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Thm 2.1.1 (Analytic Fredholm theorem)

Let Ω be an open connected subset of $\mathbb{C}$. Let $\Psi : \Omega \to K(\mathcal{H})$ be an analytic operator-valued function. Then one of the following alternative holds :

(i) $(1-\Psi(z))^{-1}$ exists for no $z \in \Omega$.

(ii) $(1-\Psi(z))^{-1}$ exists for all $z \in \Omega \setminus S$ where S is a discrete subset of Ω. In this case, $(1-\Psi(z))^{-1}$ is meromorphic in Ω, analytic in $\Omega \setminus S$, the residue at the poles are finite rank operators, and if $z \in S$ then the equation $\Psi(z)f = f$ has a nonzero solution in $\mathcal{H}$.

proof) I will prove that near any $z_0 \in \Omega$. either (i) or (ii) holds.

Because Ω is an connected subset, we can link z_0 with $z (\in \Omega)$ by a polygonal line.

We can discuss each points in this polygonal line. So, we can convert near any $z_0 \in \Omega$. into Ω.

Given $z_0 \in \Omega$, because Ψ is analytic, we can choose $r > 0$ so that

$|z - z_0| < r$ implies $\|\Psi(z) - \Psi(z_0)\| < \frac{1}{2}$

From the definition of a compact operator, we can choose a finite rank operator F so that $\|\Psi(z_0) - F\| < \frac{1}{2}$

$B_r(z_0) := \{ z \in \mathbb{C} \mid |z - z_0| < r \}$

$z \in B_r(z_0) \Rightarrow \|\Psi(z) - F\| < 1$.

By Neumann series, $(1 - \Psi(z) + F)^{-1}$ exists and is analytic.

Because F is finite rank operator, there are independent vector $\psi_1, \cdots, \psi_N$ so that

$F(\varphi) = \sum_{i=1}^{N} \alpha_i(\varphi) \psi_i$ for all $\varphi \in \mathcal{H}$. By the Riesz Lemma, $\phi_i \in \mathcal{H}$ exists

so that $\alpha_i(\varphi) = (\phi_i, \varphi)$ for all $\varphi \in \mathcal{H}$.

$\therefore \quad F(\varphi) = \sum_{i=1}^{N} (\phi_i, \varphi) \psi_i \quad (\forall \varphi \in \mathcal{H})$.

Let $\phi_n(z) := ((1 - \Psi(z) + F)^{-1})^* \phi_n$, $g(z) := F(1 - \Psi(z) + F)^{-1} = \sum_{n=1}^{N} (\phi_n(z), \cdot) \psi_n$

In this case $I - \Psi(z) = I - \Psi(z) + F - F(I - \Psi(z) + F)^{-1}(I - \Psi(z) + F)$

$$= (I - F(I - \Psi(z) + F)^{-1})(I - \Psi(z) + F) = (I - g(z))(I - \Psi(z) + F)$$

so, $(I - \Psi(z))^{-1}$ exists $(z \in D_r) \iff (I - g(z))^{-1}$ exists $\cdots (*)$

and $\exists\psi\neq 0$ s.t. $\psi=\Psi(z)\psi \Leftrightarrow \exists\varphi\neq 0$ s.t. $\varphi=g(z)\varphi$ … ①

$\because$ ① $\Rightarrow$ $\psi=\Psi(z)\psi \Rightarrow (1-\Psi(z))\psi=0 \Rightarrow (1-g(z))\underbrace{(I-\Psi(z)+F)\psi}_{=\varphi}=0$

$\Leftarrow$ $\varphi=g(z)\varphi \Rightarrow (1-g(z))\varphi=0$

Because $I-\Psi(z)+F$ is bijective, $(0\neq)\psi$ exist so that $\varphi=(I-\Psi(z)+F)\psi$

$\therefore (I-\Psi(z))\psi=(I-g(z))(I-\Psi(z)+F)\psi=(I-g(z))\varphi=0.$

If $g(z)\varphi=\varphi \Rightarrow \varphi=\sum_{n=1}^{N}\beta_n\psi_n$, $\beta_n=\sum_{m=1}^{N}(\phi_n(z),\psi_m)\beta_m$ … ②

$\because$ ② $\varphi=g(z)\varphi=\sum_{n=1}^{N}(\phi_n(z),\varphi)\psi_n=\sum_{n=1}^{N}\beta_n\psi_n$. $\beta_n=(\phi_n(z),\varphi)$ … ③

$\therefore \varphi=\sum_{n=1}^{N}\beta_n\psi_n$ and in ③, substitute $\sum_{n=1}^{N}\beta_n\psi_n$ for φ

$\sum_{n=1}^{N}(\phi_n(z),\sum_{m=1}^{N}\beta_m\varphi_m)\psi_n=\sum_{n=1}^{N}\beta_n\psi_n$

$\therefore \beta_n=(\phi_n(z),\sum_{m=1}^{N}\beta_m\varphi_m)=\sum_{m=1}^{N}(\phi_n(z),\beta_m\psi_m)$

Conversely, if ② has a solution $\{\beta_1,\cdots,\beta_n\}$, then $\varphi=\sum_{n=1}^{N}\beta_n\psi_n$ is a solution of $g(z)\varphi=\varphi$.

So, $\varphi=g(z)\varphi$ $(\varphi\neq 0)$

$\Leftrightarrow$ $\beta_n=\sum_{m=1}^{N}(\phi_n(z),\psi_m)\beta_m$ $(n=1,\cdots,N)$

$$\Leftrightarrow \begin{pmatrix}\beta_1\\ \vdots\\ \beta_N\end{pmatrix}=\begin{pmatrix}(\phi_1(z),\psi_1) & \cdots & (\phi_1(z),\psi_N)\\ (\phi_2(z),\psi_1) & \cdots & (\phi_2(z),\psi_N)\\ \vdots & & \vdots\\ (\phi_N(z),\psi_1) & \cdots & (\phi_N(z),\psi_N)\end{pmatrix}\begin{pmatrix}\beta_1\\ \vdots\\ \beta_N\end{pmatrix}$$

$$\Leftrightarrow \left(I-\begin{pmatrix}(\phi_1(z),\psi_1) & \cdots & (\phi_1(z),\psi_N)\\ \vdots & & \vdots\\ (\phi_N(z),\psi_1) & \cdots & (\phi_N(z),\psi_N)\end{pmatrix}\right)\begin{pmatrix}\beta_1\\ \vdots\\ \beta_N\end{pmatrix}=0 \quad ((\beta_1\cdots\beta_n)\neq 0)$$

$\Leftrightarrow$ $\det(\delta_{nm}-(\phi_n(z),\psi_m))=0.$

$\therefore \varphi=g(z)\varphi$ $(\varphi\neq 0)$ $\Leftrightarrow$ $\det(\delta_{nm}-(\phi_n(z),\psi_m))=0.$

$(\phi_n(z),\psi_m)$ is analytic, so $S_r(z_0)=\{z \mid z\in B_r(z_0), \det(\delta_{nm}-(\phi_n(z),\psi_m))=0\}$ is discrete set or $S_r(z_0)=B_r(z_0)$.

Let $\det(\delta_{nm} - (\phi_n(z), \psi_m)) \neq 0$. In this case, given ψ, $\varphi = \psi + \sum_{n=1}^{N} \beta_n \psi_n$ $\left(\beta_n = (\phi_n(z), \psi) + \sum_{m=1}^{N} (\phi_n(z), \psi_m)\beta_m\right)$

$$
\begin{aligned}
(I - g(z))\varphi &= \varphi - g(z)\varphi \\
&= \psi + \sum_{n=1}^{N} \beta_n \psi_n - g(z)\psi - g(z)\sum_{n=1}^{N} \beta_n \psi_n \\
&= \psi + \cancel{\sum_{n=1}^{N} (\phi_n(z), \psi)\psi_n} + \cancel{\sum_{n=1}^{N}\sum_{m=1}^{N} (\phi_n(z), \psi_m)\beta_m \psi_n} - \cancel{\sum_{n=1}^{N} (\phi_n(z), \psi)\psi_n} - \cancel{\sum_{n=1}^{N} \left(\phi_n(z), \sum_{m=1}^{N} \beta_m \psi_m\right)\psi_n} \\
&= \psi
\end{aligned}
$$

$\therefore$ $I - g(z)$ is surjective. and, $(I - g(z))\varphi = 0 \Leftrightarrow \varphi = 0$ $\therefore$ $I - g(z)$ is injection.

$\therefore$ $(I - g(z))^{-1}$ exists.

$\therefore$ $(I - g(z))^{-1}$ exists $\Leftrightarrow$ $z \notin S_r$

When $S_r = B_r(z_0)$, by $(*)$. $(I - \Psi(z))^{-1}$ do not exist for all $z \in B_r(z_0)$.

When S_r = discrete, $(I - \Psi(z))^{-1}$ exist for all $z \in B_r(z_0) \setminus S_r(z_0)$

$$\boxed{\begin{array}{l}\text{Fact } f\text{: analytic} \\ z_0 \text{ s.t. } f(z_0) = 0 \Rightarrow z_0 \text{ is pole of } \dfrac{1}{f(z)}\end{array}}$$

$(\delta_{nm} - (\phi_n(z), \psi_m))$ is analytic, so $(\delta_{nm} - (\phi_n(z), \psi_m))^{-1}$ is meromorphic.

Hence $(I - \Psi(z))^{-1}$ is meromorphic in Ω.

the residues at the poles of $(\delta_{nm} - (\phi_n(z), \psi_m))^{-1}$ are finite rank operator, so the same is true of $(I - \Psi(z))^{-1}$ //

Cor 2.3.4 Let $a, b \in \mathbb{R}^n$ be positive and ordered, and suppose that
$\prod_{j=1}^{k} b_j \leq \prod_{j=1}^{k} a_j$ for any $k \in \{1, \cdots, n\}$ $\cdots$ ①

Then, for any continuous, monotone increasing function $g : [0, \infty) \to \mathbb{R}_+$ with $t \mapsto g(e^t)$ convex, we have that

$$\sum_{j=1}^{k} g(b_j) \leq \sum_{j=1}^{k} g(a_j) \quad \text{for any } k \in \{1, \cdots, n\}.$$

In particular, $g(x) = x \Rightarrow \sum_{j=1}^{k} b_j \leq \sum_{j=1}^{k} a_j$

Exercise 2.3.5 Check the details of the previous proof

Proof of Cor 2.3.4

Because g is monotone increasing function, assume without loss of generality that a_j and b_j are all non-zero.

First, let r be positive so that all ra_j, rb_j are bigger than 1. $\tilde{a}_j := ra_j$, $\tilde{b}_j := rb_j$

By ①, $\prod_{j=1}^{k} \tilde{b}_j \leq \prod_{j=1}^{k} \tilde{a}_j$

$$\Rightarrow \sum_{j=1}^{k} \log \tilde{b}_j \leq \sum_{j=1}^{k} \log \tilde{a}_j$$

$\hat{a} := (\log \tilde{a}_1, \log \tilde{a}_2, \cdots, \log \tilde{a}_n)$

$\hat{b} = (\hat{b}_1, \hat{b}_2, \cdots, \hat{b}_n) := (\log \tilde{b}_1, \log \tilde{b}_2, \cdots, \log \tilde{b}_n)$

$\hat{a}, \hat{b}$ satisfy (2.14) in Thm 2.3.2

$f(t) := g(r^{-1} e^t)$ is convex and increasing

$\because$ convex $t, s \in \mathbb{R}$, $0 < \theta < 1$

$$f((1-\theta)t + \theta s) = g(r^{-1} e^{(1-\theta)t + \theta s})$$

$r^{-1} = e^{-\log r}$, $-\log r = (1-\theta)(-\log r) + \theta(-\log r)$

$$\therefore f((1-\theta)t + \theta s) = g(e^{(1-\theta)(t - \log r) + \theta(s - \log r)}) \leq (1-\theta) g(e^{t - \log r}) + \theta g(e^{s - \log r})$$

$\because t \mapsto g(e^t)$ is convex $= (1-\theta) g(r^{-1} e^t) + \theta g(r^{-1} e^s)$

increasing

$t \le s \Rightarrow r^{-1}e^{t} \le r^{-1}e^{s}$

$\Rightarrow g(r^{-1}e^{t}) \le g(r^{-1}e^{s})$

$\Rightarrow f(t) \le f(s)$

We define

$\Phi(x) := \sum_{j=1}^{n} f(x_j)$ $\quad x = (x_1, \cdots, x_n) \in [0, \infty)^n$ and

$\phi : \mathbb{C}^n \to \mathbb{R}^+$ $\quad \phi(c) := \Phi(c_1^*, \cdots, c_n^*)$, $c_j^* = |c_{k_j}|$

$\Rightarrow \phi(c) = \sum_{j=1}^{n} f(|c_j|)$

$t = (t_1, \cdots, t_n)$, $s = (s_1, \cdots, s_n) \in \mathbb{C}^n$, $0 < \theta < 1$,

$$\begin{aligned}
\phi(\theta t + (1-\theta)s) &= \sum_{j=1}^{n} f(|\theta t_j + (1-\theta)s_j|) \\
&\le \sum_{j=1}^{n} f(\theta|t_j| + (1-\theta)|s_j|) \\
&= \theta \sum_{j=1}^{n} f(|t_j|) + (1-\theta)\sum_{j=1}^{n} f(|s_j|) \\
&= \theta \Phi(t) + (1-\theta)\Phi(s)
\end{aligned}$$

$\therefore$ ϕ is convex

Hence, by Thm 2.3.2, $\quad \phi(\hat{b}) \le \phi(\hat{a})$

$\phi(\hat{b}) = \sum_{j=1}^{n} g(r^{-1}e^{\log r b_j}) = \sum_{j=1}^{n} g(b_j)$

$\phi(\hat{a}) = \sum_{j=1}^{n} g(r^{-1}e^{\log r a_j}) = \sum_{j=1}^{n} g(a_j)$

$\therefore \quad \sum_{j=1}^{n} g(b_j) \le \sum_{j=1}^{n} g(a_j)$ //

・$g(x) = x$. $\quad g(e^{t}) = e^{t} \Rightarrow f''(t) = e^{t} \ge 0 \Rightarrow f$: convex

Hence $\quad \sum_{j=1}^{k} b_j \le \sum_{j=1}^{k} a_j$

Exercise 3.3.5 Show that for any $A \in \mathcal{L}_1$ one has $\mathrm{Tr}_\omega(A)=0$.

Proof Fact : $a \in C_0 = \{a=(a_n) \mid \lim_{n\to\infty} a_n = 0\} \Rightarrow \omega(a)=0$.

$A \in \mathcal{L}_1 \Rightarrow \sum_{j=1}^{\infty} \mu_j(A) < \infty$

$$\therefore \lim_{n\to\infty} \frac{1}{\ln(n+1)} \sum_{j=1}^{n} \mu_j(A) = 0.$$

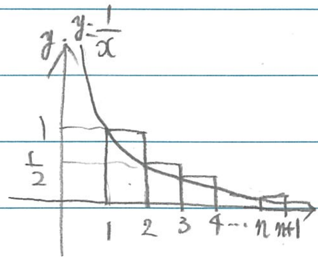
$$\therefore \left(\frac{1}{\ln(n+1)} \sum_{j=1}^{n} \mu_j(A) \right)_{n\in\mathbb{N}} \in C_0$$

$$\therefore \mathrm{Tr}_\omega(A) = \omega\left(\left(\frac{1}{\ln(n+1)} \sum_{j=1}^{n} \mu_j(A) \right)_{n\in\mathbb{N}} \right) = 0.$$ //

Exercise 3.3.3 Show that there exists an element $A \in (M_{1,\infty})_+$ which satisfies $\mathrm{Tr}_\omega(A)=1$.

Fact : $\omega(1+C_0)=1$ $\quad 1=(1,1,1,\cdots)$, $C_0=(a_n)_{n=1}^{\infty}$ s.t. $\lim_{n\to\infty} a_n = 0$

Let $A \in (M_{1,\infty})_+$ s.t. $\mu_j(A) = \frac{1}{j}$



$$\ln(n+1) = \int_1^{n+1} \frac{1}{x}dx \le \sum_{j=1}^{n} \frac{1}{j} \le 1 + \int_1^n \frac{1}{x}dx = 1+\ln n.$$

$$\therefore 1 \le \frac{1}{\ln(n+1)} \sum_{j=1}^{n} \frac{1}{j} \le \frac{1}{\ln(n+1)} + \frac{\ln n}{\ln(n+1)}$$

$\frac{1}{\ln(n+1)} \to 0$, $\frac{\ln n}{\ln(n+1)} \to 0$ as $n\to\infty$

Hence, $\frac{1}{\ln(n+1)} \sum_{j=1}^{n} \frac{1}{j} \to 1$ as $n\to\infty$ so, we set $C_0 = \frac{1}{\ln(n+1)}\sum_{j=1}^{n}\frac{1}{j} - 1$,

$$\frac{1}{\ln(n+1)} \sum_{j=1}^{n} \mu_j(A) = 1 + C_0$$

$$\therefore \mathrm{Tr}_\omega(A) = \omega(1+C_0) = 1$$ //