omega_{\cdot}^{\text{hol}}(Y^C) \). The other half, which is equally important, is carried by the subcomplex \( (\rho^C)^*\Omega_{\cdot}(Z)[1] \). We may represent this by means of the diagram:

\[ \begin{array}{ccc}
C_{(P'(Y);\text{ad})} & \xleftarrow{\text{ptr}} & \Omega_{\cdot}(Z)[1] \\
\xrightarrow{(\rho^C)^*} & \Omega_{\cdot}^{\text{hol}}(Y^C) & \xrightarrow{\text{Cauchy residue}} \Omega_{\cdot}(Z) \\
\text{Res} & & \text{Residue} \\
\end{array} \quad (11) \]

where \( \text{ptr} \) denotes the (graded) Poisson trace.
This suggests exactly two ways of constructing invariants of the graded Poisson algebra. One of them uses the canonical projection provided by the Cauchy residue while no canonical projection exists in the other case. Choosing this missing projection is equivalent essentially to fixing the differential form $dlogr$ where $r$ is an everywhere positive function on $Y$ of homogeneity 1. Apart from this difference the two approaches are roughly speaking equivalent.

We proved in 1.18 that $\text{Res}$ is a surjective map. The following lemma shows that this holds also for the graded Poisson trace.

1.26. Lemma. The map $\text{ptr}: \mathbb{C}(P'(Y);ad) + \Omega^*(Y)$ is surjective.

Proof. We have to prove that $\text{ptr}$ induces for every $k \in \mathbb{Z}$ a surjective map $\mathbb{C}^{(k)}(P'(Y);ad) + \Omega^{*,k+n}(Y)$. Obviously it suffices to do this for just one $k$, say $k = -n$. We already know that the composition

$$\mathbb{C}^{(-n)}(P'(Y);ad) \to \Omega^{*,0}(Y) \to \Omega^1(Y) \to \Omega^1(Z)$$

is surjective (see 1.18). In 1.18, in fact, we demonstrated also that $\rho^*\Omega^1(Z)[1]$, which is the kernel of the projection $\Omega^*,0(Y) + \Omega^*(Z)$, is spanned by forms $f_0 \cdots f_q \omega^n$ where $f_0' \cdots f_q'$ are homogeneous of the total order $q - n$. What remains, thus, is to show that Euler field $\Xi$ can be represented in the form

$$\Xi = \sum \Sigma \frac{h_a H_a g_a}{\alpha}$$

(12)

where $g_a, h_a \in P'$ and $\ell g_a + \ell h_a = 1$. Actually (12) is equivalent to the equality:

$$\alpha = -\sum \frac{h_a dq_a}{\alpha}$$

(13)

where $\alpha$ is the canonical 1-form introduced in 1.3. In view of 1.5 $\alpha$ is of weight 1 and thus it can be written as $hdr + r\phi$ where $r$ is everywhere non-zero and homogeneous of order 1, $h \in C^\infty(Z)$ and
\[ \phi \in \Omega^1(\mathcal{Z}). \] Since obviously both \( hrd \) and \( rs \) have required form (13), the proof is complete. (In fact, \( h = (i_r \alpha) r^{-1} = 0 \), in view of definition of \( \alpha \)). \[ \square \]

Lemma 1.26 implies the following refinement of Corollary 1.19.

1.27. **Corollary.** The graded Poisson trace induces an epimorphism

\[ H_1(C.(P'(Y));ad) \to H_1(\Omega.(Y)) \cong H^{2n-2}(\mathcal{Z}) \otimes H^{2n-1}(\mathcal{Z}). \]

1.28. **Remark.** The graded Poisson trace (9) (and its variants) play an important role in computing Hochschild homology of the associative algebras of differential operators and pseudodifferential symbols (cf. [20]). The point is that the space \( \Omega.(Y) \) is equal to the Hochschild homology of the suitable commutative algebra of homogeneous functions on \( Y \). For more details the reader is referred to [20].

For completeness sake we make a note of the variant 'with coefficients' of the symplectic residue construction.

1.29. **Symplectic residue with coefficients.** Fix a vector bundle \( \mathcal{H} \) on \( \mathcal{Z} \) and let

\[ P^\mathcal{H}_H = \{ s \in C^\infty(\mathcal{Y}, \rho^\mathcal{H}) \mid s(ty) = t^\mathcal{L}s(y); \ t \in \mathbb{R}^\mathcal{X}, \ y \in \mathcal{Y} \}, \]

and \( P^\mathcal{H}_H = \bigoplus_{l \in \mathcal{Z}} P^\mathcal{H}_H \). Choose a connection \( \nabla \) on \( \mathcal{H} \), then

\[ \text{C}_q(P'; P^\mathcal{H}_H) = P^\mathcal{H}_H \otimes \Lambda^2 P^\mathcal{H}_H \]

and

\[ \partial\nabla (s \otimes f_1 \cdots f_q) = \sum_{1 \leq h < j \leq q} (-1)^{h+j-1} s \otimes (f^h, f^j) \wedge f_1 \cdots \hat{f}_h \cdots \hat{f}_j \cdots \wedge f_q \]

\[ + \sum_{1 \leq h \leq q} (-1)^{h-1} \nabla^H f_h \otimes f_1 \wedge \cdots \wedge \hat{f}_h \wedge \cdots \wedge f_q \]

define the variant 'with coefficients' of the complex \( C.(P'; \text{ad}) \) (here \( \nabla^H f \equiv (\rho^\mathcal{H})_{\mathcal{H}_f} \)). Of course, \( C.(P'; P^\mathcal{H}_H) \) is a chain complex in the traditional sense of the word only when \( \nabla \) is integrable.
Another instance of a "complex with curvature" is supplied by the standard de Rham complex with coefficients in \( H \):

\[
\Omega_q(Z;H) = C^\infty(Z,\Omega^2_{2n-1})^*\), \quad d^\nabla = \text{id}_H \Theta d + \nabla \land \text{id}_H.
\]

We want to connect these two generalised complexes by a morphism (i.e. a graded map respecting boundary homomorphisms) similar to (4).

The definition of forms \( s; f_1, \ldots, f_q \) reproduces almost literally the definition from 1.15, so that the construction of the corresponding maps

\[
\text{Res}_q : C_q(P';P_H^') \to \Omega_q(Z;H)
\]

goes as before.

1.30. Proposition. The mappings \( \text{Res}_q(q = 0, 1, \ldots) \) form a morphism of chain complexes 'with curvature'

\[
\text{Res} : (C_*(P';P_H^'), d^\nabla) \to (\Omega_*(Z;H), d^\nabla).
\]

Proof. Notice that the two standard identities still hold for Lie derivative "with coefficients" \( L^\nabla : = \nabla \land \text{id}_H + \text{id}_H \land L^\nabla \), namely:

\[
L^\nabla = i_{\nabla} d^\nabla + d^\nabla i_{\nabla} \quad \text{and} \quad [L^\nabla, i_\zeta] = i_\nabla [\zeta].
\]

Hence the proof reduces to reproducing computation (2) with \( L^\nabla_\eta \) replaced by \( L^\nabla_\eta \).

1.31. Final remark. In the course of work with symbols of \( \psi \)DOs it will be very convenient to extend \( \text{Res} \) to a slightly bigger complex

\[
C_*(P';P_H^') \quad \text{defined as:}
\]

\[
\hat{C}_q(P';P_H^') = \left\{ \sum_{q=1}^{\infty} s^\vee \Phi_1^\vee \ldots \Phi_q^\vee \mid \text{ords}^\vee + \text{ord}_1^\vee + \ldots + \text{ord}_q^\vee \to \infty \text{ as } v \to \infty \right\}.
\]
§2. Noncommutative residue (local theory)

We will define for an arbitrary pseudodifferential operator of classical type a certain global density which will be called its noncommutative residue. Throughout this section, however, we are occupied only with local considerations. The general case is postponed to the next section.

2.1. Fix a domain \( U \subset \mathbb{R}^n \) and let \((u^1, \ldots, u^n; \xi_1, \ldots, \xi_n)\) be the corresponding coordinates in \( T^*U = U \times \mathbb{R}^n \).

Constructions of the previous section will be applied to the following special case

\[
Y = T^*_0U = T^*U \setminus U \quad \text{and} \quad Z = S^*U
\]

\((R_+^n)\) acts as a homothety in \( \xi\)-space), and

\[
\omega = \sum_{i=1}^{n} d\xi_i ^1 \wedge du^i, \quad \Xi = r \frac{\partial}{\partial r} \quad (\text{where } r = |\xi|),
\]

\[
\alpha = \sum_{i=1}^{n} \xi_i ^1 du^i,
\]

\[
\omega^n = (-1) \frac{n(n-1)}{2} n!d\xi_1 ^\wedge \cdots \wedge d\xi_n ^\wedge du^1 \wedge \cdots \wedge du^n = (-1) \frac{n(n-1)}{2} n!d\xi du^n,
\]

\[
\mu = (-1) \frac{n(n-1)}{2} (n-1)!(*rdr) \wedge du^1 \wedge \cdots \wedge du^n
\]

\[
= (-1) \frac{n(n-1)}{2} (n-1)! \left( \sum_{i=1}^{n} (-1)^{i-1} \xi_i d\xi_1 ^\wedge \cdots \wedge d\xi_i ^\wedge \cdots d\xi_n ^\wedge \right) du^1 \wedge \cdots \wedge du^n
\]

\[
= \beta_n^{-1} r d\xi (r) du
\]

where \( d\xi (r) \) is the volume form divided by \( (2\pi)^n \), of the sphere of radius \( r \) in \( \mathbb{R}_\xi^n \) and, \( \beta_n \) will hereafter denote the constant
The described situation possesses one extra feature which we will take advantage of: \( Y \) is a Lagrangean fibre bundle over \( U \) with its projection fitting into the triangle

\[
Y = T^*U
\]

\[
\pi \quad \rho
\]

\[
U \quad \tau
\]

The following lemma will be often used. Assume we are given a vector bundle \( G \) on \( U \) and a homogeneous section of \( \pi^*G \) of order \( 1 - n \).

2.2. Lemma. One has

\[
\text{Res} \left\{ \frac{2}{\beta_j} s \right\} = d_{\text{fibre}} \gamma_j \wedge du^1 \wedge \ldots \wedge du^n \quad (j = 1, \ldots, n)
\]

where \( \gamma_j \in C^\infty(U, G \otimes \Omega^{n-2} Z/U) \) is a relative \( (n-2) \)-form determined by the equality

\[
\pi^* \gamma_j = (-1)^{n(n-1)/2 + j+1} (n-1)! s \otimes \left[ \sum_{i=1}^{j-1} (-1)^i \xi_i d\xi_1 \wedge \ldots \wedge d\xi_i \wedge \ldots d\xi_j \wedge \ldots d\xi_n \right.
\]

\[
- \sum_{i=j+1}^n (-1)^i \xi_i d\xi_1 \wedge \ldots \wedge d\xi_j \wedge \ldots d\xi_i \wedge \ldots d\xi_n \right],
\]

and \( d_{\text{fibre}} \) denotes de Rham differential along fibres of the projection \( \tau \).

Proof. Take any connection \( \nabla^G \) on \( G \) and put \( H = \tau^*G, \nabla = \tau^*\nabla^G \). According to 1.30 the boundary homomorphism

\[
\partial^\nabla : C_1(P'; P^*_H) \rightarrow C_0(P^*; P^*_H)
\]
is related to de Rham differential via the equality \( \text{Res}_0 \partial^\nabla = d^\nabla \text{Res}_1 \). Since the connection \( \nabla \) is trivial along fibres of \( \tau \) and 
\[ H^j = -\partial / \partial \xi_j, \]
one obviously has 
\[ \partial^\nabla (s \otimes u^j) = -\partial / \partial \xi_j. \]
Hence
\[
\text{Res}_0(\partial / \partial \xi_j s) = -d^\nabla \text{Res}_1(s \otimes u^j) = d^\nabla(s \otimes i_\partial / \partial \xi_j u)
\]
\[
= (-1)^n (n-1)! d^\xi (s \otimes i_\partial / \partial \xi_j u)^{(rdr)} \wedge du^1 \wedge \ldots \wedge du^n
\]
and the assertion follows. \[\square\]

2.3. Corollary. \( \tau_* \bigg| \text{Res}_0 \left( \frac{\partial}{\partial \xi_j} \right) s = 0 \quad (j = 1, \ldots, n). \) \[\square\]

Here \( \tau_* \) denotes the push-out map \( A_* (\mathbb{Z}; \tau^* \mathbb{G}) \rightarrow A_* (\mathbb{U}; \mathbb{G}) \) and \( \big| \big| \) denotes the canonical identification \( \mathbb{N}_* (\mathbb{Z}; \tau^* \mathbb{G}) \cong A_* (\mathbb{Z}; \tau^* \mathbb{G}), \) (cf. 0.17 and 0.19).

Return to the scalar case. We make two elementary observations:

2.4. \( \text{Res}_q(f_0 \otimes f_1 f_2^1 \wedge f_2^2 \wedge \ldots \wedge f_q) = \text{Res}_q(f_0 f_1 \otimes f_2^1 \wedge f_2^2 \wedge \ldots \wedge f_q) + \)
\[
\text{Res}_q(f_0 f_1 \otimes f_2^1 \wedge f_2^2 \wedge \ldots \wedge f_q).
\]
2.5. \( \text{Res}_q(f_0 \otimes f_1 f_2^1 \wedge f_2^2 \wedge \ldots \wedge f_q) = \text{Res}_q(f_0 \otimes f_1 \wedge f_2^1 \wedge f_2^2 \wedge \ldots \wedge f_q) + \)
\[
\text{Res}_q(f_0 \otimes f_1^1 \wedge f_2^2 \wedge \ldots \wedge f_q).
\]
which are valid for a general symplectic cone \( Y \) and \( f_i, f'_i \in P^*(Y) \).

Identity 2.4 follows from:
\[ i_{f f'_i} = i_{f f'_i} + f'_i i_{f'}, \]
and 2.5 also from:
\[ i_{f f'_1 f'_2} = i_{f} i_{f'_1 f'_2} + i_{f'_1} i_{f'_2}. \]
The former identity implies, in particular, that:

\[ \text{Res}_q(f_0 \Theta f_1 \wedge f_2 \wedge \ldots \wedge f_q) + \text{Res}_q(f_1 \Theta f_0 \wedge f_2 \wedge \ldots \wedge f_q) = \]
\[ \text{Res}_q(l \Theta f_0 \wedge f_1 \wedge \ldots \wedge f_q) . \]

2.6. Lemma. For any set of homogeneous functions \( f_1, \ldots, f_q \) on \( T^*U \), one has

\[ \tau_*|\text{Res}_q(l \Theta f_1 \wedge \ldots \wedge f_q)| = 0 . \]

Therefore \( \tau_*|\text{Res}_q(f_0 \Theta f_1 \wedge \ldots \wedge f_q)| \) is skew-symmetric in all entries.

**Proof.** Introduce an abbreviated notation:

\[ i_j = \partial_j, \quad \partial_j f = \partial_j f. \]

It follows directly from the definition of \( \text{Res}_q \) that:

\[ \rho^*\text{Res}_q(1 \Theta f_1 \wedge \ldots \wedge f_q) = i_{f_1} \ldots i_{f_q} = \sum_{j_1, \ldots, j_q = 1}^{n} i_{j_1} \ldots i_{j_q} \partial_j f_1 \ldots \partial_j f_q \]
\[ + \sum_{k=1}^{n} i_{\partial_k \xi_k} \omega_k \]

where \( \omega_k \) are certain horizontal forms (i.e. \( i_{\xi_k} \omega_k = 0 \)) on \( Y = T^*U \) of weight 1.

The forms \( r^{-1} \omega_k \) \( (r = |\xi|) \) descend to uniquely defined forms \( \omega_k \) on \( Z = S^*U \), and the vector fields \( \partial / \partial \xi_k \) possess well-defined projections \( \Xi_k := \rho_*(r \partial / \partial \xi_k) \), hence the equalities

\[ i_{\partial / \partial \xi_k} \omega_k = \rho^*(i_{\Xi_k} \omega_k) \quad (k = 1, \ldots, n) \]

hold.

Next, let us notice that, because \( i_{k,\ell} (\partial^2_{k\ell} f_1 \ldots f_q) \) is \((k,\ell)-\)

skew-symmetric, we must have:
\[ \sum_{j_1=1}^{n} \sum_{j_2=1}^{n} \cdots \sum_{j_q=1}^{n} \left( \partial_{j_q} f_1 \partial_{j_1} f_2 \partial_{j_2} f_3 \cdots \partial_{j_q} f_q \right) \cdot \left( \partial_{j_q} f_1 \partial_{j_1} f_2 \partial_{j_2} f_3 \cdots \partial_{j_q} f_q \right) = \]
\[ \sum_{j_1=1}^{n} \sum_{j_2=1}^{n} \cdots \sum_{j_q=1}^{n} \left( \partial_{j_q} f_1 \partial_{j_1} f_2 \partial_{j_2} f_3 \cdots \partial_{j_q} f_q \right) \cdot \left( \partial_{j_q} f_1 \partial_{j_1} f_2 \partial_{j_2} f_3 \cdots \partial_{j_q} f_q \right) . \]

An inductive application of this argument yields the identity:

\[ \sum_{j_1, \ldots, j_q=1}^{n} \partial_{j_1} \cdots \partial_{j_q} (f_1 \phi_{j_1} f_2 \cdots \phi_{j_q} f_q) \cdot \left( f_1 \phi_{j_1} f_2 \cdots \phi_{j_q} f_q \right) \]
\[ = \sum_{j_1, \ldots, j_q=1}^{n} \partial_{j_1} \cdots \partial_{j_q} (f_1 \phi_{j_1} f_2 \cdots \phi_{j_q} f_q) \cdot \left( f_1 \phi_{j_1} f_2 \cdots \phi_{j_q} f_q \right) \]
\[ = \sum_{j_1, \ldots, j_q=1}^{n} \partial_{j_1} \cdots \partial_{j_q} \left( z_{j_1} \phi_{j_1} f_2 \cdots \phi_{j_q} f_q \right). \]

Thus we obtain

\[ \text{Res} (\sum_{1}^{n} f_1 \wedge \cdots \wedge f_q) = \sum_{j_1, \ldots, j_q=1}^{n} \partial_{j_1} \cdots \partial_{j_q} \left( z_{j_1} \phi_{j_1} f_2 \cdots \phi_{j_q} f_q \right) + \sum_{k=1}^{n} i_j \omega_k . \]

We will apply to both sides the operation \( \tau_* | \cdots | \). Recall that \( \tau_* \) is a composition of two operations - the projection:

\[ A_* (Z) \to A_* (U; \tau_* A_0, \text{rel}) , \]

and the integration:

\[ \int : \tau_* A_0, \text{rel} \to \partial_U^\infty . \]

(\( A_\text{rel} \) denotes the sheaf of relative integral forms \( A_{Z/U} \)). Since the fields \( z_{j_k} \) are tangent to fibres of \( \tau \) we infer the application of (3) alone kills the second sum in (2). The first sum vanishes after the integration in view of Lemma 2.2 and the observation that \( \tau_* \) commutes with \( i_j \)'s.
2.7. Remark. Lemma 2.2 allows us to write the image of \( \text{Res}(1 \circ f_1 \wedge \ldots \wedge f_q) \) under (3) as an explicit fibre-wise exact form.

2.8. Lemma 2.6 implies, in particular, commutativity of the following diagram of morphisms of chain complexes:

\[
\begin{array}{ccc}
\mathbb{C}(P'(Y); \text{ad}) & \xrightarrow{\text{Res}} & \Omega(U) \\
\tau^* \downarrow & & \downarrow \tau^* \\
\mathbb{C}(P'(Y))[-1] & \xrightarrow{R} & \Omega(U)
\end{array}
\]

(4)

where \( R \) is defined as:

\[
R_q(f_0 \wedge \ldots \wedge f_q) = \tau^* \vert \text{Res}_q(f_0 \circ f_1 \wedge \ldots \wedge f_q) \vert.
\]

Notice that (4) is precisely a higher-dimensional version of factorisation 1.24. Another important property of \( R \)-morphism is its \( C^\infty(U) \)-linearity:

2.9. \( R_q(g \circ f_0 \wedge f_1 \wedge \ldots \wedge f_q) = g R_q(f_0 \wedge f_1 \wedge \ldots \wedge f_q) \) (\( g \in C^\infty(U) \)).

Indeed, by definition we have

\[
\text{Res}_q(g \circ f_0 \wedge f_1 \wedge \ldots \wedge f_q) = g \text{Res}_q(f_0 \circ f_1 \wedge \ldots \wedge f_q)
\]

for an arbitrary \( g \in C^\infty(U) \), while the multiplication by \( g \in C^\infty(U) \) commutes with \( \tau^* \mid \ldots \mid \).

Lemmas 2.6. and 2.9. together imply:

2.10. \( R_q(f_0 \wedge \ldots \wedge f_q) = 0 \) if at least one \( f_i \) belongs to \( C^\infty(U) \).

2.11. \( \sum_{i=0}^{j} R_q(f_0 \wedge \ldots \wedge \frac{\partial f_i}{\partial \xi_j} \wedge \ldots \wedge f_q) = 0 \) (\( j \in \{1, \ldots, n\} \)).

Only the latter identity, perhaps, requires any explanation. It follows easily from:

\[
\sum_{i=0}^{j} f_0 \wedge \ldots \wedge \frac{\partial f_i}{\partial \xi_j} \wedge \ldots \wedge f_q = u_j \wedge \frac{\partial}{\partial f_0 \wedge \ldots \wedge f_q} - \frac{\partial}{\partial (u^j \wedge f_0 \wedge \ldots \wedge f_q)},
\]

(5)
identity 2.10 and the equality \( R_q^\alpha = -dR_{q+1} \). (To avoid confusion with partial derivatives or with other boundary maps the boundary in the complex \( C(\mathcal{P}') \) is denoted in (5) and in the rest of this section by \( \partial \).)

Armed with the morphism \( R \) we proceed to the definition of the residue for pseudodifferential symbols.

2.12. Consider an arbitrary \( \Psi DO \) with matric coefficients 
\[ A : C^\infty_\text{comp}(U, \theta^k) \to C^\infty_\text{comp}(U, \theta^\omega) \]
and of order \( m \). Its complete symbol is a formal series
\[ a(u, \xi) = \sum_{j=0}^{\infty} a_{m-j}(u, \xi) \]
such that each component \( a_{m-j} \in C^\infty_\text{comp}(T^*_0 U, \text{Hom}(\theta^k, \theta^\omega)) \) is homogeneous of order \( m - j \).

2.13. Definition. \( \text{res}_u A = \left\{ \int_{|\xi|=1} a_{-n}(u, \xi) |d\xi'| \right\} |du| \)
\((d\xi' = d\xi'(1) \) denotes the normalised volume form on the unit sphere).

If \( \text{ord} A \not\in \mathbb{Z} \) we put \( \text{res}_u A = 0 \).

By definition \( \text{res}_u A \) is a matric-valued density on \( U \). An alternative expression for \( \text{res}_u A \) is
\[ \text{res}_u A = \beta_n \tau_* |\text{Res}_0 a| \]
\((\tau : S^* U \to U)\)
where \( \text{Res}_0 \) is the residue map of 1.31 in the case:

\[ H = \text{Hom}(\theta^k, \theta^\omega) \]
and \( \nabla = \text{canonical flat connection on} \ H. \)

The density \( \text{res}_u A \) will be called the noncommutative residue of \( A \).

2.14. Remark. For \( n = 1 \) a slightly different density will be needed.
In this case \( S^* U \) consists of two copies of \( U \) and it can be
canonically identified with the orientation double cover of $U$. The volume form $\xi'$ reduces to

$$d\xi' = \frac{1}{2\pi} \left| d\nu \right| d\nu,$$

and $(2\pi)^{-1} a_{-1}(u,\xi)|du| = a_{-1}(u,\xi)d\xi' du$ when restricted to $\{\xi = \pm1\}$ can be viewed as a density on $U$ with coefficients in $
abla(\theta^k, \theta^\nu) \otimes S$

$(S = \tau. \theta^1_{S*U}$ is a rank 2 vector bundle on $U$, $\tau$ denotes the direct image).

This density will be denoted by $\text{res}_u A$. Choosing orientation identifies $S$ with $\theta^2_U$ and $\text{res}_u A$ with a 2-component density:

$$\left\{ \begin{array}{l} \text{res}_+, u A \\ \text{res}_-, u A \end{array} \right\}.$$

2.15. Let $f : V \rightarrow U$ be an open embedding of another domain $V \subset \mathbb{R}^n$ (possibly $V = U$, $f$ being a diffeomorphism). Then the induced operator $f^{#} A : C^\infty_{\text{comp}}(V, \theta^k) \rightarrow C^\infty(V, \theta^\nu)$ is defined (cf. 0.5).

The following lemma asserts the functoriality of $\text{res}_u A$ with respect to open embeddings of domains.

2.16. Lemma. $\text{res}_u f^{#} A = f^{*} \text{res}_u A$.

Proof. Denote the complete symbol of $f^{#} A$ by $a^f$. According to the 2nd form of the base change formula (cf. 0.14) we have

$$a^f = (T^f)^* \left\{ (1 + \sum_{j=1}^n \partial_{\xi_j} \circ \psi_j) a \right\},$$

where $\psi_j = \psi_j^f(u, \xi, \partial_{\xi})$ are some differential operators of infinite order (depending on $f$).

On the other hand, in virtue of the identities
\[(T^*f)^* \left( \sum_{i=1}^{n} d\xi_i \wedge du_i \right) = \sum_{i=1}^{n} d\eta_i \wedge dv_i, \quad (T^*f)_* \xi^V = \xi^U\]

where \(\xi^U, \xi^V\) are the Euler fields on \(T^*U\) and \(T^*V\) respectively, it follows that

\[\text{Res}_V(T^*f)^* = (S^*f)^* \text{Res}_U^0.\]

Therefore

\[\text{Res}_0^0(a^f) = (S^*f)^* \text{Res}_0^0 \left( (1 + \sum_{j=1}^{n} \partial_{\xi_j} \circ \psi_j) a \right),\]

and

\[\text{res}_V^0 a^f \# A = \beta_n \tau^V_*(S^*f)^* \left| \text{Res}_0^0 \right| (1 + \sum_{j=1}^{n} \partial_{\xi_j} \circ \psi_j) a | | = \beta_n \left( \tau^V_*(S^*f)^* \right) \left| \text{Res}_0^0 \right| (1 + \sum_{j=1}^{n} \partial_{\xi_j} \circ \psi_j) a | | = \beta_n \left( f^* \tau^U_0 \right) \left| \text{Res}_0^0 \right| (1 + \sum_{j=1}^{n} \partial_{\xi_j} \circ \psi_j) a | | = f^* \text{res}_U^0 A + \beta_n f^* \tau^U_0 | | \text{Res}_0^0 \right| \left( \sum_{j=1}^{n} \partial_{\xi_j} \circ (\psi_j a) \right) | | (6)\]

(we used here the formula for change of variables under the sign of integral: \(\tau^V_*(S^*f)^* = f^* \tau^U_0\) (cf. 0.18)). According to Lemma 2.2 the second term in (6) vanishes since \(\text{Res}_0^0 \sum_{j=1}^{n} \partial_{\xi_j} \circ (\psi_j a)\) belongs to \(\Omega^n(U, \tau^* d\text{fibre}^n S^*U/U)\), i.e. is fibre-wise exact.

2.17. Let \(r \in C^\infty(U, \text{Hom}(\theta^k, \theta^\ell))\) and \(s \in C^\infty(U, \text{Hom}(\theta^\ell, \theta^k'))\) be arbitrary matric functions. In particular, for \(A : C^\infty_{\text{comp}}(U, \theta^k) \rightarrow C^\infty(U, \theta^\ell)\) the operator \(s \cdot A \circ r : C^\infty_{\text{comp}}(U, \theta^k) \rightarrow C^\infty(U, \theta^\ell')\) is well defined.

2.18. Lemma. \(\text{res}_U^0 (s \cdot A \circ r) = s(\text{res}_U^0 A) r.\)

(\(\text{res}_U^0 A\) is a density with coefficients in \(\text{Hom}(\theta^k, \theta^\ell)\) so that its composition with \(s\) on the left and \(r\) on the right makes sense).
Proof. Let \( a(r,s) \) be the complete symbol of \( s^* A^r \), then by the composition formula (cf. 0.12) \( a(r,s) \) can be written as

\[
a(r,s) = s \sum_{j=1}^{n} \partial_{\xi_j} (\phi_j a)
\]

where \( \phi_j = \phi_j^r(u, \partial_{\xi_j}) \) are differential operators of infinite order (depending on \( r \)) defined from:

\[
\sum_{j=1}^{n} \partial_{\xi_j} \circ \phi_j \equiv \sum_{|\alpha| \geq 1} \frac{1}{\alpha!} \partial_{\xi}^\alpha a \partial^\alpha_{u^r}.
\]

Another application of Lemma 2.2 yields:

\[
\text{res}_{u}(s^* A^r) - s(\text{res}_{u} A)r = s_* \sum_{j=1}^{n} \partial_{\xi_j} (\phi_j a) = 0.
\]

2.19. Finally, let \( \phi = (f;r,s) \) be a general morphism between two triples \((V; \theta^k, \theta^k')\) and \((U; \theta^k, \theta^k)\) (cf. 0.3).

Lemmas 2.16 and 2.18 together imply the functoriality of the residue density under such morphisms.

2.20. Proposition. \( \text{res}_v(\phi^* A) = \phi^* \text{res}_u A \).

2.21. Remark. In fact the proofs of both Lemma 2.13 and 2.15 give more than just identity 2.20. They produce an explicit relative \((n-2)\)-form \( \gamma \) on \( V \) (depending on \( \phi \)) such that

\[
\text{Res}_0 a_{\phi} - (Sf)^* (\text{res}_0 A) = d_{\text{fibre}} \gamma \wedge dv^1 \wedge \ldots \wedge dv^n.
\]

(Here \( a_{\phi} \) denotes the complete symbol of \( \phi^* A \)).

2.22. The noncommutative residue via the amplitudes. Recall that a \( \psi \text{DO} \ A : C^\infty_{\text{comp}}(U\theta^k) \rightarrow C^\infty(U, \theta^k) \) possesses many different representations in the form:

\[
Af(u) = \int_{k} \int_{U} e^1(u-v, \xi) \beta(u,v, \xi) f(v) dv d\xi + Tf(u)
\]

with property that the amplitude \( \beta \) admits the standard quasi-classic expansion:
and $T$ is an operator with smooth kernel. If the dependence on $(u,v)$ is through the linear combination $(1-t)u + tv$ ($t$ being fixed real number):

$$\beta(u,v,\xi) \equiv \alpha((1-t)u + tv,\xi)$$

then $\alpha(u,\xi)$ is called a (smoothed) t-symbol of $A$. For $t = 0$ we obtain an ordinary (smoothed) symbol.

For completeness sake we should mention that even if the smoothing operator $T$ can also be made an oscillatory integral like the first term in (7) its amplitude may not be reducible to a 'symbol-like' one.

When appealing to amplitudes or symbols we shall mean, however, (if not stated otherwise) the formal amplitudes and formal symbols, which are formal series with homogeneous components as in (8). If so, then a standard assertion from the $\Psi$DO theory (cf. e.g. [15], IV.23.3-5) says that a $t$-symbol exists and is unique (for each $t$), and that the ordinary symbol $\alpha(u,\xi)$ is related to an arbitrary amplitude $b(u,v,\xi)$ by the identity

$$a = (e^{-L} b)|_{T_0^*U}$$

where $L$ denotes the second order differential operator on $T^*U$

$$L = \sqrt{-1} \sum_{i=1}^{n} \frac{\partial^2}{\partial \xi_i \partial u_i}$$

(the letter $v$ indicates that in (9) $L$ acts on the variables $(v, \xi)$), and $T_0^*U \subset U \times T_0^*U$ consists of points $(u,u,\xi)$ with $\xi \neq 0$; the exponent $e^{-L}$ in (9) should be meant as a differential operator of infinite order).

Moreover, $t$-symbols $a(t)$ for two different $t$'s are related to each other by the identity

$$a_1 = e^{(t_1-t_2)L} a_2$$

(11)
which is a particular case of (9).

In applications apart from the ordinary (left) symbol appear usually only Weyl symbol \((t = \frac{i}{4})\) and the right symbol \((t = 1)\).

Assume we are given some amplitude \(b\). Then we can apply \(\text{Res}_0\) to its restriction to \(T^*_0 \Delta\). This turns out to give another method of calculating the noncommutative residue of the operator.

### 2.23. Lemma

a) For an arbitrary amplitude \(b\) of a \(\psi\text{DO} A\) one has

\[
\text{res}_u A = \beta_n T^*_1 \text{Res}_0 (b|_{T^*_0 \Delta})
\]

b) Similarly, for an arbitrary \(t\)-symbol \(a(t)\) of \(A\) one has:

\[
\text{res}_u A = \beta_n T^*_1 \text{Res}_0 a(t)
\]  \(\quad (12)\)

In particular, the right-hand side of (12) does not depend on \(t\).

**Proof.** One has

\[
e^{-L} - 1 = \sum_{i=1}^{n} \partial_i K(i)
\]

where

\[
K(i) := D_i T(L)
\]

\[
(D_i = \frac{1}{\sqrt{-1}} \partial_i u^i)
\]

and

\[
T(L) := \frac{1 - e^{-L}}{L}
\]

is a differential operator on \(T^*_U\) of infinite order (notice that \(T(X)\) is precisely the Todd series).

In particular, the difference between the ordinary symbol of \(A\) and \(b|_{T^*_0 \Delta}\) equals (cf. (9)):

\[
a - (b|_{T^*_0 \Delta}) = (e^{-L} - 1) b|_{T^*_0 \Delta} = \sum_{i=1}^{n} \partial_i (K(i) b|_{T^*_0 \Delta})
\]
Appealing to Lemma 2.2 finishes the proof of part a). Part b) is proved similarly but with (11) used instead of (9).

2.24. Let \( B : C^\infty_{\text{comp}}(U, \theta_\mathcal{X}) \rightarrow C^\infty(U, \theta_\mathcal{K}) \) be another \( \psi\text{DO} \) and assume either \( A \) or \( B \) to be a proper \( \psi\text{DO} \) (cf. 0.10). Then the compositions \( AB \) and \( BA \) are both defined. In particular, there are two densities:

\[
\text{res}_u(AB) \in A_0(U; \text{End } \theta_\mathcal{X}) \quad \text{and} \quad \text{res}_u(BA) \in A_0(U; \text{End } \theta_\mathcal{K}) .
\]

Our aim is to demonstrate that the \textit{scalar} densities \( \text{tr res}_u(AB) \) and \( \text{tr res}_u(BA) \) actually coincide up to a certain explicit exact form. We consider first the case when both \( A \) and \( B \) are scalar operators. The matric case will follow easily from the scalar one.

Make some auxiliary remarks before we state the result. It will be convenient to extend \( \mathbb{R}\text{-morphism } \hat{C}(p'Y)[-1] \rightarrow A_0(U) \) to the formally completed complex \( \hat{C}(p'Y)[-1] \) much as we did in the case of symplectic residue morphism (cf. 2.31).

Recall that

\[
\hat{C}_q = \left\{ \sum_{\nu=1}^{\infty} f_{\nu}^{(1)} \wedge \ldots \wedge f_{\nu}^{(q)} \left| \text{ord } f_{\nu}^{(1)} + \ldots + \text{ord } f_{\nu}^{(q)} \rightarrow -\infty \quad \text{as } \nu \rightarrow \infty \right. \right\}.
\]

For basically aesthetic reasons, we shall assume orders of all operators considered to be integers (cf. however, Remark 2.26).

2.25. Lemma. Let \( A \) and \( B \) be scalar operators. Then one has:

\[
\text{res}_u[A,B] = \varphi_0(A,B)
\]

where \( \varphi_0(A,B) \in A_1(U) \) is defined as

\[
\varphi_0(A,B) = \sqrt{-1} \beta_1 R_1 \left\{ \sum_{i,j=0}^{\infty} \frac{1}{(1+i+j)!) L^i a^j b^j} \right\} . \tag{13}
\]

(\( a \) and \( b \) denote the symbols of \( A \) and \( B \) respectively, and \( L \) is the second order operator (10) on \( T^*_U \)).
2.26. Remark. We deal only with the case of integer order operators but since the differential forms \( \sum f^0, f^1, \ldots, f^q \) of 1.15 are defined under the sole condition that \( \ell_{f^0} + \cdots + \ell_{f^q} \) equals \(-n\), and crucial identity 1.16. still holds in that situation, one can easily extend both Lemma 2.25 and all considerations below to operators with any complex order. Of course, \( \rho_u(A, B) \) can be non-zero only when \( \text{ord } A + \text{ord } B \in \mathbb{Z} \).

Proof. We shall need the completed tensor square \( P^* \otimes P^* \) which consists of series \( \sum_{v=1}^{\infty} f^1_v \otimes f^2_v \) satisfying the condition
\[
\ell_{f^1_v} + \ell_{f^2_v} \to -\infty \text{ as } v \to -\infty
\]
and the similarly completed Poisson algebra \( \hat{P}^* \); the usual commutative multiplication \( f \otimes g \rightarrow fg \) defines then a map \( m : \hat{P}^* \otimes \hat{P}^* \rightarrow \hat{P}^* \). The simple identity
\[
m \circ (1 \otimes L - L \otimes 1) = \frac{1}{\sqrt{-1}} \{.,.\} + \frac{1}{\sqrt{-1}} \sum_{i=1}^{n} \delta_{\xi_i} \cdot m \cdot (\delta_{\xi_1} \otimes 1 - 1 \otimes \delta_{\xi_1})
\]
and Lemma 2.2 imply together commutativity of the diagram:
\[
\begin{array}{ccc}
P^* \otimes P^* & \longrightarrow & \hat{P}^* \\
1 \otimes L - L \otimes 1 & \downarrow & \text{R}_0 (U) \\
P^* \otimes P^* & \longrightarrow & \hat{P}^* \\
\end{array}
\]
(Here \( \{.,.\} \) is the Poisson bracket).

The operator \( AB \) possesses one especially nice amplitude:
\[
c(u, v, \xi) := a(u, \xi) b^{(1)}(v, \xi)
\]
(15)
where \( b^{(1)} \) is the right symbol (cf. 2.19) of \( B \). Similarly, \( AB - BA \) possesses the amplitude:
\[
\hat{c}(u, v, \xi) = a(u, \xi) b^{(1)}(v, \xi) - b(u, \xi) a^{(1)}(v, \xi)
\]
Restrict it to \( T^*_0 A \) and then apply formula (11) to obtain
\[ \tilde{c}_{T_0^*} = ab(1) - a(b) = m(1 \otimes e^L - e^L \otimes 1)(a \otimes b). \]

In the ring of formal power series \( \mathcal{G}[L] \otimes \mathcal{G}[L] \) the element \( 1 \otimes e^L - e^L \otimes 1 \) decomposes as
\[
1 \otimes e^L - e^L \otimes 1 = (1 \otimes L - L \otimes 1) \sum_{i,j=0}^{\infty} \frac{1}{(1+i+j)!} L^i \otimes L^j.
\]

Hence we have
\[
\text{res}_u AB - \text{res}_u BA = \beta_n R_0 \tilde{c}_{T_0^*}
\]
\[
= \beta_n R_0 \left[ m(1 \otimes L - L \otimes 1) \sum_{i,j=0}^{\infty} \frac{1}{(1+i+j)!} L^i a \otimes L^j b \right]
\]
\[
= \beta_n R_0 \left[ \sum_{i,j=0}^{\infty} \frac{1}{(1+i+j)!} L^i a \otimes L^j b \right]
\]
\[
= \sqrt{-1} \beta_n d \left[ \sum_{i,j=0}^{\infty} \frac{1}{(1+i+j)!} L^i a \otimes L^j b \right]
\]
as required (recall that in \( \hat{\mathcal{C}} (P')[-1] \) all boundary morphisms have opposite sign). \( \square \)

2.27. Let us extend the definition of the integral form (13) to general matric \( \psi \) DOs:
\[
A = (A^\lambda) , \quad B = (B^\kappa) \quad \quad (\kappa = 1, \ldots, k; \lambda = 1, \ldots, \ell)
\]
by putting
\[
\rho_u (A,B) = \sqrt{-1} \beta_n R_1 \left[ \sum_{i,j=0}^{\infty} \frac{1}{(1+i+j)!} L^i a^{\lambda \kappa} \otimes L^j b^{\kappa \lambda} \right].
\]

Lemma 2.25 implies then immediately

2.28. Proposition. \( \text{tr res}_u AB - \text{tr res}_u BA = d \rho_u (A,B). \)
2.29. Remark. Let \( F(X,Y) \in \mathbb{C}[[X,Y]] \) denote the formal series:

\[
F(X,Y) = \frac{e^X - e^Y}{X - Y}.
\]  

(16)

Make substitutions \( X = 1 \otimes L \) and \( Y = L \otimes 1 \) and regard

\[ F: = F(1 \otimes L, L \otimes 1) \]

as a linear map \( P' \otimes P' \rightarrow P' \otimes P' \). Then (13) reads as

\[
\rho_u(A,B) = \sqrt{-1} \beta_n(R_1 \circ F)(a \otimes b).
\]

(17)

Since

\[
F(X,Y) = A(Z)e^{T/2} \quad \text{(Z:= X - Y, T:= X + Y)}
\]

and

\[
e^{\frac{i}{2}(1 \otimes L + L \otimes 1)} (a \otimes b) = a^W \otimes b^W
\]

where \( a^W \) and \( b^W \) are Weyl symbols of operators \( A \) and \( B \) respectively (cf. (11)) and \( \hat{A}(Z) \) is the standard \( \hat{A} \)-series

\[
\hat{A}(Z) = \frac{\sinh (Z/2)}{(Z/2)}.
\]

one can rewrite (17) also in the form:

\[
\rho_u(A,B) = \sqrt{-1} \beta_n(R_1 \circ \hat{A})(a^W \otimes b^W)
\]

\( (\hat{A} = \hat{A}(1 \otimes L - L \otimes 1)) \). In particular, Lemma 2.25 asserts nothing more than commutativity of the following diagram:

\[
\begin{array}{ccc}
\hat{P}' & \overset{[\cdot, \cdot]_{\text{Weyl}}}{\longrightarrow} & \hat{P}' \\
\sqrt{-1} R_1 \circ \hat{A} & \overset{}{\longrightarrow} & R_0 \\
A_1(U) & \overset{d}{\longrightarrow} & A_0(U)
\end{array}
\]

(18)

Here \([\cdot, \cdot]_{\text{Weyl}}\) denotes commutator defined by the composition law of Weyl symbols.
Recall that we have a similar commutative diagram but with \([,]\) replaced by Poisson bracket:

\[
\begin{array}{ccc}
\mathbb{P}^* \Phi \mathbb{P}^* & \{,\} & \hat{\mathbb{P}}^* \\
R_1 & \downarrow & R_0 \\
A_1(U) & \longrightarrow & A_0(U)
\end{array}
\]

If one interprets vertical arrows in (18) and (19) as "non-commutative integrations" along fibres of the projection \( S^* U \times S^1 \to U \) (integration over the circle corresponds to taking Cauchy residue, cf. diagram (11) of previous section) which are defined either in terms of the full commutator of symbols or of only its principal part (i.e. in terms of Poisson bracket) then one reaches the conclusion that the difference between the two is roughly speaking, the "\(\hat{A}\)-genus".

In light of still another evidence (referring to Hochschild homology of differential operators and pseudodifferential symbols, cf. [20]) it is inspiring to regard the observed phenomenon as a manifestation of the "noncommutative" Riemann-Roch.

2.30. Functorial properties of the form \( \rho_U \). Assume there are given two matrix operators \( A \) and \( B \) and two matrix functions \( s \) and \( r \) such that the compositions \( s \circ A \), \( A \circ r \), \( r \circ B \) and \( B \circ s \) are all well defined. In other words:

\[
s \in \text{Mat}_{\ell^2_2,\ell^1_1}(C^\infty(U)), \quad r \in \text{Mat}_{k^1_1,k^2_2}(C^\infty(U)), \quad (k^1_1,k^2_2,\ell^1_1,\ell^2_2 \in \mathbb{Z}_+)
\]

and

\[
A : C^\infty_{\text{comp}}(U,\theta^{k^1_1}) \to C^\infty(U,\theta^{\ell^1_1}), \quad B : C^\infty_{\text{comp}}(U,\theta^{k^2_2}) \to C^\infty(U,\theta^{\ell^2_2})
\]
(as before we assume either A or B to be proper).

2.31. Proposition. \( \rho_u (s \circ A \circ r, B) = \rho_u (A, r \circ B \circ s) \).

Proof. It is clearly enough to prove the statement in the scalar case (i.e. when \( k_1 = k_2 = l_1 = l_2 = 1 \)). Because for scalar operators \( \rho_u (A, B) \) is skew-symmetric in A and B and its value at every point depends only on finite jets of their symbols at that point all we need is, actually, to prove 2.31 for \( r \equiv 1 \) and \( s = s(u) \) - a polynomial function of \( u \)'s. This last statement is an immediate consequence of the following lemma.

2.32. Lemma. \( \rho_u (M_j \circ A, B) = \rho_u (A, B \circ M_j) \) where \( M_j \) is the operator of multiplication by a \( j \)-th coordinate \( u^j \).

Proof. Using the identities:

\[
L^h(u^j \circ a) = u^j L^h a + \sqrt{-1} \xi_j L^{-1} a
\]

and

\[
L^i (b \circ u^j) = u^j L^i b + \sqrt{-1} \xi_j L^{-1} (i - L) b,
\]

the following equalities are easily verified:

\[
\sum_{h, i = 0}^{\infty} \frac{1}{(1 + h + i)!} L^h (u^j \circ a) \wedge L^i b = \sum_{h, i = 0}^{\infty} \frac{1}{(1 + h + i)!} [u^j + \sqrt{-1} \frac{1}{2 + h + i} \partial_{\xi_j}] L^h a \wedge L^i b
\]

\[
\sum_{h, i = 0}^{\infty} \frac{1}{(1 + h + i)!} L^h a \wedge L^i (b \circ u^j) = \sum_{h, i = 0}^{\infty} \frac{1}{(1 + h + i)!} L^h a \wedge [u^j - \sqrt{-1} \frac{1}{2 + h + i} \partial_{\xi_j}] L^i b.
\]

Apply \( R_1 \) to both (21) and (22) and then evoke 2.11 to obtain

\[
\rho_u (M_j \circ A, B) - \rho_u (A, B \circ M_j) = \sqrt{-1} \xi_j \left\{ \sum_{h, i = 0}^{\infty} \frac{1}{(1 + h + i)!} [u^j L^h a \wedge L^i b - L^h a \wedge L^i (b \circ u^j)] \right\}.
\]

In view of \( C^\infty(U) \)-linearity of the \( \mathbb{R} \)-morphism (cf. 2.9) the right-hand side of (23) vanishes.
As a corollary of Proposition 2.31 we obtain invariance of the form $ρ_u(A,B)$ with respect to gauge transformations (the gauge group is $GL_k(ℚ) × GL_ℓ(ℚ)$; for the meaning of $k$ and $ℓ$ see 2.27).

To avoid complicated expressions we adopt the following notation. For a formal series $G ∈ C[[X,Y]]$ let $\tilde{G} = G(1 ⊙ L,L ⊙ 1)$ denote the corresponding linear endomorphism of $P' ⊗ P'$. By $G_i(i = 1,2)$ we denote the partial derivatives $∂G/∂X$ and $∂G/∂Y$ respectively.

Finally, for $f ∈ P'$, $\tilde{f}$ will denote the operation on $P'$ of composition with $f$ on the left (in the sense of symbols), i.e. $\tilde{f}(a) = f ◦ a$, $\hat{f}$ – the corresponding composition on the right, and the plain $f$ will denote the ordinary commutative multiplication by $f$.

In this notation equalities (21) and (22) read as the following identities in $\text{End}(P' ⊗ P')$:

$F ⊙ (u^j ⊙ 1) = (u^j ⊙ 1)F + (\sqrt{-1} \, \partial_{\tilde{z}})E_2$

and

$F ⊙ (1 ⊙ u^j) = (1 ⊙ u^j)F - (1 ⊙ \sqrt{-1} \, \partial_{\tilde{z}})E_2 \quad (j = 1, \ldots, n)$

($F$ will always denote the series (16)).

Let $G = G(X,Y)$ be an arbitrary series and $τ$ denote the transposition of factors in $P' ⊗ P'$. The following lemmas prove to be useful.

**Lemma.** One has

$τG_2τ = G_1^o$ and $τG_1τ = G_2^o$

where $G^o(X,Y) = G(Y,X)$. 

2.36. Lemma. \[ G(u^j_0 1) = (u^j_0 1)G + (\sqrt{-1} \delta_{j}^{\xi_j} 1)(G_2-G), \quad (j=1, \ldots, n). \]

(Analogously for \( G(1 \otimes u^j_0) \)).

The former assertion reflects the fact that \( \tau \) corresponds to the involution of \( \mathbb{C}[[X,Y]] \) which transposes \( X \) and \( Y \). The identity of 2.36 is an immediate consequence of (20).

2.37. Since the form \( \rho_u(A,B) \), as was mentioned, depends at every point only on jets at that point of symbols of \( A \) and \( B \) it is clear that in order to examine the effect on \( \rho_u \) of an (orientation preserving) open embedding \( V + U \) it suffices to consider solely self-embeddings of the form \( \exp(\eta) \) where \( \eta \) is a vector field\(^(*)\).

In that case the functoriality of \( \rho_u \) would mean the identity

\[ I_u(\eta;A,B) := L_{\eta} \rho_u(A,B) - \rho_u([L_{\eta}; A], B) - \rho_u(A, [L_{\eta}; B]) = 0 \quad (24) \]

to hold for an arbitrary pair of \( \psi \)DOs \( A \) and \( B \) and any vector field \( \eta \).

Actually, it turns out that \( I_u(\eta;A,B) \) need not vanish, but, as we shall demonstrate below, it can be written as an explicit exact form:

\[ I_u(\eta;A,B) = d\sigma_u(\eta;A,B) \quad (25) \]

with \( \sigma_u \in A_2(U) \) depending at each point on finite jets at that point of \( \eta \) and of symbols of \( A \) and \( B \). It is not clear, however, whether it is possible to modify the form \( \rho_u \) in order that (24) be satisfied.

It should be added, perhaps, that namely the partial failing of functoriality prevented us from denoting \( \rho_u(A,B) \) more naturally as \( \text{res}_{1,u}(A,B) \) or \( \text{res}_{1}(u;A,B) \), and calling it the higher noncommutative residue form.

\(^(*)\) The instance of an orientation reversing embedding also reduces to the one because \( \rho_u \) is clearly functorial with respect to e.g. the involution \( u^1 \to -u^1, \quad u^i \to u^i \) \((i = 2, \ldots, n)\).
2.38. Proposition. Equality (25) holds with

\[ \sigma_{\alpha}(\eta; A, B) = \sqrt{-1} \sum_{j=1}^{n} i_{\beta/j} R_{1}^{\eta j} (a \circ b - b \circ a) \]

where

\[ \phi^f \equiv (f \circ 1)_{\mathcal{P}^2} - \mathcal{P}^2 (f \circ 1) , \]

(\( F \) is the series (16) and \( \eta = \sum_{j=1}^{n} \eta^j \beta/j \)).

The proof relies on the lemma which is of an independent interest.

2.39. Lemma. \( R_{1}\mathcal{P}(a \circ \xi_j \circ b - a \circ \xi_j \circ b + b \circ a \circ \xi_j) = \sqrt{-1} \delta_{\beta/j} R_{1}\mathcal{P}^2(a \circ b) \)

(\( j = 1, \ldots, n \)). \( \square \)

2.39. Remark. The expression in brackets on the left is the boundary of the Hochschild 2-chain \( a \circ \xi_j \circ b \) in the algebra of symbols.

Since, however, \( R_{1}\mathcal{P} \) is skew-symmetric it might be more proper to speak about the boundary of the cyclic 2-chain \( a \circ \xi_j \circ b \):

\[ R_{1}\mathcal{P}^{cyc}(a \circ \xi_j \circ b) = \sqrt{-1} \delta_{\beta/j} R_{1}\mathcal{P}^2(a \circ b) . \]

Proof of 2.39. Much as we did in the proof of 2.32 we establish the equalities

\[ \mathcal{P}(\xi_j \circ 1) = (\xi_j \circ 1)\mathcal{P} + (\sqrt{-1} \delta_{u/j}) \mathcal{P}^2 \quad (26) \]

and

\[ \mathcal{P}(1 \circ \xi_j) = (1 \circ \xi_j)\mathcal{P} + (1 \circ \sqrt{-1} \delta_{u/j}) (\mathcal{P}^1 - \mathcal{P}) . \]

Since one has easily verifiable identity \( \mathcal{P} \equiv \mathcal{P}^1 + \mathcal{P}^2 \) the latter equality can be written also as

\[ \mathcal{P}(1 \circ \xi_j) = (1 \circ \xi_j)\mathcal{P} - (1 \circ \sqrt{-1} \delta_{u/j}) \mathcal{P}^2 . \quad (27) \]

We want to calculate \( R_{1}\mathcal{P}(\xi_j \circ 1 - 1 \circ \xi_j) \). In order to do this let us notice first that:
\[ R_1(\partial_j f \wedge g + f \wedge \partial_j g) = -R_1(\partial_j (f \wedge \xi_j + g) - R_1(\{f, g\} \wedge \xi_j) \]

\[ = dR_2(\partial_j (f \wedge \xi_j + g) - i_{\partial j} R_0(f, g) \]

\[ = d_{\partial j} R_1(f \wedge g) - i_{\partial j} R_0(f, g), \]

and then recall (cf. diagram (14)) that \( R_0(\{\ldots\} = \sqrt{-1} R_0 m(X - Y) \).

Hence we obtain from \((26)\) and \((27)\) that

\[ R_1(F(\xi_j \Theta l - 1 \Theta \xi_j)) = R_1(\xi_j \Theta l - 1 \Theta \xi_j) F + i_{\partial j} R_0 m(X - Y) F_2 \]

\[ + \sqrt{-1} d_{\partial j} R_1 F_2. \tag{28} \]

We have another general equality (cf. 2.4)

\[ R_1(\xi_j \Theta l - 1 \Theta \xi_j)(f \Theta g) = -R_1(f \Theta \xi_j) = -i_{\partial j} R_0(f g) \]

\[ = -i_{\partial j} R_0 m(f \Theta g) \].

In particular,

\[ R_1(\xi_j \Theta l - 1 \Theta \xi_j) F + i_{\partial j} R_0 m(X - Y) F_2 = i_{\partial j} R_0 m(-F + (X - Y) F_2). \tag{29} \]

Since

\[ -F + (X - Y) F_2 = ((X - Y) F)_2 = (e^X - e^Y)_2 = -e^Y \]

and \( R_0 m(e^L f \Theta g) = R_0 (g \circ f) \) (cf. \((15)\); the circle denotes the composition of symbols) we obtain that the right-hand side of \((29)\) evaluated on the tensor \( a \Theta b \) equals:

\[ -i_{\partial j} R_0(b \circ a) = -R_1(b \circ a \Theta \xi_j). \]

Desired equality 2.39 follows then from \((28)\) and from the following general lemma.
2.40. Lemma. The identity

$$R_1(G - g_{00})(f \otimes \xi_j) = 0 \quad (j = 1, \ldots, n)$$

holds for arbitrary series $G = \sum_{k, l=0}^{\infty} g_{kl} X^k Y^l$ and $f \in \mathcal{P}$. 

Proof. One has obviously

$$(G - g_{00})(f \otimes \xi_j) = \sum_{k=1}^{\infty} g_{k0} L^k f \otimes \xi_j.$$ 

Hence:

$$R_1(G - g_{00})(f \otimes \xi_j) = i \partial \partial u^j R_0 \left( \sum_{k=1}^{\infty} g_{k0} L^k f \right),$$

and the latter expression apparently vanishes (cf. 2.11). \[ \square \]

2.41. Proposition.

$$\rho_u(A \circ L_{\bar{\eta}}, B) = \rho_u(A, L_{\bar{\eta}} \circ B) + \rho_u(BA, L_{\bar{\eta}})$$

$$= \frac{1}{\sqrt{-1}} \bar{\rho} \sum_{j=1}^{n} \rho \left( a \otimes \eta^j \xi_j \right) \otimes b$$

for an arbitrary vector field $\eta = \sum_{j=1}^{n} \eta^j \partial \partial u^j$.

(This is a generalisation of 2.39).

Proof. To simplify notation we shall adopt in this and other proofs the summing convention with respect to repeating indices.

Direct computation using (17), Lemma 2.39 and Proposition 2.31 yields:

$$\rho_u(A \circ L_{\bar{\eta}}, B) = \rho_u(A, L_{\bar{\eta}} \circ B) + \rho_u(BA, L_{\bar{\eta}})$$

$$= \frac{1}{\sqrt{-1}} \bar{\rho} \sum_{j=1}^{n} \rho \left( a \otimes \eta^j \xi_j \right) \otimes b$$

$$= \beta_n \rho \left( -a \otimes \eta^j \xi_j \circ b + b \circ a \otimes \eta^j \xi_j \right)$$

$$- \sqrt{-1} \beta_n \bar{\rho} \sum_{j=1}^{n} \partial \partial u^j R_1 F_2 \left( a \otimes \eta^j \xi_j \right) \otimes b. \quad (30)$$
and the first term on the right-hand side of (30) equals

$$\rho_u(A, L_\eta \cdot B) - \rho_u(BA, L_\eta) .$$

Before we prove Proposition 2.38, we need one more useful lemma.

2.42. Lemma. $\rho_u(L_\eta \cdot C) = i_\eta \text{res}_u C$.

Proof. Use Proposition 2.31 and Lemma 2.40 to obtain:

$$\rho_u(L_\eta \cdot C) = \rho_u(L_{\partial/\partial u}^L \cdot \text{res}_u C) = - \beta_n R_1 F(\xi_j \otimes \eta_j)$$

$$= \beta_n R_1 F(\text{res}_u^j \otimes \eta_j) = \beta_n i_\eta R_0(\text{res}_u^j)$$

$$= \beta_n \text{res}_u^j \eta R_0(c) \equiv i_\eta \text{res}_u C .$$

Now the proof of Proposition 2.38 goes as follows.

The skew-symmetry of $R_1$ together with 2.35 and the identity $F = F_1 + F_2$ yield the equality

$$R_1 F(f \otimes g) = R_1 F_2(f \otimes g - g \otimes f) .$$

(31)

Then 2.38 is verified by the following computation:

$$I_u(\eta; A, B)$$

2.25, 2.42

$$= \delta_\eta \rho_u(A, B) + \rho_u(L_\eta, [A, B]) - \rho_u([L_\eta, A], B) - \rho_u(A, [L_\eta, B])$$

2.41

$$= \sqrt{-1} \beta_n \delta_\eta R_1 F(a \otimes b) - \sqrt{-1} \beta_n \delta_\eta \partial_j (R_1 F_2(a \otimes \eta_j \partial b) - R_1 F_2(b \otimes \eta_j \partial a))$$

$$= \sqrt{-1} \beta_n \delta_\eta \partial_j (R_1 F_2(a \otimes b - b \otimes a) - R_1 F_2(a \otimes \eta_j \partial b - b \otimes \eta_j \partial a))$$

(31)

$$= \sqrt{-1} \beta_n \delta_\eta \partial_j R_1 F(a \otimes b - b \otimes a) .$$
The following proposition establishes vanishing of $\sigma_u(n; \cdot, \cdot)$ for vector fields with constant and linear coefficients, and shows that this fails already for quadratic fields.

2.43. Proposition. a) The form $\sigma_u(n; A, B)$ vanishes for vector fields $n = \sum_{j=1}^{n} n_j \partial / \partial u_j$ with $n_j$'s being linear functions of $u$.

b) For $n = u^p u^q \partial / \partial u_j$ ($j, p, q = 1, \ldots, n$; $p$ and $q$ may coincide) the actual value of $\sigma_u(n; A, B)$ is

$$\sigma_u(n; A, B) = \sqrt{-1} n^j \partial \partial u_j \partial R_1 \left( \partial^2 \partial \xi_j \partial \xi_j \right) F_{12} (a \otimes b)$$

where $F_{12} \equiv \partial^2 F / \partial X \partial Y$.

Proof. a) We have to show that $R_1 \phi^f (1 - \tau)$ vanishes for $f$'s which are linear functions. For $f = \text{const}$ even $\phi^f \equiv 0$, so there is nothing to do. For $f = u^j$ the actual value of $\phi^{u^j}$ is given by Lemma 2.36 (as applied to $G = F_2$):

$$\phi^{u^j} = (\sqrt{-1} \partial \xi_j \otimes 1) (F_2 - F_{22})$$

($F_{22} \equiv \partial^2 F / \partial X \partial Y$). Hence, by making use of 2.35, we obtain:

$$R_1 \phi^{u^j} (1 - \tau) = R_1 (\sqrt{-1} \partial \xi_j \otimes 1) (F_2 - F_{22}) - R_1 (\sqrt{-1} \partial \xi_j \otimes 1) (F_1 - F_{11})$$

$$= R_1 (\sqrt{-1} \partial \xi_j \otimes 1) (F_2 - F_{22}) + R_1 (1 \otimes \sqrt{-1} \partial \xi_j \otimes 1) (F_1 - F_{11})$$

$$= R_1 (\sqrt{-1} \partial \xi_j \otimes 1) (F_2 - F_{22} - F_{11}) + F_{11}$$

It can be easily verified that:

$$F_1 - F_2 = F_{11} - F_{22} \quad \text{in} \quad \mathbb{C}[[X, Y]], \quad (32)$$

and thereby we reach required conclusion.

b) Use Lemma 2.36 twice in order to obtain the general identity that is valid for an arbitrary series $G$:
In particular, if for \( f = u^p u^q \) equals (one puts \( G = F_2 \))

\[
\phi^f = \left[ \sqrt{-1} (u^q \xi_p + u^p \xi_q) \otimes 1 \right] (F_2 - F_{22}) + (\xi_p \xi_q) (F_{222} - 2F_{22} + F_2)
\]

and, by playing with \( \tau \) and basic properties of \( R_1 \) (like 2.9 and 2.11), we get immediately

\[
R_1 \phi^f (1 - \tau) = R_1 [\sqrt{-1} (u^q \xi_p + u^p \xi_q) \otimes 1] (F_2 - F_{22} - F_1 + F_{11})
\]

\[
+ (\xi_p \xi_q) (F_{222} - 2F_{22} + F_2 + F_{111} - 2F_{11} + F_1) \right) .
\]

Let \( F_1^{(k)} \equiv \xi^k F / \partial X^k \) and \( F_2^{(k)} \equiv \xi^k F / \partial Y^k \). Note the identities:

\[
F_1^{(k)} + F_2^{(k)} = F - k \frac{F_1^{(k-1)} - F_2^{(k-1)}}{X - Y}
\]

and

\[
F_{12} = \frac{F_1 - F_2}{X - Y}
\]

(both easily verifiable).

From (35), (32) and (36) one easily gets that

\[
F_{222} - 2F_{22} + F_2 + F_{111} - 2F_{11} + F_1 = F_{12},
\]

and (34) turns, in view of (32), into

\[
R_1 \phi^f (1 - \tau) = R_1 (\xi_p \xi_q) \otimes 1) F_{12} .
\]
2.44. Example. Part b) of Proposition 2.43 provides probably simplest instances when not only $\sigma_u$ itself but also $I_u = d\sigma_u$ do not vanish. Take e.g. operators with symbols

$$a(u, \xi) = \phi(\xi) \quad \text{and} \quad b(u, \xi) = u^j \phi(\xi).$$

Then it can be easily verified that

$$I_u(u^P \partial^j_\alpha \partial^j_\beta; A, B) = \sqrt{-1} \beta_n d \frac{\partial^i}{\partial u^j} R^1 \left( \partial^2 \xi_p \partial^2 \xi_q \right) \left( \partial^2 \xi_p \partial^2 \xi_q \right)$$

$$= \sqrt{-1} \beta_n \frac{d(u^j)}{\partial u^j} \left( \partial^2 \xi_p \partial^2 \xi_q \right)$$

and the latter expression is a general constant form on $U$ divisible by $du^j$.

2.45. Remark. Simply by iterating Lemma 2.36 one can generalise (33) to polynomial (and hence all smooth) $f$'s in the following manner:

$$\phi^f = \left( \sqrt{-1} \frac{\partial^f}{\partial u^p} \partial^2 \xi_p \right) \left( F_2 - F_{22} \right) + \frac{1}{2} \left( \frac{\partial^f}{\partial u^p} \partial^2 \xi_p \right) \left( F_{222} - 2F_{222} + F_2 \right)$$

+ higher order terms

(higher order - means terms which include higher than second derivatives with respect to $\xi$'s).

Therefore one can deduce exactly as in the proof of 2.43b) that

$$\sigma_u \left( f \partial^j_\alpha \partial^j_\beta; A, B \right) = \sqrt{-1} \beta_n \left( \partial^i \frac{\partial^f}{\partial u^p} \partial^2 \xi_p \right) R^1 \left( \frac{1}{2} \left( \frac{\partial^f}{\partial u^p} \partial^2 \xi_p \right) \left( \partial^2 \xi_p \partial^2 \xi_q \right) \right)$$

+ 'higher order terms'.

In particular, if one of the two operators is a differential operator of order $\leq 2$ the form $\sigma_u(\eta; A, B)$ vanishes for every vector field $\eta$. 
2.46. Corollary. If one of the two \( \psi \)DOs is a differential operator of order \( \leq 2 \) the form \( \rho_u(A, B) \) is functorial with respect to arbitrary morphisms \( (V; \theta^k, \theta^\xi) \to (U; \theta^k, \theta^\xi) \). \( \square \)

2.47. (Comment on 2.46). Using Proposition 2.41 and Lemma 2.42 it is not difficult to obtain the actual value of \( \rho_u(A, B) \). For example, for \( A = L_{\eta_1}^L, L_{\eta_2}^L \) Proposition 2.41 yields:

\[
\rho_u(L_{\eta_1}^L, L_{\eta_2}^L, B) = \rho_u(L_{\eta_1}^L, L_{\eta_2}^\circ B) + \rho_u(L_{\eta_2}^L, B=L_{\eta_1}^L) \\
+ \frac{\beta}{\sqrt{-1}} \frac{d}{d\xi} R_1 F_2(\sqrt{-1} L_{\eta_1}^L \xi \xi \hat{\theta} B).
\]

Notice that \( F_2 = \frac{1}{2} F \mod (X \mathcal{O}[X,Y]) + Y \mathcal{O}[X,Y]) \).

Hence, by using 2.11 and 2.10 (twice) we can rewrite third term on the right-hand side of (33) as:

\[
\frac{\beta}{\sqrt{-1}} \frac{d}{d\xi} R_1 F_2(\sqrt{-1} L_{\eta_1}^L \xi \hat{\theta} B) = \frac{1}{2} \frac{d}{d\xi} L_{\eta_1}^L \xi \hat{\theta} B.
\]

and, finally, obtain from (38) the required expression:

\[
\rho_u(L_{\eta_1}^L, L_{\eta_2}^L, B) = i_{\eta_1} \text{res}_u(L_{\eta_2}^\circ B) + i_{\eta_2} \text{res}_u(B=L_{\eta_1}^L) + \frac{1}{2} \frac{d}{d\xi} L_{\eta_1}^L \xi \hat{\theta} B.
\]

whose right-hand side is transparently functorial. We thereby reprove Corollary 2.46.

2.48. Real structures and adjoint operators. For completeness sake we shall determine the residue form of a complex conjugated \( A^c \) and adjoint \( A^* \) operators.

The symbol of \( A^c \) is equal to \( \overline{a(u, -\xi)} \). In particular,

\[
\text{res}_u A^c = \text{res}_u A.
\]
In order to define $A^*$ one has to specify hermitian metrics $g_1$ on $\theta^k$ and $g_2$ on $\theta^l$ (viewed invariantly as $\mathbb{C}$-linear morphisms $\theta^k \to (\theta^k)^*$ and $\theta^l \to (\theta^l)^*$) and a positive density $\nu$. Then $A^*$ is defined by the requirement of fulfilling the equality

$$\left\{ \begin{array}{l}
g_1(s,A^*t)\nu = \int_U g_2(As,t)\nu \\
\end{array} \right.$$  \hspace{1cm} (40)

for every $s \in C^\infty_{\text{comp}}(U;\theta^k)$ and $t \in C^\infty_{\text{comp}}(U;\theta^l)$.

It is clear from (40) that $A^* = (g_1m)^{-1}\circ A^*\circ (g_2m)$ where $m = m(u) \equiv \nu/|du|$ and $A^*$ denotes the complex conjugated transpose of $A$. Thus, we obtain by using Lemma 2.18 that

$$\text{res}_u A^* = (g_1m)^{-1}\cdot \text{res}_u A^*\cdot (g_2m) = g_1^{-1}\cdot \text{res}_u A^*\cdot g_2.$$  

On the other hand, by using standard formula for the symbol of $A^+$ (cf. [9], Thm. 4.2 or [15], formula 1.3.37) we obtain that

$$\text{res}_u A^+ = \beta_n\tau_* |\text{Res}_0 \sum_a \frac{1}{a!} \partial_a \partial^\ast a^+ | 2\pi^2 \beta_n\tau_* |\text{Res}_0 a^+ | = (\text{res}_u A)^+.$$  

Thus we proved

**2.49. Lemma.** a) $\text{res}_u A^c = \overline{\text{res}_u A}$.  

b) $\text{res}_u A^* = g_1^{-1}(\text{res}_u A)^+ g_2$.  

\[ \square \]

\hspace{1cm} § 3. Noncommutative residue (global case)

Throughout this section $X$ denotes a fixed smooth open (i.e. without boundary but not necessarily compact) manifold, and $E$ and $F$ - two vector bundles on it of ranks equal to $k$ and $\ell$ respectively.

**3.1. Lemma-Definition.** For an arbitrary $\psi$DO $A : C^\infty_{\text{comp}}(X;E) \to C^\infty(X;F)$ there exists a unique density $\text{res}_x A \in A_0(X;\text{Hom}(E,F))$ such that
\[ \phi^* \text{res}_\chi A = \text{res}_\chi \phi^\# A \]

whenever \( \phi : (U; \theta^k, \theta^l) \to (X; E, F) \) is a local coordinate patch (cf. 0.6).

This density will be called the (homomorphism valued) residue density (or residue form) of \( A \).

(Lemma 3.1. is an immediate corollary of Lemma 2.4).

The basic properties of residue density are summed up in

3.2. Proposition

(I) (Linearity) For \( A \in \text{CL}^k(X; E, F), B \in \text{CL}^m(X; E, F) \) (with \( k - m \in \mathbb{Z} \)) and \( a, \beta \in \mathbb{C} \) one has

\[ \text{res}_\chi (aA + \beta B) = a \text{res}_\chi A + \beta \text{res}_\chi B \]

(II) (Functoriality) For any morphism \( \psi : (Y; E', F', \theta) \to (X; E, F) \) (cf. 0.3) one has

\[ \text{res}_\chi \psi^\# A = \psi^* \text{res}_\chi A \]

(III) (Locality) If at a neighbourhood of a point \( x_0 \in X \) the complete symbol of \( A \) vanishes then \( \text{res}_{x_0} A = 0 \).

(IV) If \( \text{ord } A < -\text{dim } X \) or \( \text{ord } A \notin \mathbb{Z} \) then \( \text{res}_{x_0} A \equiv 0 \).

(V) (Real structures) If \( E \) and \( F \) both admit real structures \( \rho_E : E \to E \) and \( \rho_F : F \to F \) respectively, and \( A^C := \rho_F A \rho_E \) is the corresponding complex conjugated operator then

\[ \text{res}_{x_0} A^C = (\text{res}_{x_0} A)^C = \rho_F (\text{res}_{x_0} A) \rho_E. \]

(VI) (Adjoints) \( \text{res}_{x_0} A^* = g^{-1}_E (\text{res}_{x_0} A)^+ g_F \)

where \( A^* \) is an operator adjoint to \( A \) with respect to hermitian metrics \( g_E : E \to \overline{E}^* \) and
\text{gr}_F : F \to \overline{F}^* \text{ and a certain positive density on } X \text{ (the sign } + \text{ denotes the complex conjugated transpose).}

In particular, \( \text{res}_x A^* \) does not depend on the choice of volume density in question.

(VII) If \( E = F \) and \( \nabla_\eta \) denotes covariant differentiation of sections of \( E \) along a vector field \( \eta \) then

\[
\text{res}_x [\nabla_\eta, A] = \nabla_\eta \text{ res}_x A
\]

where \( \nabla_\eta \) denotes the "Lie derivative" on \( A_0(X; \text{End } E) \) defined by the connection \( \nabla \) on \( \text{End } E \) induced by \( \nabla \).

Comments on proof: Assertions (I), (III) and (IV) follow directly from the definition; (II), (V) and (VI) follow from the corresponding local assertions: 2.20, 2.49 and from Lemma 3.1. Finally, (VII) is verified by a brief local computation using Lemma 2.2 (details are left as an easy exercise to the reader).

3.3. Corollary. Assume \( E = F \).

(a) If \( E \) admits a real structure and \( A^C \) is the corresponding complex conjugated operator then

\[
\text{tr } \text{res}_x A^C = \overline{\text{tr } \text{res}_x A}
\]

(the bar denotes the usual complex conjugation).

(b) \( \text{tr } \text{res}_x A^* = \overline{\text{tr } \text{res}_x A} \).

(c) For an endomorphism \( r : E \to E \) one has

\[
\text{tr } \text{res}_x [A, r] = 0.
\]

(d) \( \text{tr } \text{res}_x [\nabla_\eta, A] = \nabla_\eta \text{ tr res}_x A \)

(in particular, it does not depend on the connection).
Parts (c) and (d) admit the following simple generalisation. Let
\( A : C^\infty_{\text{comp}}(X,E) \to C^\infty(X,F) \) be an arbitrary \( \psi \)-DO and
\( D : C^\infty(X,F) \to C^\infty(X,E) \) be a first order differential operator. For a
density \( \sigma \in A^0_0(X;\text{Hom}(E,F)) \) there is an invariantly defined scalar
density \( <L_D,\sigma> \in A^0_0 \) which is the image under the composition of
sheaf-morphisms:

\[
(\text{contraction}) \quad L_X \otimes \text{Hom}(F,E) \otimes (A_{X,0} \otimes \text{Hom}(E,F)) \to L_X \otimes A_{X,0} \to A_{X,0}
\]

\( (L \) denotes Lie derivative) of \( p_D \otimes \sigma \) where \( p_D \) is the
principal symbol of \( D \) regarded as a vector field with coefficients in \( \text{Hom}(F,E) \).

A direct computation using Lemma 2.2 yields

\[<L_D,\sigma> = \text{tr} \ \text{res}_x [A,B] = <L_D,\text{res}_x A>_x \]

(Notice that as in the scalar case \( <L_D,\sigma> \) is an explicit exact
density:

\[<L_D,\sigma> = d<i_D,\sigma>\]

where \( <i_D,\sigma> \) is defined similarly to \( <L_D,\sigma> \) but with Lie
derivative replaced in (1) by interior product.)

3.5. Let \( A : C^\infty_{\text{comp}}(X,E) \to C^\infty(X,F) \) and \( B : C^\infty_{\text{comp}}(X,F) \to C^\infty(X,E) \)
be a pair of \( \psi \)-DOs similar to that considered in 2.24 (i.e. \( A \) or
\( B \), or both, are assumed to be proper \( \psi \)-DOs, and both are assumed
to have integer orders (but also cf. 3.8 below)). We want to express
\( \text{res}_x [A,B] \) as an explicit exact form.

Were \( A \) supported by the range of some local coordinate patch
\( \phi = (i : U \to X;r,s) \) (i.e. \( A|_{X \setminus i(U)} = 0 \)) we would have:

\[\text{tr} \ \text{res}_x [A,B] = \int_U \phi^*(A,\phi^* B).\]
In general, one has to proceed as follows. It is always possible to represent \( A \) as a locally finite sum \( \sum_{\chi} A_{\chi} \) (i.e. for any \( x \in X \) the restriction of \( \sum_{\chi} A_{\chi} \) to some neighbourhood of \( x \) has only finitely many non-zero entries) such that each \( A_{\chi} \) is supported by the range of some local coordinates \( \phi^X \). Then the bi-linearity and locality of the form \( \rho_u \) imply the equality:

\[
\text{tr res}_{\chi} [A, B] = d \sum_{\chi} i_{\chi}^X \rho_u((\phi^X)^\# A_{\chi}, (\phi^X)^\# B).
\]

(3)

Let us consider the expression under the sign of differential in (3) as an element of \( A_1(X)/dA_2(X) \) and denote this element as

\[
\overline{\rho}(A, B; \{ A_{\chi} \}, \{ \phi^X \}).
\]

3.6. Lemma. \( \overline{\rho}(A, B; \{ A_{\chi} \}, \{ \phi^X \}) \) depends neither on \( \{ \phi^X \} \), nor on the particular choice of the representation \( A = \sum_{\chi} A_{\chi} \).

Proof. Independence of \( \{ \phi^X \} \) is clear from Proposition 2.31. In order to prove the second assertion we take such a partition of unity \( \{ f^\lambda \} \) that the sum \( \sum_{\chi, \lambda, \mu} f^\lambda \circ A_{\chi} \circ f^\mu \) is locally finite. Then, clearly, we have

\[
\sum_{\chi} i_{\chi}^X \rho_u((\phi^X)^\# A_{\chi}, (\phi^X)^\# B) = \sum_{\chi, \lambda, \mu} i_{\chi}^X \rho_u((\phi^X)^\# (f^\lambda \circ A_{\chi} \circ f^\mu), (\phi^X)^\# B).
\]

(4)

All we need is to prove that for \( A = 0 \) the left-hand side of (4) belongs to \( dA_2 \) irrespective of the representation \( \sum_{\chi} A_{\chi} = 0 \).

Actually, Proposition 2.38 implies that every summand under the sign of the first sum on the right-hand side of (4) is the differential of a form with support in \( \text{supp } f^\lambda \cap \text{supp } f^\mu \). Hence, the right-hand side, and therefore the left-hand side of (4) belong to \( dA_2 \), as required.

\[\square\]

As the class \( \overline{\rho} \) does not depend on extra choices we shall use notation \( \overline{\rho}(A, B) \).
3.7. Proposition. \( \text{tr } \text{res}_x[A,B] = d\overline{\rho}(A,B) \) where \( \overline{\rho}(A,B) \) is the canonical element of \( A_1(X)/dA_2(X) \) defined above. □

3.8. Remark. The assertion of Proposition 3.7 extends easily to the more general case of operators with arbitrary complex orders (cf. Remark 2.26).

An important property of the class \( \overline{\rho} \) is stated in the following

3.9. Proposition. \( \overline{\rho}(A_0A_1,A_2) - \overline{\rho}(A_0,A_1A_2) + \overline{\rho}(A_2A_0,A_1) \equiv 0 \).

3.10. Remarks. 1) Since \( \overline{\rho} \) is obviously invariant with respect to the 'stabilisation':

\[
\begin{align*}
A &\mapsto \begin{pmatrix} 0 & 0 \\ A & 0 \end{pmatrix}, \\
B &\mapsto \begin{pmatrix} 0 & B \\ 0 & 0 \end{pmatrix} \quad \text{E} \\
&\quad \text{F}
\end{align*}
\]

3.9. is equivalent to:

\[
\overline{\rho}_{\text{cycl}}(A_0 \otimes A_1 \otimes A_2) = 0 ,
\]
i.e. to saying that \( \overline{\rho} \) is a cyclic 1-cocycle with values in \( A_1/dA_2 \) on the algebra of symbols \( CS'(X,E) \).

Thus Propositions 3.7 and 3.9 can be restated as the commutativity of the diagram:

\[
\begin{array}{cccc}
CC_0(CS'(X,E)) & \xrightarrow{\rho} & CC_1(CS'(X,E)) & \xrightarrow{\rho} & CC_2(CS'(X,E)) \\
\text{tr res}_x & \downarrow & \overline{\rho} & \downarrow & \\
A_0(x) & \xrightarrow{d} & A_1(x)/dA_2 & \leftarrow & 0 \\
\end{array}
\]

Diagram (5) is obviously an initial piece of the hypothetical noncommutative residue morphism. It is not clear yet how to construct the 'higher residues'. Quite probably, they arise in the setting of equivariant cyclic (or Chevalley) chain complexes. One can prove at least the existence of canonical homomorphisms in homology.
380
HC(\text{CS}'(X,E)) \to H(A_\circ(X)) \text{ and } H^{\text{Lie}}(\text{CS}'(X,E))[-1] \to H(A_\circ(X)) \text{ (recall that } H_q(A_\circ(X)) = H_{q,\text{cl}}(X), \text{ i.e. the homology of } X \text{ with closed supports), cf. [20].}

2) There are several particular cases with a canonical choice of the representative for the class \( \overline{\rho} \). This holds e.g. for all pairs \((A,B)\) on a manifold with linear transition functions (arbitrary domains in \( \mathbb{R}^n \), tori \( T^n \) etc.), as implied by Propositions 2.31 and 2.43a).

On the other hand it follows directly from formula (37) and its corollary (see 2.46) that the form
\[
\rho_X(A,B;\{A\},\{\phi_X\}) = \sum \int_X \rho_{\chi}((\phi_X)^\# A_\chi, (\phi_X)^\# B)
\]
depends neither on particular representation \( A_\chi = \int \chi A \) nor on local coordinates \( \{\phi_X\} \) if one of the two operators is a differential one and of order \( \leq 2 \). And this holds irrespective of geometry of an underlying manifold.

**Proof of 3.9.** The assertion reduces easily to the following local question. We want to prove for symbols of scalar \( \psi \text{DOs} \) in a contractible domain \( U \) that
\[
R_1F(a^0 \otimes a^1 \otimes a^2 - a^0 \otimes a^1 \otimes a^2 + a^2 \otimes a^0 \otimes a^1) = d\phi(a^0, a^1, a^2)
\]
provided \( \text{supp } a^0 \) is compact. The form \( \phi(a^0, a^1, a^2) \) is required to have support contained in \( \text{supp } a^0 \).

Let \( \chi \) denote the form \( R_1F^{\text{cyl}}(a^0 \otimes a^1 \otimes a^2) \). Clearly, \( \text{supp } \chi \subset \text{supp } a^0 \), and \( d\chi = 0 \). The latter follows, in view of Lemma 2.25, from
\[
d\chi = \frac{1}{\sqrt{-1}} R_0(\omega^{\text{cyl}})^2(a^0 \otimes a^1 \otimes a^2) = 0.
\]
Since \( \chi \in A_{1,\text{comp}}(U) \), and \( U \) is contractible, there must be a form \( \phi \in A_{2,\text{comp}}(U) \) with \( \text{supp } \phi \subset \text{supp } \chi \) such that \( \chi = d\phi \).

3.11. Identity 3.9 provides for a differential operator \( A \) a flexible way to produce a specific representative of the class
\[ \bar{\rho}(A,B). \quad \text{Indeed, represent } A \text{ as a locally finite sum } \sum_{\alpha} A_{j}^{\alpha} \ldots A_{p(a)}^{\alpha} \]

where each \( A_{j}^{\alpha} \) is of order \( \leq 1 \). Then the recurrent use of Proposition 3.9 yields the equality:

\[ \bar{\rho}(A,B) = \sum_{\alpha} \sum_{j=1}^{p(a)} \bar{\rho}(A_{j}^{\alpha}, A_{j+1}^{\alpha} \ldots A_{p(a)}^{\alpha}) \bar{A}_{1}^{\alpha} \ldots \bar{A}_{j-1}^{\alpha} , \]

and, as each class on the right possesses the canonical representative

\[ \langle i_{A_{j}^{\alpha}}, \text{res}_{x}(A_{j+1}^{\alpha} \ldots A_{p(a)}^{\alpha}) \bar{A}_{1}^{\alpha} \ldots \bar{A}_{j-1}^{\alpha} \rangle \]

(cf. Proposition 3.4 and (2)) we obtain the following

3.12. **Proposition.** For a differential operator

\[ A = \sum_{\alpha} A_{1}^{\alpha} \ldots A_{p(a)}^{\alpha} \]

the class \( \text{mod } dA_{2}(X) \) of the form

\[ \sum_{\alpha} \sum_{j=1}^{p(a)} \langle i_{A_{j}^{\alpha}}, \text{res}_{x}(A_{j+1}^{\alpha} \ldots A_{p(a)}^{\alpha}) \bar{A}_{1}^{\alpha} \ldots \bar{A}_{j-1}^{\alpha} \rangle \]

is equal to \( \bar{\rho}(A,B) \).

In particular, it does not depend on the specific representation (6).

\[ \square \]

3.13. **Remark.** As we already know (cf. Remark 3.10.2) for an operator \( A \) of order \( \leq 2 \) there is another canonical way to write down a form representing \( \bar{\rho}(A,B) \). As demonstrates equality (39) of Section 2 the two differ!

The same equality also provides the simplest instance of dependence of the actual form (7) on the specific choice of the presentation (6). Assume e.g. both operators to be scalar ones, and let for \( A \) (that is our operator of the second order)

\[ A = \sum_{\alpha, \beta} L_{\beta} \eta_{1}^{\alpha} \eta_{2}^{\alpha} + \sum_{\gamma, \delta} L_{\gamma} \eta_{1}^{\gamma} \eta_{2}^{\gamma} + \sum_{\zeta} L_{\zeta} \]

be its two different presentations (\( \eta \)'s and \( \zeta \)'s denote vector fields). Then the difference between the two forms (7) is
In the particular case of $A = \sum \eta_1 \eta_2$ this reduces to $d_1 \eta_1 \eta_2$.


Apart from its noncommutative residue density every $\psi DO A : C_\text{comp}^\infty(X,E) \to C^\infty(X,F)$ determines yet another invariant local quantity which is a density with coefficients in $\Omega^1_X \otimes \text{Hom}(E,F)$. The relevant construction depends solely on $A$ if $F = \mathcal{O}$ is a trivial bundle. For general $F$ one has, however, to specify also a connection $\nabla$ on $F$.

Consider first the local case

$$X = U \subset \mathbb{R}^n, \quad E = \mathcal{O}^k, \quad F = \mathcal{O}^l, \quad \text{and} \quad \nabla = \sum_{j=1}^n \Gamma_j du^j$$

where $\Gamma_j \in \text{Mat}_j(\mathcal{O}^\infty(U))$.

### 3.15. Definition.

$$\text{sub res}_u^\nabla(A) = \sum_{j=1}^n du^j \left\{ \right. (\nabla_j a - \sqrt{-1} \xi_j a_{-1}) \left| d\xi_j \right| \left| du \right|$$

(we retain the notation of Definition 2.13; $\nabla_j = \partial/\partial u^j + \Gamma_j \otimes \text{id}(\mathcal{O}^k)$).

By definition $\text{sub res}_u^\nabla(A)$ is a density on $U$ with coefficients in the sheaf of matrix-valued differential $1$-forms. An alternative definition is:

$$\text{sub res}_u^\nabla(A) = \frac{\partial a}{\partial \tau} \left| \text{Res}_0(d(\nabla) + a) a \right|$$

$$(d(\nabla) = \partial_H \otimes d + \nabla \otimes \partial_H, \quad \text{where} \quad H = \mathcal{O}^k \otimes (\mathcal{O}^k)^*,$$

$$\alpha = \sum_{j=1}^n \xi_j du^j$$

is the canonical $1$-form on $T^*U$ (cf. 1.3) and $\tau$, as usual, denotes the projection $S^*U \to U$.

The quantity $\text{sub res}_u^\nabla(A)$ will be called the subresidue of $A$ (with respect to the connection $\nabla$).
3.16. Lemma. Subresidue is functorial with respect to such morphisms \( \phi = (f;r,s) : (V;\theta^k,\theta^\ell) \rightarrow (U;\rho^k,\rho^\ell) \) that \( s \) is an isomorphism.

3.17. Corollary. For an arbitrary \( \psi DO \ A : C^\infty_{\text{comp}}(X,E) \rightarrow C^\infty(X,F) \) and a connection \( \nabla \) on \( F \) there exists a unique section \( \text{subres}_x^\nabla(A) \) of the bundle \( \Omega^1_X \otimes A_{X,0} \otimes \text{Hom}(E,F) \) with property that:

\[
\phi \ast \text{subres}_x^\nabla(A) = \text{subres}_{u}^{\phi \ast \nabla} (\phi \ast A)
\]

whenever \( \phi \) is a local coordinate patch (as in 3.1).

Proof-Explanation. The subresidue is so defined that it corresponds under the natural isomorphism of sheaves

\[
\Omega^1_X \otimes A_{X,0} \otimes \text{Hom}(E,F) = \text{Hom}(T^*_X, A_{X,0} \otimes \text{Hom}(E,F)) \rightarrow C^\infty(X)\text{-linear and transparently functorial homomorphism}
\]

\[
T^*_X \otimes \eta \mapsto \text{res}_x^\eta (V \circ A).
\]

When \( E = F \) we obtain the global section \( \text{tr subres}_x^\nabla(A) \) of the vector bundle \( \Omega^1_X \otimes A_{X,0} \). The holomorphic realization of the latter and its \((n-1)\)-st cohomology are the well known ingredients of deformation theory of complex structures. This coincidence is likely to indicate some deeper connection between \( \psi DOs \) and variations of complex structures.

3.18. The relation to zeta function. Let \( X \) be closed (i.e. compact and without boundary) and \( E = F \). Assume that

\[ A : C^\infty(X,E) \rightarrow C^\infty(X,E) \]

is an elliptic \( \psi DO \) of positive order which admits complex powers \( A_\theta^{-S} \) (the subscript \( \theta \) means that the branch \( \theta < \text{Arg}\lambda < \theta + 2\pi \) of the argument is chosen). Let \( A_\theta^{-S}(x,x) \) be the restriction to the diagonal of the Schwartz kernel of \( A_\theta^{-S} \). This is a density on \( X \) with coefficients in \( \text{End } E \) which is defined only for \( \text{Res} > \dim X/\text{ord } A \). The classical nowadays theory of the zeta-
function of an elliptic $\psi$DO due to R.T. Seeley (cf. [14], and also [15] and [17]) asserts that the holomorphic in a half-plane function

$$s \mapsto A_\theta^{-s}(x,x) \quad (\text{Res } > \dim X/\text{ord } A)$$

with values in $A_0(X,\text{End } E)$ admits an analytic continuation to a function $A_\theta^{(-s)}(x)$ that is meromorphic in the whole complex plane. The sole singularities of it are simple poles located at points of the arithmetic progression $s_j = (\dim X - j)/\text{ord } A$ (except the origin where the function is always regular). The theory also gives explicit local formulae for all residues and the value at $s = 0$. Finally, if $A$ happens to be a differential operator the residues at other negative integers vanish too and the theory also gives local formulae for values at those points.

Using the language of noncommutative residue the results of Seeley become extremely clear and the formulae - brief.

**Formulae for residues.**

$$\text{Res}_{s=s_j} A_\theta^{(-s)}(x) = \frac{\text{res}_x(A_\theta^{-s_j})}{\text{ord } A}.$$ \hspace{1cm} (8)

In particular, $A_\theta^{(-s)}(x)$ can possess a pole exactly there where $A_\theta^{-s}$ can have a priori non-vanishing noncommutative residue form. For $s = 0$ we have

$$A_\theta^0 = I - P_0$$

($P_0$ is the projector on the zero-eigenspace, it has a finite rank). Hence $\text{res}_x A_\theta^0$ always vanishes identically. At other negative integers $s = -\ell$ one has $A = A \circ \ldots \circ A$ ($\ell$ times). Hence for a differential operator all $A_\theta^{-\ell}$'s ($\ell = 1, 2, \ldots$) are also differential operators and $\text{res}_x A_\theta^\ell$ vanishes identically.

**Formula for $A_\theta^{(0)}(x)$**. Let us consider first a local situation. For an elliptic $\psi$DO $A : C_\text{comp}^\infty(U,\mathbb{R}^k) \to C^\infty(U,\mathbb{R}^k)$ let $b(\lambda) = \sum_{j=0}^{\infty} b^{m-j}(\lambda)$ be defined by requiring that:
1° \quad \text{condition 1° defines what is called sometimes a "symbol with a parameter". Notice that this is not a classical symbol which happens to depend on a parameter unless } m = 0). \text{ Divide } b(\lambda) \text{ into the sum of its leading term } b_m(\lambda) \text{ and the rest } b^-(\lambda), \text{ and suppose that for no } (u,\xi) \in T^*_0 U \text{ there is an eigenvalue of the principal symbol } a_m(u,\xi) \text{ lying on the ray } \text{Arg} \lambda = \theta. \text{ Then the integral}

\int_0^\infty b^-(\lambda) d\lambda

\text{is well defined (notice that } \int_0^\infty b_m(\lambda) d\lambda \text{ diverges). It follows easily from 1° that } \lambda_\theta \text{ where each } \lambda_\theta, -j \text{ is a homogeneous of order } -j \text{ matric function on } T^*_0 U. \text{ In order to indicate its dependence on } A \text{ it will be also denoted } \lambda_\theta^A \text{ and called the "logarithmic symbol" of } A. \text{ It has obviously a different transformation rule from standard symbols (or even amplitudes) under local morphisms } \psi : (V;\theta^k) \to (U;\theta^K). \text{ It shares with them, however, the property that}

\lambda_\psi^A = (T^*\psi)^* (\lambda_\theta^A) = \sum_{i=1}^n \theta_{\eta i} \lambda(i)

\text{where } \lambda(i) \in \mathcal{P}^* (T^*_0 V) \otimes \text{Mat}_k (\mathbb{C}) \text{ depend microlocally on the symbol of } A \text{ and on } \psi, \text{ and } (\nu,\eta) \in T^*_0 V.

\text{Thus Lemma 2.2 implies, as before (cf. 2.20), the functoriality of } \tau^* |\text{Res}_0^{\lambda^A} |. \text{ Lemma 3.19. The matric density } \zeta_{\theta, u}(A) := -\text{Res}_0^{\lambda^A} | \text{ is functorial with respect to morphisms } (V;\theta^k) \to (U;\theta^K). \Box

\text{Corollary 3.20. For an arbitrary elliptic operator } A : C^\infty_{\text{comp}} (X,E) \to C^\infty (X,E) \text{ with the property that no eigenvalue of its}
principal symbol lies on the ray \( \text{Arg} \lambda = \theta \) there is a well defined density \( Z_{\theta,x}(A) \) with coefficients in \( \text{End} \ E \) such that
\[
\phi^* Z_{\theta,x}(A) = Z_{\theta,u}(\phi \# A)
\]
for every local coordinate patch \( \phi : (U; \theta^k) \to (V; E) \).

The density \( Z_{\theta,x}(A) \) is, in particular, functorial with respect to global morphisms \( \psi : (Y; F) \to (X; F) \).

The promised formula for \( A^{(0)}_\theta(x) \) reads then as
\[
A^{(0)}_\theta(x) = \frac{Z_{\theta,x}(A)}{\text{ord} \ A} - P_0(x,x)
\]
(11)

where \( P_0(x,x) \) is the restriction of the kernel of the projector \( P_0 \) to the diagonal.

3.21. Remark. Notice that \( Z_{\theta,x}(A) \) is well defined even if \( \text{ord} \ A \leq 0 \). For an operator of zeroth order \( b(\lambda) \) is simply a classical symbol of \( (A - \lambda)^{-1} \) and \( Z_{\theta,x}(A) \) turns to be equal
\[
Z_{\theta,x}(A) = - \int_{0}^{i \theta} \text{res}_{\chi}(A - \lambda)^{-1} d\lambda.
\]
(12)

This equality suggests that \( Z_{\theta,x}(A) \) is roughly speaking the "noncommutative residue of <\log \approx_m> - \log A" where \( \approx_m \) is a figurative notation for the "infinite operator of order \( m = \text{ord} \ A \); \( \theta^A \) is just the symbol of that non-existing operator with its divergent principal symbol thrown out.

It is easy to deduce from (12) that one has indeed the equality
\[
\int_X \text{tr} Z_{\theta,x}(A) = - \text{res} \log^{(\theta)} A
\]
(13)
for operators of zeroth order \( \log^{(\theta)} A \) is the logarithm of \( A \) defined by integration of \( - \frac{\log \lambda}{2\pi i} (A - \lambda)^{-1} d\lambda \) along the closed contour in \( \mathbb{C} \setminus \{\text{Arg} \lambda = \theta \} \) enclosing the spectrum of the principal symbol of \( A \). For more details see [19]. A simple comparison of (13) with (11) suggests that
ord $A(\zeta_\theta(0;A) + h_0(A)) \equiv \text{ord} A \left( \int_X \text{tr}(A_\theta^0(x)) + P_0(x,x) \right)$

should possess certain multiplicative properties (analogous to tr log functional on matrices). This is indeed the case (see [19]).

Finally, let us consider the case when $A$ is a differential operator. Let $q \geq 1$ and $b(q)(\lambda) := \prod_{j=q+1}^{\infty} b_j(\lambda)$, then the integral

$$
\zeta(q), A = \int_0^{\infty} \lambda^q b(q)(\lambda) d\lambda
$$

is well defined provided $A$ is a differential operator. As before one verifies easily that $z(q), A := -\partial_{n^*} \text{Res}_{0^*} z(q), A$ gives rise to a global functorial density $z(q), A$. Then the formula for $\zeta(q), A$ reads as:

$$
\zeta(q), A = \frac{z(q), A}{\text{ord} A}.
$$

Because $\zeta(s), A \equiv \int_X \text{tr} A\langle -s \rangle (x)$, the integration of fibre traces of (6), (11) and (15) yields the corresponding formulae for residues and values of the zeta function of $A$.

3.22. Generalized zeta-functions. The ordinary and more general zeta-functions all arise as a special example of the following general picture. On the space $\bigcup_{z \in \mathbb{C}} CL^2(X,E)$ there is a suitable topology making it a holomorphic fibre bundle over $A^1(\mathbb{C})$ so that $<\Delta_E>^z \equiv (1 + \Delta_E)^{z/2}$ provides a global invertible section of it ($\Delta_E$ denotes the Laplacian with coefficients in $E$). For a holomorphic section $\Phi$ let $\Phi(a;x,x)$ be the restriction of the Schwartz kernel of $\Phi(z)$ to the diagonal where it is defined (i.e. for $\text{Re} z < -\text{dim} X$). Then $\Phi(z;x,x)$ can be analytically continued to a meromorphic function $\Phi(z;x)$ with values in $A_X, 0(\text{End} E)$. The sole singularities are simple poles located at the points $z_j = j - \text{dim} X$ ($j = 0,1,\ldots$). The relevant residues are given by the simple formula
\[ \text{Res}_{z=z} \phi(z;x) = -\text{res}_x \phi(z) \]  

which is a generalization of (8). It follows from (16) that the residue of \( \phi(z;x) \) depends only on the value of section \( \phi \) at the relevant point and not, for instance, on the whole germ. It follows also that \( \phi(z;x) \) is regular at those points where \( \phi(z) \) happens to be a differential operator.

Formula (16) finds an immediate application to the zeta-functions \( \zeta_\theta(s;Q|A) \) which are analytic continuations of the "Dirichlet series" \( \text{Tr}QA^{-s} \).

Sometimes it can be also possible to obtain formulae for values of \( \phi(z;x) \) at certain points (generalizing (11) and (15)). This happens, for instance, for the section \( \phi(z) = QA^W \) \((w \equiv (z - \text{ord } Q)/\text{ord } A)\) with \( Q \) being a differential operator. Let \( q \) be the symbol of \( Q \) and \( b^-_Q(\lambda) \) be \( q \circ b(\lambda) \) with first few divergent terms thrown out. Then we put (in similarity to (10))

\[ x^{Q,A}_\theta = \int_0^\infty b^-_Q(\lambda) d\lambda \]  

and

\[ Z_{\theta,u}(Q|A) = -\lambda \tau_\theta \text{Res}_0 x^{Q,A}_\theta \]  

Much as before we derive the existence of the global functorial density \( Z_{\theta,x}(Q|A) \), and the desired formula has the form:

\[ \phi(\text{ord } Q;x) = \frac{Z_{\theta,x}(Q|A)}{\text{ord } A} \]  

For \( Q = A^Q \) \((q = 1,2,\ldots)\) formula (19) is equivalent to Seeley's formula (14).

3.23. Relation to the heat kernel expansions. Let \( D : C^\infty(X,E) \to C^\infty(X,F) \) be an arbitrary elliptic \( \psi DO \) (of positive
order) and \( D^* : \mathcal{C}^\infty(X,F) \to \mathcal{C}^\infty(X,E) \) be the corresponding adjoint operator (with respect to certain hermitian metrics on \( E \) and \( F \), and a positive density on \( X \)). Put \( \square_E = D^*D \) and \( \square_F = DD^* \), and let \( \zeta(s,x;\square) := \text{tr} \nabla(-s)(x) \) be the corresponding local zeta-function.

Since one obviously has the equalities

\[
D(\square_E - \lambda)^{-1} = (\square_F - \lambda)^{-1}D \quad \text{and} \quad (\square_E - \lambda)^{-1}D^* = D^*(\square_F - \lambda)^{-1}
\]

it follows easily from (8) and from Proposition 3.7 that

\[
\text{Res}_{s=s_j} \left( \zeta(s,x;\square_E) - \zeta(s,x;\square_F) \right) = \frac{1}{2\text{ord}A} \left( \text{tr} \text{res}_x D^* \square_F^{-s_j-1} D - \text{tr} \text{res}_x \square_F^{-s_j-1} D D^* \right)
\]

or by interchanging \( D \) and \( D^* \):

\[
\text{Res}_{s=s_j} \left( \zeta(s,x;\square_E) - \zeta(s,x;\square_F) \right) = \frac{1}{2\text{ord}A} d_{\rho}(\square_F^{s_j-1} D^*,D) \quad \text{or} \quad \frac{1}{2\text{ord}A} d_{\rho}(\square_E^{s_j-1} D^*,D)
\]

(If the order of \( A \) is not an integer we have to use \( \tilde{\rho} \) extended as mentioned in Remark 3.8).

Recall that the inverse Mellin transform translates the singularities of \( \Gamma(s)^{-s}(x) \) into the "high temperature" expansion

\[
e^{-t\square}(x) \sim \sum_{j=0}^{\infty} a_j(x) t^{2m} \sum_{q=1}^{\infty} \beta_q(x) t^q \log t \quad (t \searrow 0)
\]

(\( n = \text{dim} X, \quad m = \text{ord} D \)) so that

\[
a_j(x) = \Gamma(s_j) \text{Res}_{s=s_j} (-s)(x)
\]

(\( s_j = \frac{n-j}{2m}; \quad j \neq n, \ n+2m, \ n+4m, \ldots \)),

\[
\beta_q(x) = \frac{(-1)^{q-1}}{q!} \text{Res}_{s=0} (-s)(x) \quad (q = 1,2,\ldots)
\]

and

\[
\alpha_0(x) = \square^{(0)}(x) + P_0(x,x;\square) \quad \text{(cf. (11))}
\]
depend locally on the symbol of $\square$. By comparing (23) and (24) with (21) we obtain

3.24. Proposition. One has

$$\text{tr} \alpha_j^F(x) - \text{tr} \alpha_j^F(x) = \frac{\Gamma(s_j)}{2^m} d\bar{\rho} \left( \square_j^{-s_j-1} D^*, D \right)$$

where $s_j = \frac{n-j}{2m}$; $j \neq n, n + 2m, n + 4m, \ldots$), and

$$\text{tr} \beta_q^F(x) - \text{tr} \beta_q^F(x) = \frac{(-1)^{q-1}}{2^mq!} d\bar{\rho} \left( \square_q^{-1} D^*, D \right).$$

(Also to this situation extends the remark after (21)).

3.25. Remark. If $D$ is a differential operator all $\beta_q(x)$'s vanish, and then also

$$\alpha_{n+2mq}(x) \equiv \frac{(-1)^{q}}{q!} \square^{(-q)}(x) \quad (q = 1, 2, \ldots)$$

depend locally on the symbol of $\square$ (cf. (15)).

Denote by $d^*$ and $c^q$ the symbols of $D^*$ and $\square^q$ respectively. Then the second equality in (20) implies that

$$c_F^{(q)} \circ b(\lambda; \square_F) = d \circ (c_E^{(q-1)} \circ b(\lambda; \square_E) \circ d^*)$$

On the other hand one has the obvious equality

$$c_E^{(q)} \circ b(\lambda; \square_E) = (c_E^{(q-1)} \circ b(\lambda; \square_E) \circ d^*) \circ d$$

This yields (via formulae (17) - (19) applied to $A = \square$ and $Q = \square^q$)

$$\text{tr} \alpha_n^{F \square}(u) - \text{tr} \alpha_n^{F \square}(u) = \frac{(-1)^{q-1}}{2mq!} \underset{\text{Res}_0}{\text{Res}_x} \int_0^\infty \left\{ \text{tr}(b^{(q-1)}(\lambda)) \circ d \right\}$$

$$- \text{tr}(d \circ b^{(q-1)}(\lambda)) \circ d$$

where $b^{(q-1)}(\lambda) \equiv c_E^{(q-1)} \circ b(\lambda; \square_E) \circ d^*$ and the upper "minus" means the "non-divergent part of" obtained by throwing out terms $O(\varnothing^{-1})$ and bigger (as $\lambda \to \infty$). Equality (28) suggests that some refinement
of the constructions of Section 1 might lead also in this case to expressing the left-hand side of (28) as an explicit exact form. This purpose is served by the formalism of a suitably "twisted" symplectic residue which is sketched briefly below.

We are guided by the idea of entering with residue under the sign of integral \( \int_0 d\lambda \) (in formulae like 3.19 or (28)).

3.26. \( \Lambda \)-twisted symplectic residue (A sketch). Let us return to the situation of Section 1. We were dealing there with a symplectic cone \( Y^{2n} \) whose base was denoted by \( Z^{2n-1} \). Let \( \Lambda \) be another space acted by the multiplicative group \( \mathbb{R}^+_x \). (\( \Lambda \) is assumed generally to be a manifold with boundary, but the boundary is allowed to have corners, etc.).

Let \( \phi^* \in C^\infty(Y,\Omega^\cdot_x(\Lambda)) \subset \Omega^\cdot_x(Y \times \Lambda) \) be a linear subspace which satisfies the following two conditions:

1) \( d_\lambda \phi^* = 0 \),

2) \( \phi^* = \bigoplus_{k \in \mathbb{Z}} \phi^k \) and \( L_{\Xi} = k \cdot \text{id} \) on \( \phi^k \)

(\( d_\lambda \) denotes de Rham differential in the \( \Lambda \)-direction, and \( \Xi \) is the Euler field determined by the product \( \mathbb{R}^+_x \)-action on \( Y \times \Lambda \)).

It should be clear that \( \phi^* \) is closed with respect to \( L_f \)'s for \( f \in \mathcal{P}^* \). Indeed, write \( \Xi \) as the sum \( \Xi + \Psi \) where \( \Xi \) and \( \Psi \) are the corresponding Euler fields on \( Y \) and \( \Lambda \). Then

\[
L_{\Xi}L_f = L_fL_{\Xi} + L_{[\Xi,H_f]} = L_f(L_{\Xi} + H_f - 1).
\]

In particular, \( \phi^* \) is made a \( \mathcal{P}^* \)-module. Let \( C_\cdot(\mathcal{P}^*;\phi^*) \) be the relevant chain complex. As before it is graded:

\[
C_\cdot(\mathcal{P}^*;\phi^*) = \bigoplus_{k \in \mathbb{Z}} C^{(k)}(\mathcal{P}^*;\phi^*).
\]

(cf. 1.17)

Without loss of generality we can assume that \( \phi^* \) is concentrated in a fixed codimension \( p \), i.e. \( \phi^* \in C^\infty(Y,\Omega^p_x(\Lambda)) \). Then a slight
modification of 1.12-13 and (2) of Section 1 shows that

\[
\sum_{i=1}^{q} \frac{\omega_n}{n} \land i_q \phi = q - n \quad (30)
\]

descends to a form \( \mu \phi f_1, \ldots, f_q \in \Omega_{p+q}(Y^\Lambda) \) where \( Y^\Lambda := (Y \times \Lambda)/\mathbb{R}_+^\times \),

and that the analogue of 1.16 holds for \( \mu \). Similarly, the collection of maps \( \phi \phi f_1 \land \ldots \land f_q \mapsto \mu \phi f_1, \ldots, f_q \) gives rise to a morphism of chain complexes

\[
\text{Res}^\Lambda : C_\cdot (P'; \phi') \to \Omega_\cdot (Y^\Lambda)[-p].
\]

The space \( Y^\Lambda \) is naturally fibred over \( Z \). Any singular chain \( Y \) of the codimension \( p \) in \( \Lambda \) which is preserved by the multiplicative group defines a family \( Y_Y \) of chains in fibres of the projection \( Y^\Lambda \to Z \). By \( Y_* \), we shall denote the map \( \Omega_\cdot (Y^\Lambda)[-p] \to \Omega_\cdot (Z) \) which is the composition of the restriction to \( Y^\Lambda \) with integration along fibres of \( Y^\Lambda \to Z \).

If \( Y \) is not closed then \( Y_* \) is not a morphism. It may happen, moreover, that \( Y \) is not even compact. Then \( Y_* \) is only a partial map defined on "\( Y \)-integrable" forms. For both reasons simultaneously some exact forms on \( Y^\Lambda \) need not remain exact when pushed down to \( Z \). Precisely this phenomenon stands behind the fact that \( a_0 = \text{tr} a_0 F(x) - \text{tr} a_0 F(x) \) may not be exact. We shall exploit it to propose in one of subsequent papers(*) a fresh approach to index theory.

Much as we did in 1.29 the above construction can be extended to the case of coefficients in a vector bundle on \( Z \). The next step is to introduce twisted symplectic residue into the constructions of Section 2 (we omit the details).

(*) Say, in 'Chapter V'.
3.27. Example. Let \( \Lambda \) be a closed sector of the complex plane.

For a fixed \( m \in \mathbb{R} \) we define the action of \( \mathbb{R}_+^X \) on \( \Lambda \) as:

\[
\chi^{(m)}_t : \lambda \mapsto t^m \lambda.
\]

Put \( Y = T_0^* X \) and \( \varphi^\Lambda = \{ \varphi \in C^\infty(\mathfrak{g}, \hat{\Omega}_{\text{hol}}(\Lambda)) | (L^\infty + m L | L | \frac{d}{d L} - \lambda) \varphi = 0 \} \).

The ray \( \text{Arg } \lambda = \theta \) (contained in \( \Lambda \)) will play a role of \( Y \). It is neither compact nor a cycle.

Suppose we have a \( \Lambda \)-elliptic \( \psi\text{DO} \) \( C^\infty_{\text{comp}}(X, E) \to C^\infty(X, E) \) of the order \( m \) (that will be assumed to be positive; recall that \( \Lambda \)-elliptic means that no eigenvalue of the principal symbol of \( \Lambda \) can lie in the sector \( \Lambda \)). Then for every local coordinate patch \( U \) the corresponding '\( \lambda \)-symbol' \( \varphi^\Lambda \) (cf. (9)) is defined, and it should be clear that \( \varphi^\Lambda = b(\lambda) d\lambda \) determines an element of the formally completed space \( \hat{\varphi}^\Lambda \) (in fact, each \( b^\Lambda_{m-j}(\lambda) d\lambda \) belongs to \( \varphi^\Lambda \) \( \text{Mat}_k(\mathbb{C}) \) where \( k = \text{rank } E \)). Since all \( b^\Lambda_{m-j}(\lambda) \)'s (except \( b^\Lambda_0(\lambda) \)) decrease for a fixed \((u, \xi)\) as \( O(|\lambda|^{-2}) \), \( \text{Res}^\Lambda_0(b(\lambda) d\lambda) \) is apparently \( Y \)-integrable, and 3.19 can be rewritten as

\[
Z_{\theta, u}(A) = -\beta_{n^X} Y^X_\ast |\text{Res}^\Lambda_0(b(\lambda) d\lambda)|. \tag{32}
\]

Similarly, (18) can be rewritten as

\[
Z_{\theta, u}(Q|A) = -\beta_{n^X} Y^X_\ast |\text{Res}^\Lambda_0(q \circ b(\lambda) d\lambda)|. \tag{33}
\]

Notice that since \( q \) has no negative components, \( \text{Res}^\Lambda_0(q \circ b(\lambda) d\lambda) \) is still \( Y \)-integrable (the component of \( q \circ b(\lambda) d\lambda \) of weight \(-n \) behaves like \( O(|\lambda|^{-2} d\lambda) \)).

Finally, \( Z_{\theta, u}^{(q)}(A) \) for a differential \( A \) can be also expressed as

\[
Z_{\theta, u}^{(q)}(A) = -\beta_{n^X} Y^X_\ast |\text{Res}^\Lambda_0(\lambda^q b(\lambda) d\lambda)|. \tag{34}
\]

(in this case the component of \( \lambda^q b(\lambda) d\lambda \) of weight \(-n \) behaves like \( O(|\lambda|^{-1-n-(m-1)} q d\lambda) \).
Now we can return to the differences
\[ \tilde{\alpha}_{n+2mq}(x) \equiv \text{tr} \alpha_{n+2mq}^F(x) - \text{tr} \alpha_{n+2mq}^E(x) \]
of the coefficients of two heat kernel expansions when \( D \) is a differential operator. We shall sketch below why for \( q > 0 \) \( \tilde{\alpha}_{n+2mq}(x) \) can be expressed as fine explicit exact forms whereas for \( q = 0 \) this fails completely.

First observe that for all \( q \) (i.e. \( q = 0, 1, 2, \ldots \)) the form
\[ \tilde{\alpha}_{n+2mq}(u) = \frac{(-1)^{q-1} \gamma}{2mq!} \tau_0 \gamma \left| \text{Res} \frac{L}{\lambda^2} \left( \text{tr}(\phi(q-1) \circ d) - \text{tr}(d \circ \phi(q-1)) \right) \right| (35) \]
written explicitly as an exact form on \( (T^*_0U) \). This follows from the corresponding '\( \Lambda \)-twisted version' of Proposition 2.28. That form for \( q \geq 1 \) is undoubtedly \( \gamma \)-integrable while for \( q = 0 \) (and for general \( D \)) it isn't. Notice also that there is no 'boundary term' in this case after the integration along \( \gamma \) was performed. This is related to the fact that \( \lambda = 0 \) is the fixed point of the \( R^+ \)-action on \( \Lambda \) and hence the Euler field \( \Psi \) vanishes at \( \lambda = 0 \) assuring thereby vanishing of (30) after restriction on \( T^*_0U \times \{0\} \).

This gives the required representation locally. In order to obtain it globally one can use either the partition of unity argument (since \( \phi(q-1) \) and \( d \) in (35) can be regarded as being independent without destroying the mechanism of exactness described above or to try to establish the '\( \Lambda \)-twisted' analogues of Proposition 3.7 and its more precise version for operators of order \( \leq 2 \) (cf. Remark 3.10.2)). The details will be presented elsewhere.

3.28. Secondary classes. As is well known, for operators arising in Riemannian geometry such as Euler, signature and Cauchy-Riemann operators, or more physically significant Dirac operator, the differences \( \tilde{\alpha}_j(x) \equiv \text{tr} \alpha_j^F(x) - \text{tr} \alpha_j^E(x) \) (for \( j < n \)) have in a number of cases tendency to vanish pointwise. This is called a local cancellation of divergencies (\( \alpha_j \)'s with \( j < \dim X \) correspond to
divergent terms in (22)), and is usually considered as a manifestation of some super-symmetries or integrability conditions (cf. e.g. local cancellation for the Cauchy-Riemann operator on a Kähler manifold \[5],[7]. Such cancellations often rely on methods of Invariant Theory of classical groups (cf. [1], [6] and other papers by P.B. Gilkey).

In any case, whenever a local cancellation occurs Proposition 3.24 tells us that in place of the coefficient \( \bar{a}_j(x) \) which has vanished arises a certain cohomology class (namely the class \( \bar{p}(\sum_{s,j}^{-1} D^*,D) \)). In general, it seems to be no reason a priori for these classes to vanish (cf. Example 3.29 below) subject to the impression that the mechanism of local cancellations is rather some extra feature and probably should be regarded as being imposed from the outside on the fairly universal formalism of noncommutative residue.

The same applies to the differences \( \bar{\beta}_q(x) \equiv \text{tr} \bar{\beta}_q^E(x) - \text{tr} \bar{\beta}_q^F(x) \) in case they vanish.

3.29. Example. Let \( E = F \) and \( D = A + \sqrt{-1}B \) where \( A \) and \( B \) are self-adjoint. Equality (27) yields

\[
\bar{\beta}_1(x) = \frac{1}{2m} \text{d}\bar{p}(D^*,D) = \frac{\sqrt{-1}}{m} \text{d}\bar{p}(A,B). 
\]

Assume that \( X = T^n \) is an n-dimensional torus and \( D \) is a \( \psi \text{DO} \) with constant coefficients (in particular, \( E \) is assumed to be a trivial bundle). Then \( \bar{p}(A,B) \) possesses a canonical representative (cf. Remark 3.10.2)) and this representative is clearly a constant form, and hence \( \bar{\beta}_q(x) \) can be obtained in this way. Let e.g. \( A \) be the 1-st order operator

\[
A = \text{c}B^\alpha + \sum_{j=1}^n \alpha_j \frac{\partial}{\partial x_j} 
\]
where $\mathcal{D} = -\sqrt{-1} \sum_{j=1}^{n} \gamma^j \partial_j x^j$ is the standard Dirac operator, $\alpha_j$'s are arbitrary complex scalars and $c$ is a constant big enough to assure ellipticity of (37). As $B$ we take an arbitrary scalar $\psi\mathcal{D}O$ with principal symbol $|\xi|^{-n}$ (e.g. $B = (1 + \Delta)^{-n/2}$). Note that $B$ can be chosen so that it commute with $A$.

According to Proposition 3.12 and Definition 2.13 $\bar{\rho}(A,B)$ is represented by

$$\rho_x(A,B) = \sum_{j=1}^{n} \text{tr}(-\sqrt{-1}c \gamma^j + \alpha_j) \frac{1}{\sqrt{2\pi}} \text{res}_x B$$

$$= \frac{d_n \nu_n}{(2\pi)^n} \sum_{j=1}^{n} (-1)^{j-1} \alpha_j |dx^1 \wedge \ldots \wedge dx^j \wedge \ldots \wedge dx^n|$$

if $n > 1$ ($\nu_n$ is the volume of the unit sphere in $\mathbb{R}^n$ and $d_n = 2^{[n/2]}$ is the rank of the spinor bundle), and

$$\rho_x(A,B) = -\frac{\sqrt{-1}c + \alpha}{\pi} \left| \frac{dx}{dx} \right|$$

if $n = 1$. (The asymmetry between the two cases is related to the fact that for $n = 2v + 1$ and $v \geq 1$ the matrix $\gamma^{2v+1}$ is up to a power of $\sqrt{-1}$ the product of all $\gamma^j$'s with $j \leq 2v$, and hence is traceless, whereas for $n = 1$, $\gamma^1 = 1$).

Equalities (38) and (39) show that every cohomology class from $H^{n-1}(T^n,\mathcal{E})$ serves as the secondary class $\bar{\rho}(D^*,D)$ for a certain suitably rescaled and perturbed Dirac operator $D$. In the considered example not only all coefficients $\alpha_j(x)$'s and $\beta_q(x)$'s vanish but also $e^{-tD^*D} - e^{-tDD^*} \equiv 0$.

3.30. Remark. The above example shows even more. Recall that a $\psi DO D \in \text{CL}_\text{prop}^*(X,\mathcal{E})$ (for notation cf. 0.10) is called normal if $D^* \wedge D$ is a 2-cycle on the Lie algebra $g \equiv \text{CL}_\text{prop}^*(X,\mathcal{E})$. According to diagram (5) $\bar{\rho}$ defines a map $H_2(g) \rightarrow H^{n-1}(X,\mathcal{E}_X)$. Example 3.29 shows that for $X = T^n$ and $\mathcal{E}$ - the (trivial) bundle of spinors this map is surjective and, moreover, that any class from
$H^{n-1}(X, \mathbb{C}_\mathcal{X})$ is, actually, the image of the cycle $D^* \wedge D$ where $D$ is a normal elliptic PDO of order 1. In [18] we shall demonstrate that at least the former assertion remains valid for arbitrary bundle and closed manifold.

3.31. Let us return once more to differential operators arising in Riemannian geometry. In a 'non-integrable' situation even the coefficients $\tilde{g}_j(x), (j < n)$ corresponding to divergent terms of the high-temperature expansion may not vanish (e.g. for non-Kähler Hermitian manifolds, cf. e.g. [4], [5]). In certain cases, at least, methods of Invariant Theory of classical groups yield specific forms $q_j(x)$ such that $\tilde{g}_j(x) = dq_j$ for all $j \neq \dim X$ (cf. [4],[7]). The order of considered operators rarely exceeds 2 (usually equals 1). Recall that for differential operators of order $\leq 2$ we possess just not only the class $\bar{\varrho}$ but also its canonical representative $\rho_X$ (cf. Remark 3.13). In particular, the forms

$$\frac{\Gamma(s_j)}{2m} \rho_X(\mathcal{L}_E^{-s_j-1} D^* D) - q_j(x) \quad (j \neq n, n + 2m, n + 4m)$$

are closed (as was pointed out cf. e.g. Example 3.27, it is possible to obtain analogues of the forms $\rho_X(\mathcal{L}_E^{-s_j-1} D^* D)$ also for $j = n + 2m, n + 4m, \ldots$). It would be very interesting to know more about the associated cohomology classes.
References


