Chapter 4

Heat kernel and ζ-function

In this chapter we present the links between the Dixmier traces and two other functions which are also quite well-known. Some additional definitions or results skipped in Chapter 3 will be introduced on the way. First of all, the notion of *symmetric* or *fully symmetric* linear functional can not be avoided any further. Recall that for any $A \in \mathscr{K}(\mathcal{H})$ the function $\mu(\cdot, A)$ has been introduced in (3.14).

Definition 4.0.19. *Let $\mathscr{M}_\psi$ be a Lorentz ideal introduced in* (3.16)*, and let φ be a linear functional on $\mathscr{M}_\psi$.*

(i) φ is symmetric *if for any $A, B \in \mathscr{M}_\psi$ with $A \geq 0$, $B \geq 0$ and satisfying $\mu(\cdot, B) = \mu(\cdot, A)$ one has $\varphi(B) = \varphi(A)$,*

(ii) φ is fully symmetric *if for any $A, B \in \mathscr{M}_\psi$ with $A \geq 0$, $B \geq 0$ and satisfying $\int_0^x \mu(y, B)\,\mathrm{d}y \leq \int_0^x \mu(y, A)\,\mathrm{d}y$ for any $x > 0$ one has $\varphi(B) \leq \varphi(A)$.*

Note that for the notion of a symmetric norm on ℓ_∞ had already been introduced in Definition 2.3.8 and coincide with the previous one in the discrete setting. On the other hand, the notion of fully symmetric functional was only mentioned in Section 3.4.3 but was not further developed at this place. However, the inequality $\int_0^x \mu(y, B)\,\mathrm{d}y \leq \int_0^x \mu(y, A)\,\mathrm{d}y$ corresponds to the notation $B \lll A$ in the discrete setting of Section 2.3. Note finally that a fully symmetric linear functional φ is automatically positive since $0 \leq A$ implies that $0 = \int_0^x \mu(y, 0)\,\mathrm{d}y \leq \int_0^x \mu(y, A)\,\mathrm{d}y$ for any $x > 0$, from which one infers that $0 = \varphi(0) \leq \varphi(A)$.

4.1 ζ-function residue

For a positive operator A the corresponding ζ-function is defined by the map

$$s \mapsto \zeta(s) := \mathrm{Tr}(A^s)$$

whenever this expression is meaningful. For example if there exists $s_0 > 1$ such that A^{s_0} belongs to the trace class ideal $\mathscr{J}_1$, then the previous expression is well-defined for any

$s \geq s_0$. A rather common assumption on A is to assume that $A^s \in \mathscr{J}_1$ for any $s > 1$ and to study the asymptotic behavior of $(s-1)\zeta(s)$ as $s \searrow 1$. For example if $A \in \mathscr{J}_1$, then the limit clearly exists and is equal to 0. The aim of this section is to consider more general positive operator A and to relate the limits (suitably defined) at $s = 1$ with some Dixmier traces. Here suitable means that we shall consider the limits in a broad sense, namely with the notion of extended limits already used in the previous chapter. Note that for convenience and in order to stay closer to the notations introduced so far, we shall replace the parameter s by $1 + 1/x$ and consider the limit $x \to \infty$.

First of all, recall that an extended limit γ on $L^\infty(\mathbb{R}_+)$ is a positive element of $L^\infty(\mathbb{R}_+)^*$ satisfying $\gamma(\mathbf{1}) = 1$ and such that $\gamma(f) = 0$ whenever $f \in L^\infty(\mathbb{R}_+)$ has compact support. Then, for any extended limit γ on $L^\infty(\mathbb{R}_+)$ one can define the function $\zeta_\gamma : \big(\mathscr{M}_{1,\infty}\big)_+ \to \mathbb{R}_+$ by

$$\zeta_\gamma(A) := \gamma\Big(x \mapsto \frac{1}{x}\mathrm{Tr}\big(A^{1+1/x}\big)\Big). \tag{4.1}$$

Our first duty is to check that this expression is well-defined. For that purpose, we shall need a result of which can be useful in other context. Its proof can be found in [Fac, Lem. 4.1].

Lemma 4.1.1. *Let $\mu_1, \mu_2 : \mathbb{R}_+ \to \mathbb{R}$ be two decreasing and upper-bounded functions satisfying for any $x > 0$*

$$\int_0^x \mu_1(y)\,\mathrm{d}y \leq \int_0^x \mu_2(y)\,\mathrm{d}y.$$

Then, for any convex and increasing function $f : \mathbb{R} \to \mathbb{R}$ and for any $x > 0$ one has

$$\int_0^x f\big(\mu_1(y)\big)\,\mathrm{d}y \leq \int_0^x f\big(\mu_2(y)\big)\,\mathrm{d}y.$$

Note that if μ_1 and μ_2 takes values in $\mathbb{R}_+$ an important example for the function f consists in the map $\mathbb{R}_+ \ni x \mapsto x^t$ for any $t \geq 1$.

Lemma 4.1.2. *For any extended limit γ on $L^\infty(\mathbb{R}_+)$ and for any $A \in \big(\mathscr{M}_{1,\infty}\big)_+$ one has $\zeta_\gamma(A) < \infty$.*

Proof. Observe first that if C is trace class and positive, then

$$\mathrm{Tr}(C) = \sum_j \lambda_j(C) = \sum_j \mu_j(C) = \int_0^\infty \mu(y, C)\,\mathrm{d}y$$

where the function $\mu(\cdot, C)$ was introduced in (3.14). Thus, for any $C \geq 0$ such that $C^{1+1/x} \in \mathscr{J}_1$ for some $x > 0$, one deduces by functional calculus that

$$\mathrm{Tr}(C^{1+1/x}) = \int_0^\infty \mu(y, C)^{1+1/x}\,\mathrm{d}y. \tag{4.2}$$

On the other hand, for any $A \in \big(\mathscr{M}_{1,\infty}\big)_+$ one has

$$\sup_{z>0} \frac{1}{\ln(z+1)} \int_0^z \mu(y,A)\,\mathrm{d}y =: \|A\|_{1,\infty}$$

which implies that for any $z > 0$

$$\int_0^z \mu(y,A)\,\mathrm{d}y \leq \|A\|_{1,\infty} \ln(1+z) = \int_0^z \frac{\|A\|_{1,\infty}}{1+y}\,\mathrm{d}y. \tag{4.3}$$

Now, by taking these results into account as well as the content of the previous lemma one infers that

$$\begin{aligned}\mathrm{Tr}(A^{1+1/x}) &= \int_0^\infty \mu(y,A)^{1+1/x}\,\mathrm{d}y \leq \int_0^\infty \Big(\frac{\|A\|_{1,\infty}}{1+y}\Big)^{1+1/x}\mathrm{d}y \\ &= \|A\|_{1,\infty}^{1+1/x} \int_0^\infty \frac{1}{(1+y)^{1+1/x}}\,\mathrm{d}y = x\,\|A\|_{1,\infty}^{1+1/x}.\end{aligned}$$

As a consequence of this inequality and since γ is an extended limit one directly gets that $\zeta_\gamma(A) \leq \|A\|_{1,\infty}$. □

The main property of the map ζ_γ is summarized in the following statement whose proof can be find either in [LSZ, Thm. 8.6.4] or in [SZ, Thm. 8].

Theorem 4.1.3. *For any extended limit γ on $L^\infty(\mathbb{R}_+)$ the map ζ_γ extends by linearity to a fully symmetric linear functional on $\mathscr{M}_{1,\infty}$.*

Let us just mention that for the linearity it is sufficient to show that ζ_γ is a weight on $\big(\mathscr{M}_{1,\infty}\big)_+$, namely that it is positive homogeneous and additive, see Definition 2.6.7. The map ζ_γ is sometimes called *a ζ-function residue*.

As already mentioned at the end of Chapter 3, the set of all normalized (*i.e.* of norm 1) fully symmetric linear functionals on $\mathscr{M}_{1,\infty}$ is in bijective correspondence with the set of all Dixmier traces, as defined in Definition 3.4.11. This statement corresponds to the main result of [KSS]. We are naturally led to the following result.

Corollary 4.1.4. *For any extended limit γ on $L^\infty(\mathbb{R}_+)$ there exists a dilation invariant extended limit ω on $L^\infty(\mathbb{R}_+)$ such that*

$$\zeta_\gamma = \mathrm{Tr}_\omega.$$

It is then natural to wonder about the relation between γ and ω. In fact, a simple relation has been exhibited only in a restricted setting, see [SZ, Thm.] or [LSZ, Thm. 8.6.8]. For stating the result, let us recall from Section 3.4.1 that starting from a translated invariant extended limit ω on $L^\infty(\mathbb{R})$ we have defined a dilatation invariant extended limit $\exp^*\omega$ on $L^\infty(\mathbb{R}_+)$. Conversely, starting from a dilation invariant extended limit ω on $L^\infty(\mathbb{R}_+)$ one easily observes that defining $\ln^*\omega$ by $[\ln^*\omega](f) = \omega(f\circ\ln)$ we get a translation invariant extended limit on $L^\infty(\mathbb{R})$. Note that this extended limit is often denoted by $\omega\circ\ln$ but we shall avoid this ambiguous notation.

Theorem 4.1.5. *Let ω be a dilatation invariant extended limit on $L^\infty(\mathbb{R}_+)$ and assume that the extended limit $\ln^*\omega$ is also dilation invariant on $\mathbb{R}_+$. Then one has*

$$\zeta_{\ln^*\omega} = \mathrm{Tr}_\omega.$$

Remark 4.1.6. *In Corollary 4.1.4 it is claimed that one can associate to every ζ-function residue constructed with an extended limit γ on $L^\infty(\mathbb{R}_+)$ a Dixmier trace Tr_ω. However, let us mention that the converse is not true, namely the set of Dixmier traces on $\mathscr{M}_{1,\infty}$ is strictly larger than the set of ζ-function residues. We refer to [LSZ, Sec. 8.7] for more explanations and for a concrete counterexample.*

Let us close this section with a result about the uniqueness of the values taken by the ζ-function residues. This result will complement the one already mentioned in Theorem 3.4.18. Its proof can be found in [CS2, Thm. 7]. Recall that the notion of Dixmier measurable has been introduced in Definition 3.4.16 and means that all values obtained by $\mathrm{Tr}_\omega(A)$ are the same, for all dilation invariant extended limits ω.

Theorem 4.1.7. *For any $A \in \big(\mathscr{M}_{1,\infty}\big)_+$ the following conditions are equivalent:*

(i) A is Dixmier measurable,

(ii) The limit $\lim_{x\to\infty} \frac{1}{\ln(x+1)} \int_0^x \mu(y,A)\,\mathrm{d}y$ exits,

(iii) The limit $\lim_{x\to\infty} \frac{1}{x}\mathrm{Tr}\big(A^{1+1/x}\big)$ exits,

(iv) The limit $\lim_{s\searrow 1}(s-1)\mathrm{Tr}\big(A^s\big)$ exits,

Furthermore, if any of these conditions is satisfied, all limiting values exist and coincide with $\mathrm{Tr}_\omega(A)$ for any dilation invariant extended limit on $L^\infty(\mathbb{R}_+)$. These values also coincide with the limit $\lim_{s\searrow 1}(s-1)\zeta_\gamma(s)$ for any extended limit γ on $L^\infty(\mathbb{R}_+)$.

4.2 The heat kernel functional

The ζ-function mentioned in the previous section shares many properties with the heat kernel functional that we shall briefly introduce here. For a positive operator A the corresponding heat kernel function is defined by the map

$$s \mapsto \mathrm{Tr}\big(\exp(-sA^{-1})\big)$$

whenever such an expression is meaningful. Since the behavior of this function is usually studied around 0, we shall replace the parameter s by $1/x$ and consider the map $x \mapsto \frac{1}{x}\mathrm{Tr}\big(\exp(-(xA)^{-1})\big)$.

In order to study this function, we introduce the logarithmic mean $M : L^\infty(\mathbb{R}_+) \to L^\infty(\mathbb{R}_+)$ defined for $f \in L^\infty(\mathbb{R}_+)$ and any $x > 1$ by

$$[Mf](x) := \frac{1}{\ln(x)} \int_1^x f(y)\frac{\mathrm{d}y}{y}. \tag{4.4}$$

With this definition at hand, we define for any extended limit ω on $L^\infty(\mathbb{R}_+)$ the functional $\xi_\omega : \big(\mathscr{M}_{1,\infty}\big)_+ \to \mathbb{R}_+$ by

$$\xi_\omega(A) := (\omega \circ M)\Big(x \mapsto \frac{1}{x}\mathrm{Tr}\big(\exp(-(xA)^{-1})\big)\Big). \tag{4.5}$$

In the next statement we ensure that the above expression is well defined.

Lemma 4.2.1. *Let $A \in \big(\mathscr{M}_{1,\infty}\big)_+$ and consider ω an extended limit on $L^\infty(\mathbb{R}_+)$*

(i) The image by M of the function $x \mapsto \frac{1}{x}\mathrm{Tr}\big(\exp(-(xA)^{-1})\big)$ belongs to $L^\infty(\mathbb{R}_+)$,

(ii) The following equality holds

$$\xi_\omega(A) = \omega\Big(x \mapsto \frac{1}{\ln(x+1)}\mathrm{Tr}\big(A\exp(-(xA)^{-1})\big)\Big)$$

where $\xi_\omega(A)$ is defined by (4.5).

Proof. 1) Let us first consider $A \in \big(\mathscr{M}_{1,\infty}\big)_+$ and $\mu(y) := \|A\|_{1,\infty}\frac{1}{1+y}$ for any $y > 0$. Then, by the inequality (4.3) one has for any $z > 0$

$$\int_0^z \mu(y,A)\,\mathrm{d}y \le \int_0^z \frac{\|A\|_{1,\infty}}{1+y}\,\mathrm{d}y = \int_0^z \mu(y)\,\mathrm{d}y. \tag{4.6}$$

For any fixed $x > 0$, since the function $f : \mathbb{R}_+ \to \mathbb{R}_+$ defined by $f_x(z) := z\,\mathrm{e}^{-(xz)^{-1}}$ is convex and increasing on $\mathbb{R}_+$ we infer from Lemma 4.1.1 and from the functional calculus of self-adjoint operators that for any $x > 0$ one has

$$\begin{aligned}
\mathrm{Tr}\big(A\exp(-(xA)^{-1})\big) &= \int_0^\infty f_x\big(\mu(y,A)\big)\,\mathrm{d}y \\
&\le \int_0^\infty f_x\big(\mu(y)\big)\,\mathrm{d}y \\
&= \int_0^\infty \mu(y)\,\mathrm{e}^{-(x\mu(y))^{-1}}\,\mathrm{d}y \\
&= \int_0^\infty \|A\|_{1,\infty}\frac{1}{1+y}\,\mathrm{e}^{-(x\|A\|_{1,\infty})^{-1}(1+y)}\,\mathrm{d}y \\
&= \|A\|_{1,\infty}\int_{(x\|A\|_{1,\infty})^{-1}}^\infty \frac{1}{z}\,\mathrm{e}^{-z}\,\mathrm{d}z \\
&< \infty.
\end{aligned}$$

One thus deduces that $A\exp(-(xA)^{-1}) \in \mathscr{J}_1$ and that

$$\mathrm{Tr}\big(A\exp(-(xA)^{-1})\big) \in \mathcal{O}\big(\ln(x+1)\big) \quad \text{for } x \to \infty. \tag{4.7}$$

2) By definition we have

$$M\Big(x \mapsto \frac{1}{x}\mathrm{Tr}\big(\exp(-(xA)^{-1})\big)\Big) = \Big(x \mapsto \frac{1}{\ln(x)}\int_1^x \mathrm{Tr}\big(\,\mathrm{e}^{-(yA)^{-1}}\,\big)\frac{\mathrm{d}y}{y^2}\Big).$$

However, since

$$\int_1^x \mathrm{e}^{-(yA)^{-1}}\,\frac{\mathrm{d}y}{y^2} = \int_{1/x}^1 \mathrm{e}^{-uA^{-1}}\,\mathrm{d}u = A\,\mathrm{e}^{-(xA)^{-1}} - A\,\mathrm{e}^{-A^{-1}},$$

it follows that

$$M\Big(x \mapsto \frac{1}{x}\mathrm{Tr}\big(\exp(-(xA)^{-1})\big)\Big) = \Big(x \mapsto \frac{1}{\ln(x)}\big(\mathrm{Tr}(A\,\mathrm{e}^{-(xA)^{-1}}) - \mathrm{Tr}(A\,\mathrm{e}^{-A^{-1}})\big)\Big). \quad (4.8)$$

3) By taking the previous expression into account as well as the estimate obtained in (4.7), one infers that $M\Big(x \mapsto \frac{1}{x}\mathrm{Tr}\big(\exp(-(xA)^{-1})\big)\Big)$ is bounded for x large. In addition, since the r.h.s. of (4.8) is continuous and vanishes when $x \searrow 0$, one deduces the statement (i). Since ω is an extended limit and thus vanishes on $C_0(\mathbb{R}_+)$, the statement (ii) easily follows from the expression obtained in (4.8). □

The next statement is the analogue of Theorem 4.1.3 but for the heat kernel. Its proof can be found in [LSZ, Thm. 8.2.4].

Theorem 4.2.2. *For any dilation invariant extended limit γ on $L^\infty(\mathbb{R}_+)$ the map ξ_γ extends by linearity to a fully symmetric linear functional on $\mathscr{M}_{1,\infty}$.*

By the previous result one directly infers that a statement similar to the content of Corollary 4.1.4 holds for the functional ξ_γ. However, an even stronger result holds in this case.

Theorem 4.2.3. *(i) If ω is a dilation invariant extended limit on $L^\infty(\mathbb{R}_+)$ satisfying $\omega \circ M = \omega$, then $\xi_\omega = \mathrm{Tr}_\omega$,*

(ii) For any normalized fully symmetric linear functional φ on $\mathscr{M}_{1,\infty}$ there exists a dilation invariant extended limit ω on $L^\infty(\mathbb{R}_+)$ such that $\varphi = \xi_\omega$.

These two results can be found in [LSZ, Thm. 8.2.9 & Thm. 8.3.6] to which we refer for the proofs and for more information.