Chapter 1

Hilbert space and linear operators

The purpose of this first chapter is to introduce (or recall) many standard definitions related to the study of operators on a Hilbert space. Its content is mainly based on the first two chapters of the book [Amr].

1.1 Hilbert space

Definition 1.1.1. A (complex) Hilbert space $\mathcal{H}$ *is a vector space on* $\mathbb{C}$ *with a strictly positive scalar product (or inner product) which is complete for the associated norm*[1] *and which admits a countable orthonormal basis. The scalar product is denoted by* $\langle \cdot, \cdot \rangle$ *and the corresponding norm by* $\| \cdot \|$.

In particular, note that for any $f, g, h \in \mathcal{H}$ and $\alpha \in \mathbb{C}$ the following properties hold:

(i) $\langle f, g \rangle = \overline{\langle g, f \rangle}$,

(ii) $\langle f, g + \alpha h \rangle = \langle f, g \rangle + \alpha \langle f, h \rangle$,

(iii) $\|f\|^2 = \langle f, f \rangle \geq 0$, and $\|f\| = 0$ if and only if $f = 0$.

Note that $\overline{\langle g, f \rangle}$ means the complex conjugate of $\langle g, f \rangle$. Note also that the linearity in the second argument in (ii) is a matter of convention, many authors define the linearity in the first argument. In (iii) the norm of f is defined in terms of the scalar product $\langle f, f \rangle$. We emphasize that the scalar product can also be defined in terms of the norm of $\mathcal{H}$, this is the content of the *polarisation identity*:

$$4\langle f, g \rangle = \|f + g\|^2 - \|f - g\|^2 - i\|f + ig\|^2 + i\|f - ig\|^2. \tag{1.1}$$

From now on, the symbol $\mathcal{H}$ will always denote a Hilbert space.

[1] Recall that $\mathcal{H}$ is said to be complete if any Cauchy sequence in $\mathcal{H}$ has a limit in $\mathcal{H}$. More precisely, $\{f_n\}_{n \in \mathbb{N}} \subset \mathcal{H}$ is a Cauchy sequence if for any $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that $\|f_n - f_m\| < \varepsilon$ for any $n, m \geq N$. Then $\mathcal{H}$ is complete if for any such sequence there exists $f_\infty \in \mathcal{H}$ such that $\lim_{n \to \infty} \|f_n - f_\infty\| = 0$.

Examples 1.1.2. *(i)* $\mathcal{H} = \mathbb{C}^d$ *with* $\langle \alpha, \beta \rangle = \sum_{j=1}^d \overline{\alpha_j} \beta_j$ *for any* $\alpha, \beta \in \mathbb{C}^d$*,*

(ii) $\mathcal{H} = \ell_2(\mathbb{Z})$ *with* $\langle a, b \rangle = \sum_{j\in\mathbb{Z}} \overline{a_j} b_j$ *for any* $a, b \in \ell_2(\mathbb{Z})$*,*

(iii) $\mathcal{H} = L^2(\mathbb{R}^d)$ *with* $\langle f, g \rangle = \int_{\mathbb{R}^d} \overline{f(x)}\, g(x)\, \mathrm{d}x$ *for any* $f, g \in L^2(\mathbb{R}^d)$*.*

Let us recall some useful inequalities: For any $f, g \in \mathcal{H}$ one has

$$|\langle f, g \rangle| \leq \|f\|\, \|g\| \qquad \text{Schwarz inequality,} \tag{1.2}$$
$$\|f + g\| \leq \|f\| + \|g\| \qquad \text{triangle inequality,} \tag{1.3}$$
$$\|f + g\|^2 \leq 2\|f\|^2 + 2\|g\|^2, \tag{1.4}$$
$$\big|\|f\| - \|g\|\big| \leq \|f - g\|. \tag{1.5}$$

The proof of these inequalities is standard and is left as a free exercise, see also [Amr, p. 3-4]. Let us also recall that $f, g \in \mathcal{H}$ are said to be *orthogonal* if $\langle f, g \rangle = 0$.

Definition 1.1.3. *A sequence* $\{f_n\}_{n\in\mathbb{N}} \subset \mathcal{H}$ *is* strongly convergent *to* $f_\infty \in \mathcal{H}$ *if* $\lim_{n\to\infty} \|f_n - f_\infty\| = 0$*, or is* weakly convergent *to* $f_\infty \in \mathcal{H}$ *if for any* $g \in \mathcal{H}$ *one has* $\lim_{n\to\infty} \langle g, f_n - f_\infty \rangle = 0$*. One writes* $\mathrm{s-}\lim_{n\to\infty} f_n = f_\infty$ *if the sequence is strongly convergent, and* $\mathrm{w-}\lim_{n\to\infty} f_n = f_\infty$ *if the sequence is weakly convergent.*

Clearly, a strongly convergent sequence is also weakly convergent. The converse is not true.

Exercise 1.1.4. *In the Hilbert space* $L^2(\mathbb{R})$*, exhibit a sequence which is weakly convergent but not strongly convergent.*

Lemma 1.1.5. *Consider a sequence* $\{f_n\}_{n\in\mathbb{N}} \subset \mathcal{H}$*. One has*

$$\mathrm{s-}\lim_{n\to\infty} f_n = f_\infty \quad \Longleftrightarrow \quad \mathrm{w-}\lim_{n\to\infty} f_n = f_\infty \;\; \text{and} \;\; \lim_{n\to\infty} \|f_n\| = \|f_\infty\|.$$

Proof. Assume first that $\mathrm{s-}\lim_{n\to\infty} f_n = f_\infty$. By the Schwarz inequality one infers that for any $g \in \mathcal{H}$:

$$|\langle g, f_n - f_\infty \rangle| \leq \|f_n - f_\infty\|\, \|g\| \to 0 \qquad \text{as } n \to \infty,$$

which means that $\mathrm{w-}\lim_{n\to\infty} f_n = f_\infty$. In addition, by (1.5) one also gets

$$\big|\|f_n\| - \|f_\infty\|\big| \leq \|f_n - f_\infty\| \to 0 \quad \text{as} \quad n \to \infty,$$

and thus $\lim_{n\to\infty} \|f_n\| = \|f_\infty\|$.

For the reverse implication, observe first that

$$\|f_n - f_\infty\|^2 = \|f_n\|^2 + \|f_\infty\|^2 - \langle f_n, f_\infty \rangle - \langle f_\infty, f_n \rangle. \tag{1.6}$$

If $\mathrm{w-}\lim_{n\to\infty} f_n = f_\infty$ and $\lim_{n\to\infty} \|f_n\| = \|f_\infty\|$, then the right-hand side of (1.6) converges to $\|f_\infty\|^2 + \|f_\infty\|^2 - \|f_\infty\|^2 - \|f_\infty\|^2 = 0$, so that $\mathrm{s-}\lim_{n\to\infty} f_n = f_\infty$. □

Let us also note that if $\text{s}-\lim_{n\to\infty} f_n = f_\infty$ and $\text{s}-\lim_{n\to\infty} g_n = g_\infty$ then one has

$$\lim_{n\to\infty} \langle f_n, g_n\rangle = \langle f_\infty, g_\infty\rangle$$

by a simple application of the Schwarz inequality.

Exercise 1.1.6. *Let $\{e_n\}_{n\in\mathbb{N}}$ be an orthonormal basis of an infinite dimensional Hilbert space. Show that* $\text{w}-\lim_{n\to\infty} e_n = 0$*, but that* $\text{s}-\lim_{n\to\infty} e_n$ *does not exist.*

Exercise 1.1.7. *Show that the limit of a strong or a weak Cauchy sequence is unique. Show also that such a sequence is bounded,* i.e. *if $\{f_n\}_{n\in\mathbb{N}}$ denotes this Cauchy sequence, then* $\sup_{n\in\mathbb{N}} \|f_n\| < \infty$.

For the weak Cauchy sequence, the boundedness can be obtained from the following quite general result which will be useful later on. Its proof can be found in [Kat, Thm. III.1.29]. In the statement, Λ is simply a set.

Theorem 1.1.8 (Uniform boundedness principle)**.** *Let $\{\varphi_\lambda\}_{\lambda\in\Lambda}$ be a family of continuous maps*[2] $\varphi_\lambda : \mathcal{H} \to [0,\infty)$ *satisfying*

$$\varphi_\lambda(f+g) \leq \varphi_\lambda(f) + \varphi_\lambda(g) \qquad \forall f, g \in \mathcal{H}.$$

If the set $\{\varphi_\lambda(f)\}_{\lambda\in\Lambda} \subset [0,\infty)$ is bounded for any fixed $f \in \mathcal{H}$, then the family $\{\varphi_\lambda\}_{\lambda\in\Lambda}$ is uniformly bounded, i.e. *there exists $c > 0$ such that* $\sup_\lambda \varphi_\lambda(f) \leq c$ *for any $f \in \mathcal{H}$ with $\|f\| = 1$.*

In the next definition, we introduce the notion of subspace of a Hilbert space.

Definition 1.1.9. A subspace $\mathcal{M}$ *of a Hilbert space $\mathcal{H}$ is a linear subset of $\mathcal{H}$, or more precisely $\forall f, g \in \mathcal{M}$ and $\alpha \in \mathbb{C}$ one has $f + \alpha g \in \mathcal{M}$. The subspace $\mathcal{M}$ is* closed *if any Cauchy sequence in $\mathcal{M}$ converges strongly in $\mathcal{M}$.*

Note that if $\mathcal{M}$ is closed, then $\mathcal{M}$ is a Hilbert space in itself, with the scalar product and norm inherited from $\mathcal{H}$.

Examples 1.1.10. *(i) If $f_1, \dots, f_n \in \mathcal{H}$, then $\mathsf{Span}(f_1, \dots, f_n)$ is the closed vector space generated by the linear combinations of $f_1, \dots f_n$. $\mathsf{Span}(f_1, \dots, f_n)$ is a closed subspace.*

(ii) If $\mathcal{M}$ is a subset of $\mathcal{H}$, then

$$\mathcal{M}^\perp := \{f \in \mathcal{H} \mid \langle f, g\rangle = 0, \forall g \in \mathcal{M}\} \tag{1.7}$$

is a closed subspace of $\mathcal{H}$.

Exercise 1.1.11. *Check that in the above example the set $\mathcal{M}^\perp$ is a closed subspace of $\mathcal{H}$.*

[2] φ_λ is continuous if $\varphi_\lambda(f_n) \to \varphi_\lambda(f_\infty)$ whenever $\text{s}-\lim_{n\to\infty} f_n = f_\infty$.

Exercise 1.1.12. *Check that a subspace $\mathcal{M} \subset \mathcal{H}$ is dense in $\mathcal{H}$ if and only if $\mathcal{M}^\perp = \{0\}$.*

If $\mathcal{M}$ is a subset of $\mathcal{H}$ the closed subspace $\mathcal{M}^\perp$ is called *the orthocomplement of $\mathcal{M}$ in $\mathcal{H}$*. The following result is important in the setting of Hilbert spaces. Its proof is not complicated but a little bit lengthy, we thus refer to [Amr, Prop. 1.7].

Proposition 1.1.13 (Projection Theorem)**.** *Let $\mathcal{M}$ be a closed subspace of a Hilbert space $\mathcal{H}$. Then, for any $f \in \mathcal{H}$ there exist a unique $f_1 \in \mathcal{M}$ and a unique $f_2 \in \mathcal{M}^\perp$ such that $f = f_1 + f_2$.*

Let us close this section with the so-called Riesz Lemma. For that purpose, recall first that the dual $\mathcal{H}^*$ of the Hilbert space $\mathcal{H}$ consists in the set of all bounded linear functionals on $\mathcal{H}$, *i.e.* $\mathcal{H}^*$ consists in all mappings $\varphi : \mathcal{H} \to \mathbb{C}$ satisfying for any $f, g \in \mathcal{H}$ and $\alpha \in \mathbb{C}$

(i) $\varphi(f + \alpha g) = \varphi(f) + \alpha\varphi(g)$, (linearity)

(ii) $|\varphi(f)| \le c\|f\|$, (boundedness)

where c is a constant independent of f. One then sets

$$\|\varphi\|_{\mathcal{H}^*} := \sup_{0 \ne f \in \mathcal{H}} \frac{|\varphi(f)|}{\|f\|}.$$

Clearly, if $g \in \mathcal{H}$, then g defines an element φ_g of $\mathcal{H}^*$ by setting $\varphi_g(f) := \langle g, f\rangle$. Indeed φ_g is linear and one has

$$\|\varphi_g\|_{\mathcal{H}^*} := \sup_{0 \ne f \in \mathcal{H}} \frac{1}{\|f\|}|\langle g, f\rangle| \le \sup_{0 \ne f \in \mathcal{H}} \frac{1}{\|f\|}\|g\|\,\|f\| = \|g\|.$$

In fact, note that $\|\varphi_g\|_{\mathcal{H}^*} = \|g\|$ since $\frac{1}{\|g\|}\varphi_g(g) = \frac{1}{\|g\|}\|g\|^2 = \|g\|$.

The following statement shows that any element $\varphi \in \mathcal{H}^*$ can be obtained from an element $g \in \mathcal{H}$. It corresponds thus to a converse of the previous construction.

Lemma 1.1.14 (Riesz Lemma)**.** *For any $\varphi \in \mathcal{H}^*$, there exists a unique $g \in \mathcal{H}$ such that for any $f \in \mathcal{H}$*

$$\varphi(f) = \langle g, f\rangle.$$

In addition, g satisfies $\|\varphi\|_{\mathcal{H}^*} = \|g\|$.

Since the proof is quite standard, we only sketch it and leave the details to the reader, see also [Amr, Prop. 1.8].

Sketch of the proof. If $\varphi \equiv 0$, then one can set $g := 0$ and observe trivially that $\varphi = \varphi_g$.

If $\varphi \ne 0$, let us first define $\mathcal{M} := \{f \in \mathcal{H} \mid \varphi(f) = 0\}$ and observe that $\mathcal{M}$ is a closed subspace of $\mathcal{H}$. One also observes that $\mathcal{M} \ne \mathcal{H}$ since otherwise $\varphi \equiv 0$. Thus, let $h \in \mathcal{H}$ such that $\varphi(h) \ne 0$ and decompose $h = h_1 + h_2$ with $h_1 \in \mathcal{M}$ and $h_2 \in \mathcal{M}^\perp$ by Proposition 1.1.13. One infers then that $\varphi(h_2) = \varphi(h) \ne 0$.

For arbitrary $f \in \mathcal{H}$ one can consider the element $f - \frac{\varphi(f)}{\varphi(h_2)} h_2 \in \mathcal{H}$ and observe that $\varphi\big(f - \frac{\varphi(f)}{\varphi(h_2)} h_2\big) = 0$. One deduces that $f - \frac{\varphi(f)}{\varphi(h_2)} h_2$ belongs to $\mathcal{M}$, and since $h_2 \in \mathcal{M}^\perp$ one infers that

$$\varphi(f) = \frac{\varphi(h_2)}{\|h_2\|^2} \langle h_2, f\rangle.$$

One can thus set $g := \frac{\overline{\varphi(h_2)}}{\|h_2\|^2} h_2 \in \mathcal{H}$ and easily obtain the remaining parts of the statement. □

As a consequence of the previous statement, one often identifies $\mathcal{H}^*$ with $\mathcal{H}$ itself.

Exercise 1.1.15. *Check that this identification is not linear but* anti-linear.

1.2 Bounded linear operators

First of all, let us recall that a linear map B between two complex vector spaces $\mathcal{M}$ and $\mathcal{N}$ satisfies $B(f + \alpha g) = Bf + \alpha Bg$ for all $f, g \in \mathcal{M}$ and $\alpha \in \mathbb{C}$.

Definition 1.2.1. *A map $B : \mathcal{H} \to \mathcal{H}$ is* a bounded linear operator *if $B : \mathcal{H} \to \mathcal{H}$ is a linear map, and if there exists $c > 0$ such that $\|Bf\| \le c\|f\|$ for all $f \in \mathcal{H}$. The set of* all bounded linear operators on $\mathcal{H}$ *is denoted by $\mathscr{B}(\mathcal{H})$.*

For any $B \in \mathscr{B}(\mathcal{H})$, one sets

$$\begin{aligned} \|B\| &:= \inf\{c > 0 \mid \|Bf\| \le c\|f\| \ \forall f \in \mathcal{H}\} \\ &= \sup_{0 \ne f \in \mathcal{H}} \frac{\|Bf\|}{\|f\|}. \end{aligned} \tag{1.8}$$

and call it *the norm of* B. Note that the same notation is used for the norm of an element of $\mathcal{H}$ and for the norm of an element of $\mathscr{B}(\mathcal{H})$, but this does not lead to any confusion. Let us also introduce the *range* of an operator $B \in \mathscr{B}(\mathcal{H})$, namely

$$\mathsf{Ran}(B) := B\mathcal{H} = \{f \in \mathcal{H} \mid f = Bg \text{ for some } g \in \mathcal{H}\}. \tag{1.9}$$

This notion will be important when the inverse of an operator will be discussed.

Exercise 1.2.2. *Let $\mathcal{M}_1, \mathcal{M}_2$ be two dense subspaces of $\mathcal{H}$, and let $B \in \mathscr{B}(\mathcal{H})$. Show that*

$$\|B\| = \sup_{f \in \mathcal{M}_1, g \in \mathcal{M}_2 \text{ with } \|f\| = \|g\| = 1} |\langle f, Bg\rangle|. \tag{1.10}$$

Exercise 1.2.3. *Show that $\mathscr{B}(\mathcal{H})$ is a complete normed algebra and that the inequality*

$$\|AB\| \le \|A\| \, \|B\| \tag{1.11}$$

holds for any $A, B \in \mathscr{B}(\mathcal{H})$.

An additional structure can be added to $\mathscr{B}(\mathcal{H})$: an involution. More precisely, for any $B \in \mathscr{B}(\mathcal{H})$ and any $f, g \in \mathcal{H}$ one sets

$$\langle B^* f, g\rangle := \langle f, Bg\rangle. \tag{1.12}$$

Exercise 1.2.4. *For any $B \in \mathscr{B}(\mathcal{H})$ show that*

(i) B^* is uniquely defined by (1.12) and satisfies $B^* \in \mathscr{B}(\mathcal{H})$ with $\|B^*\| = \|B\|$,

(ii) $(B^*)^* = B$,

(iii) $\|B^*B\| = \|B\|^2$,

(iv) If $A \in \mathscr{B}(\mathcal{H})$, then $(AB)^* = B^*A^*$.

The operator B^* is called *the adjoint of B*, and the proof the unicity in (i) involves the Riesz Lemma. A complete normed algebra endowed with an involution for which the property (iii) holds is called a C^*-algebra. In particular $\mathscr{B}(\mathcal{H})$ is a C^*-algebra. Such algebras have a well-developed and deep theory, see for example [Mur]. However, we shall not go further in this direction in this course.

We have already considered two distinct topologies on $\mathcal{H}$, namely the strong and the weak topology. On $\mathscr{B}(\mathcal{H})$ there exist several topologies, for the time being we consider only three of them.

Definition 1.2.5. *A sequence $\{B_n\}_{n\in\mathbb{N}} \subset \mathscr{B}(\mathcal{H})$ is* uniformly convergent *to $B_\infty \in \mathscr{B}(\mathcal{H})$ if $\lim_{n\to\infty} \|B_n - B_\infty\| = 0$, is* strongly convergent *to $B_\infty \in \mathscr{B}(\mathcal{H})$ if for any $f \in \mathcal{H}$ one has $\lim_{n\to\infty} \|B_n f - B_\infty f\| = 0$, or is* weakly convergent *to $B_\infty \in \mathscr{B}(\mathcal{H})$ if for any $f, g \in \mathcal{H}$ one has $\lim_{n\to\infty}\langle f, B_n g - B_\infty g\rangle = 0$. In these cases, one writes respectively $u - \lim_{n\to\infty} B_n = B_\infty$, $s - \lim_{n\to\infty} B_n = B_\infty$ and $w - \lim_{n\to\infty} B_n = B_\infty$.*

Clearly, uniform convergence implies strong convergence, and strong convergence implies weak convergence. The reverse statements are not true. Note that if $\{B_n\}_{n\in\mathbb{N}} \subset \mathscr{B}(\mathcal{H})$ is weakly convergent, then the sequence $\{B_n^*\}_{n\in\mathbb{N}}$ of its adjoint operators is also weakly convergent. However, the same statement does not hold for a strongly convergent sequence. Finally, we shall not prove but often use that $\mathscr{B}(\mathcal{H})$ is also weakly and strongly closed. In other words, any weakly (or strongly) Cauchy sequence in $\mathscr{B}(\mathcal{H})$ converges in $\mathscr{B}(\mathcal{H})$.

Exercise 1.2.6. *Let $\{A_n\}_{n\in\mathbb{N}} \subset \mathscr{B}(\mathcal{H})$ and $\{B_n\}_{n\in\mathbb{N}} \subset \mathscr{B}(\mathcal{H})$ be two strongly convergent sequence in $\mathscr{B}(\mathcal{H})$, with limits A_∞ and B_∞ respectively. Show that the sequence $\{A_nB_n\}_{n\in\mathbb{N}}$ is strongly convergent to the element $A_\infty B_\infty$.*

Let us close this section with some information about the inverse of a bounded operator.

Definition 1.2.7. *An operator* $B \in \mathscr{B}(\mathcal{H})$ *is* invertible *if the equation* $Bf = 0$ *only admits the solution* $f = 0$*. In such a case, there exists a linear map* $B^{-1} : \mathsf{Ran}(B) \to \mathcal{H}$ *which satisfies* $B^{-1}Bf = f$ *for any* $f \in \mathcal{H}$*, and* $BB^{-1}g = g$ *for any* $g \in \mathsf{Ran}(B)$*. If* B *is invertible and* $\mathsf{Ran}(B) = \mathcal{H}$*, then* $B^{-1} \in \mathscr{B}(\mathcal{H})$ *and* B *is said to be* invertible in $\mathscr{B}(\mathcal{H})$ *(or* boundedly invertible*).*

Note that the two conditions B invertible and $\mathsf{Ran}(B) = \mathcal{H}$ imply $B^{-1} \in \mathscr{B}(\mathcal{H})$ is a consequence of the Closed graph Theorem[3]. In the sequel, we shall use the notation $\mathbf{1} \in \mathscr{B}(\mathcal{H})$ for the operator defined on any $f \in \mathcal{H}$ by $\mathbf{1}f = f$, and $\mathbf{0} \in \mathscr{B}(\mathcal{H})$ for the operator defined by $\mathbf{0}f = 0$.

The next statement is very useful in applications, and holds in a much more general context. Its proof is classical and can be found in every textbook.

Lemma 1.2.8 (Neumann series)**.** *If* $B \in \mathscr{B}(\mathcal{H})$ *and* $\|B\| < 1$*, then the operator* $(\mathbf{1}-B)$ *is invertible in* $\mathscr{B}(\mathcal{H})$*, with*

$$(\mathbf{1} - B)^{-1} = \sum_{n=0}^{\infty} B^d,$$

and with $\big\|(\mathbf{1} - B)^{-1}\big\| \leq (1 - \|B\|)^{-1}$*. The series converges in the uniform norm of* $\mathscr{B}(\mathcal{H})$*.*

Note that we have used the identity $B^0 = \mathbf{1}$.

1.3 Special classes of bounded linear operators

In this section we provide some information on some subsets of $\mathscr{B}(\mathcal{H})$. We start with some operators which will play an important role in the sequel.

Definition 1.3.1. *An operator* $B \in \mathscr{B}(\mathcal{H})$ *is called* self-adjoint *if* $B^* = B$*, or equivalently if for any* $f, g \in \mathcal{H}$ *one has*

$$\langle f, Bg\rangle = \langle Bf, g\rangle. \tag{1.13}$$

For these operators the computation of their norm can be simplified (see also Exercise 1.2.2) :

Exercise 1.3.2. *If* $B \in \mathscr{B}(\mathcal{H})$ *is self-adjoint and if* $\mathcal{M}$ *is a dense subspace in* $\mathcal{H}$*, show that*

$$\|B\| = \sup_{f\in\mathcal{M},\ \|f\|=1} |\langle f, Bf\rangle|. \tag{1.14}$$

A special set of self-adjoint operators is provided by the set of orthogonal projections:

[3]Closed graph theorem: If $(B, \mathcal{H})$ is a closed operator (see further on for this definition), then $B \in \mathscr{B}(\mathcal{H})$, see for example [Kat, Sec. III.5.4]. This can be studied as an Extension.

Definition 1.3.3. *An element $P \in \mathscr{B}(\mathcal{H})$ is* a projection *if $P = P^2$. This projection is* orthogonal *if in addition $P = P^*$. The set of all orthogonal projections is denoted by $\mathscr{P}(\mathcal{H})$.*

It not difficult to check that there is a one-to-one correspondence between the set of closed subspaces of $\mathcal{H}$ and the set of orthogonal projections in $\mathscr{B}(\mathcal{H})$. Indeed, any orthogonal projection P defines a closed subspace $\mathcal{M} := P\mathcal{H}$. Conversely by taking the projection Theorem (Proposition 1.1.13) into account one infers that for any closed subspace $\mathcal{M}$ one can define an orthogonal projection P with $P\mathcal{H} = \mathcal{M}$.

We gather in the next exercise some easy relations between orthogonal projections and the underlying closed subspaces. For that purpose we use the notation $P_\mathcal{M}, P_\mathcal{N}$ for the orthogonal projections on the closed subspaces $\mathcal{M}$ and $\mathcal{N}$ of $\mathcal{H}$.

Exercise 1.3.4. *Show the following relations:*

(i) If $P_\mathcal{M} P_\mathcal{N} = P_\mathcal{N} P_\mathcal{M}$, then $P_\mathcal{M} P_\mathcal{N}$ is an orthogonal projection and the associated closed subspace is $\mathcal{M} \cap \mathcal{N}$,

(ii) If $\mathcal{M} \subset \mathcal{N}$, then $P_\mathcal{M} P_\mathcal{N} = P_\mathcal{N} P_\mathcal{M} = P_\mathcal{M}$,

(iii) If $\mathcal{M} \perp \mathcal{N}$, then $P_\mathcal{M} P_\mathcal{N} = P_\mathcal{N} P_\mathcal{M} = \mathbf{0}$, and $P_{\mathcal{M}\oplus\mathcal{N}} = P_\mathcal{M} + P_\mathcal{N}$,

(iv) If $P_\mathcal{M} P_\mathcal{N} = \mathbf{0}$, then $\mathcal{M} \perp \mathcal{N}$.

Note that the operators introduced so far are special instances of normal operators:

Definition 1.3.5. *An operator $B \in \mathscr{B}(\mathcal{H})$ is* normal *if the equality $BB^* = B^*B$ holds.*

Clearly, bounded self-adjoint operators are normal. Other examples of normal operators are unitary operators, as considered now. In fact, we introduce not only unitary operators, but also isometries and partial isometries. For that purpose, we recall that $\mathbf{1}$ denotes the identify operator in $\mathscr{B}(\mathcal{H})$.

Definition 1.3.6. *An element $U \in \mathscr{B}(\mathcal{H})$ is* a unitary operator *if $UU^* = \mathbf{1}$ and if $U^*U = \mathbf{1}$.*

Note that if U is unitary, then U is invertible in $\mathscr{B}(\mathcal{H})$ with $U^{-1} = U^*$. Indeed, observe first that $Uf = 0$ implies $f = U^*(Uf) = U^*0 = 0$. Secondly, for any $g \in \mathcal{H}$, one has $g = U(U^*g)$, and thus $\mathsf{Ran}(U) = \mathcal{H}$. Finally, the equality $U^{-1} = U^*$ follows from the unicity of the inverse.

More generally, an element $V \in \mathscr{B}(\mathcal{H})$ is called *an isometry* if the equality

$$V^*V = \mathbf{1} \tag{1.15}$$

holds. Clearly, a unitary operator is an instance of an isometry. For isometries the following properties can easily be obtained.

Proposition 1.3.7. *a) Let $V \in \mathscr{B}(\mathcal{H})$ be an isometry. Then* *(i) V preserves the scalar product, namely $\langle Vf, Vg\rangle = \langle f, g\rangle$ for any $f, g \in \mathcal{H}$,*

(ii) V preserves the norm, namely $\|Vf\| = \|f\|$ for any $f \in \mathcal{H}$,

(iii) If $\mathcal{H} \neq \{0\}$ then $\|V\| = 1$,

(iv) VV^* is the orthogonal projection on $\mathsf{Ran}(V)$,

(v) V is invertible (in the sense of Definition 1.2.7),

(vi) The adjoint V^* satisfies $V^*f = V^{-1}f$ if $f \in \mathsf{Ran}(V)$, and $V^*g = 0$ if $g \perp \mathsf{Ran}(V)$.

b) An element $W \in \mathscr{B}(\mathcal{H})$ is an isometry if and only if $\|Wf\| = \|f\|$ for all $f \in \mathcal{H}$.

Exercise 1.3.8. *Provide a proof for the previous proposition (as well as the proof of the next proposition).*

More generally one defines *a partial isometry* as an element $W \in \mathscr{B}(\mathcal{H})$ such that

$$W^*W = P \tag{1.16}$$

with P an orthogonal projection. Again, unitary operators or isometries are special examples of partial isometries.

As before the following properties of partial isometries can be easily proved.

Proposition 1.3.9. *Let $W \in \mathscr{B}(\mathcal{H})$ be a partial isometry as defined in* (1.16). *Then*

(i) One has $WP = W$ and $\langle Wf, Wg\rangle = \langle Pf, Pg\rangle$ for any $f, g \in \mathcal{H}$,

(ii) If $P \neq \mathbf{0}$ then $\|W\| = 1$,

(iii) WW^* is the orthogonal projection on $\mathsf{Ran}(W)$.

For a partial isometry W one usually calls *initial set projection* the orthogonal projection defined by W^*W and by *final set projection* the orthogonal projection defined by WW^*.

Let us now introduce a last subset of bounded operators, namely the ideal of *compact operators*. For that purpose, consider first any family $\{g_j, h_j\}_{j=1}^N \subset \mathcal{H}$ and for any $f \in \mathcal{H}$ one sets

$$Af := \sum_{j=1}^N \langle g_j, f\rangle\, h_j. \tag{1.17}$$

Then $A \in \mathscr{B}(\mathcal{H})$, and $\mathsf{Ran}(A) \subset \mathsf{Span}(h_1, \dots, h_N)$. Such an operator A is called *a finite rank operator*. In fact, any operator $B \in \mathscr{B}(\mathcal{H})$ with $\dim\big(\mathsf{Ran}(B)\big) < \infty$ is a finite rank operator.

Exercise 1.3.10. *For the operator A defined in* (1.17), *give an upper estimate for $\|A\|$ and compute A^*.*

Definition 1.3.11. *An element $B \in \mathscr{B}(\mathcal{H})$ is* a compact operator *if there exists a family $\{A_n\}_{n\in\mathbb{N}}$ of finite rank operators such that* $\lim_{n\to\infty}\|B - A_n\| = 0$. *The set of all compact operators is denoted by $\mathscr{K}(\mathcal{H})$.*

The following proposition contains sone basic properties of $\mathscr{K}(\mathcal{H})$. Its proof can be obtained by playing with families of finite rank operators.

Proposition 1.3.12. *The following properties hold:*

(i) $B \in \mathscr{K}(\mathcal{H}) \Longleftrightarrow B^* \in \mathscr{K}(\mathcal{H})$,

(ii) $\mathscr{K}(\mathcal{H})$ *is a $*$-algebra, complete for the norm $\|\cdot\|$,*

(iii) If $B \in \mathscr{K}(\mathcal{H})$ and $A \in \mathscr{B}(\mathcal{H})$, then AB and BA belong to $\mathscr{K}(\mathcal{H})$.

As a consequence, $\mathscr{K}(\mathcal{H})$ is a C^*-algebra and an ideal of $\mathscr{B}(\mathcal{H})$. In fact, compact operators have the nice property of improving some convergences, as shown in the next statement.

Proposition 1.3.13. *Let $K \in \mathscr{K}(\mathcal{H})$.*

(i) If $\{f_n\}_{n\in\mathbb{N}} \subset \mathcal{H}$ is a weakly convergent sequence with limit $f_\infty \in \mathcal{H}$, then the sequence $\{Kf_n\}_{n\in\mathbb{N}}$ strongly converges to Kf_∞,

(ii) If the sequence $\{B_n\}_{n\in\mathbb{N}} \subset \mathscr{B}(\mathcal{H})$ strongly converges to $B_\infty \in \mathscr{B}(\mathcal{H})$, then the sequences $\{B_nK\}_{n\in\mathbb{N}}$ and $\{KB_n^*\}_{n\in\mathbb{N}}$ converge in norm to $B_\infty K$ and KB_∞^*, respectively.

Proof. a) Let us first set $\varphi_n := f_n - f_\infty$ and observe that $\mathrm{w-}\lim_{n\to\infty}\varphi_n = 0$. By an application of the uniform boundedness principle, see Theorem 1.1.8, it follows that $\{\|\varphi_n\|\}_{n\in\mathbb{N}}$ is bounded, *i.e.* there exists $M > 0$ such that $\|\varphi_n\| \leq M$ for any $n \in \mathbb{N}$. Since K is compact, for any $\varepsilon > 0$ there exists a finite rank operator A of the form given in (1.17) such that $\|K - A\| \leq \frac{\varepsilon}{2M}$. Then one has

$$\|K\varphi_n\| \leq \|(K - A)\varphi_n\| + \|A\varphi_n\| \leq \frac{\varepsilon}{2} + \sum_{j=1}^{N} |\langle g_j, \varphi_n\rangle|\,\|h_j\|.$$

Since $\mathrm{w-}\lim_{n\to\infty}\varphi_n = 0$ there exists $n_0 \in \mathbb{N}$ such that $|\langle g_j, \varphi_n\rangle|\,\|h_j\| \leq \frac{\varepsilon}{2N}$ for any $j \in \{1, \dots, N\}$ and all $n \geq n_0$. As a consequence, one infers that $\|K\varphi_n\| \leq \varepsilon$ for all $n \geq n_0$, or in other words $\mathrm{s-}\lim_{n\to\infty} K\varphi_n = 0$.

b) Let us set $C_n := B_n - B_\infty$ such that $\mathrm{s-}\lim_{n\to\infty} C_n = \mathbf{0}$. As before, there exists $M > 0$ such that $\|C_n\| \leq M$ for any $n \in \mathbb{N}$. For any $\varepsilon > 0$ consider a finite rank operator A of the form (1.17) such that $\|K - A\| \leq \frac{\varepsilon}{2M}$. Then observe that for any

$f \in \mathcal{H}$

$$\begin{aligned}\|C_n K f\| &\leq M\|(K-A)f\| + \|C_n A f\| \\ &\leq M\|K-A\|\,\|f\| + \sum_{j=1}^{N} |\langle g_j, f\rangle|\,\|C_n h_j\| \\ &\leq \Big\{M\|K-A\| \ + \sum_{j=1}^{N} \|g_j\|\,\|C_n h_j\|\Big\}\|f\|.\end{aligned}$$

Since C_n strongly converges to $\mathbf{0}$ one can then choose $n_0 \in \mathbb{N}$ such that $\|g_j\|\,\|C_n h_j\| \leq \frac{\varepsilon}{2N}$ for any $j \in \{1, \dots N\}$ and all $n \geq n_0$. One then infers that $\|C_n K\| \leq \varepsilon$ for any $n \geq n_0$, which means that the sequence $\{C_n K\}_{n\in\mathbb{N}}$ uniformly converges to $\mathbf{0}$. The statement about $\{KB_n^*\}_{n\in\mathbb{N}}$ can be proved analogously by taking the equality $\|KB_n^* - KB_\infty^*\| = \|B_n K^* - B_\infty K^*\|$ into account and by remembering that K^* is compact as well. □

Exercise 1.3.14. *Check that an orthogonal projection P is a compact operator if and only if $P\mathcal{H}$ is of finite dimension.*

There are various subalgebras of $\mathscr{K}(\mathcal{H})$, for example the algebra of Hilbert-Schmidt operators, the algebra of trace class operators, and more generally the Schatten classes. Note that these algebras are not closed with respect to the norm $\|\cdot\|$ but with respect to some stronger norms $|||\cdot|||$. These algebras are ideals in $\mathscr{B}(\mathcal{H})$. In the following chapter these subalgebras will be extensively studied.

1.4 Unbounded, closed, and self-adjoint operators

Even if unbounded operators will not play an important role in the sequel, they might appear from time to time. For that reason, we gather in this section a couple of important definitions related to them. Obviously, the following definitions and results are also valid for bounded linear operators.

Definition 1.4.1. *A* linear operator *on $\mathcal{H}$ is a pair $\big(A, \mathsf{D}(A)\big)$, where $\mathsf{D}(A)$ is a subspace of $\mathcal{H}$ and A is a linear map from $\mathsf{D}(A)$ to $\mathcal{H}$. $\mathsf{D}(A)$ is called* the domain of A. *One says that the operator $\big(A, \mathsf{D}(A)\big)$ is* densely defined *if $\mathsf{D}(A)$ is dense in $\mathcal{H}$.*

Note that one often just says *the linear operator A*, but that its domain $\mathsf{D}(A)$ is implicitly taken into account. For such an operator, its range $\mathsf{Ran}(A)$ is defined by

$$\mathsf{Ran}(A) := A\,\mathsf{D}(A) = \{f \in \mathcal{H} \mid f = Ag \text{ for some } g \in \mathsf{D}(A)\}.$$

In addition, one defines the kernel $\mathsf{Ker}(A)$ of A by

$$\mathsf{Ker}(A) := \{f \in \mathsf{D}(A) \mid Af = 0\}.$$

Let us also stress that the sum $A + B$ for two linear operators is *a priori* only defined on the subspace $\mathsf{D}(A) \cap \mathsf{D}(B)$, and that the product AB is *a priori* defined only on the subspace $\{f \in \mathsf{D}(B) \mid Bf \in \mathsf{D}(A)\}$. These two sets can be very small.

Example 1.4.2. *Let* $\mathcal{H} := L^2(\mathbb{R})$ *and consider the operator* X *defined by* $[Xf](x) = xf(x)$ *for any* $x \in \mathbb{R}$. *Clearly,* $\mathsf{D}(X) = \{f \in \mathcal{H} \mid \int_{\mathbb{R}} |xf(x)|^2 \,\mathrm{d}x < \infty\} \subsetneq \mathcal{H}$. *In addition, by considering the family of functions* $\{f_y\}_{y\in\mathbb{R}} \subset \mathsf{D}(X)$ *with* $f_y(x) := 1$ *in* $x \in [y, y+1]$ *and* $f_y(x) = 0$ *if* $x \notin [y, y+1]$, *one easily observes that* $\|f_y\| = 1$ *but* $\sup_{y\in\mathbb{R}} \|Xf_y\| = \infty$, *which can be compared with* (1.8).

Clearly, a linear operator A can be defined on several domains. For example the operator X of the previous example is well-defined on the Schwartz space $\mathcal{S}(\mathbb{R})$, or on the set $C_c(\mathbb{R})$ of continuous functions on $\mathbb{R}$ with compact support, or on the space $\mathsf{D}(X)$ mentioned in the previous example. More generally, one has:

Definition 1.4.3. *For any pair of linear operators* $\big(A, \mathsf{D}(A)\big)$ *and* $\big(B, \mathsf{D}(B)\big)$ *satisfying* $\mathsf{D}(A) \subset \mathsf{D}(B)$ *and* $Af = Bf$ *for all* $f \in \mathsf{D}(A)$, *one says that* $\big(B, \mathsf{D}(B)\big)$ *is* an extension *of* $\big(A, \mathsf{D}(A)\big)$ *to* $\mathsf{D}(B)$, *or that* $\big(A, \mathsf{D}(A)\big)$ *is the restriction of* $\big(B, \mathsf{D}(B)\big)$ *to* $\mathsf{D}(A)$.

Let us now note that if $\big(A, \mathsf{D}(A)\big)$ is densely defined and if there exists $c \in \mathbb{R}$ such that $\|Af\| \le c\|f\|$ for all $f \in \mathsf{D}(A)$, then there exists a natural continuous extension $\overline{A}$ of A with $\mathsf{D}(\overline{A}) = \mathcal{H}$. This extension satisfies $\overline{A} \in \mathscr{B}(\mathcal{H})$ with $\|\overline{A}\| \le c$, and is called *the closure of the operator* A.

Exercise 1.4.4. *Work on the details of this extension.*

Let us now consider a similar construction but in the absence of a constant $c \in \mathbb{R}$ such that $\|Af\| \le c\|f\|$ for all $f \in \mathsf{D}(A)$. More precisely, consider an arbitrary densely defined operator $\big(A, \mathsf{D}(A)\big)$. Then for any $f \in \mathcal{H}$ there exists a sequence $\{f_n\}_{n\in\mathbb{N}} \subset \mathsf{D}(A)$ strongly converging to f. Note that the sequence $\{Af_n\}_{n\in\mathbb{N}}$ will not be Cauchy in general. However, let us assume that this sequence is strongly Cauchy, *i.e.* for any $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that $\|Af_n - Af_m\| < \varepsilon$ for any $n, m \ge N$. Since $\mathcal{H}$ is complete, this Cauchy sequence has a limit, which we denote by h, and it would then be natural to set $\overline{A}f = h$. In short, one would have $\overline{A}f := \mathrm{s-}\lim_{n\to\infty} Af_n$. It is easily observed that this definition is meaningful if and only if by choosing a different sequence $\{f'_n\}_{n\in\mathbb{N}} \subset \mathsf{D}(A)$ strongly convergent to f and also defining a Cauchy sequence $\{Af'_n\}_{n\in\mathbb{N}}$ then $\mathrm{s-}\lim_{n\to\infty} Af'_n = \mathrm{s-}\lim_{n\to\infty} Af_n$. If this condition holds, then $\overline{A}f$ is well-defined. Observe in addition that the previous equality can by rewritten as $\mathrm{s-}\lim_{n\to\infty} A(f_n - f'_n) = 0$, which leads naturally to the following definition.

Definition 1.4.5. *A linear operator* $\big(A, \mathsf{D}(A)\big)$ *is* closable *if for any sequence* $\{f_n\}_{n\in\mathbb{N}}$ *in* $\mathsf{D}(A)$ *satisfying* $\mathrm{s-}\lim_{n\to\infty} f_n = 0$ *and such that* $\{Af_n\}_{n\in\mathbb{N}}$ *is strongly Cauchy, then* $\mathrm{s-}\lim_{n\to\infty} Af_n = 0$.

As shown before this definition, in such a case one can define an extension $\overline{A}$ of A with $\mathsf{D}(\overline{A})$ given by the sets of $f \in \mathcal{H}$ such that there exists $\{f_n\}_{n\in\mathbb{N}} \subset \mathsf{D}(A)$ with $\mathrm{s-}\lim_{n\to\infty} f_n = f$ and such that $\{Af_n\}_{n\in\mathbb{N}}$ is strongly Cauchy. For such an element f one sets $\overline{A}f = \mathrm{s-}\lim_{n\to\infty} Af_n$, and the extension $\big(\overline{A}, \mathsf{D}(\overline{A})\big)$ is called the closure of A.

In relation with the previous construction the following definition is now natural:

Definition 1.4.6. *An linear operator* $\big(A, \mathsf{D}(A)\big)$ *is* closed *if for any sequence* $\{f_n\}_{n\in\mathbb{N}} \subset \mathsf{D}(A)$ *with* $\mathrm{s-}\lim_{n\to\infty} f_n = f \in \mathcal{H}$ *and such that* $\{Af_n\}_{n\in\mathbb{N}}$ *is strongly Cauchy, then one has* $f \in \mathsf{D}(A)$ *and* $\mathrm{s-}\lim_{n\to\infty} Af_n = Af$.

Let us now come back to the notion of the adjoint of an operator. This concept is slightly more subtle for unbounded operators than in the bounded case.

Definition 1.4.7. *Let* $\big(A, \mathsf{D}(A)\big)$ *be a densely defined linear operator on* $\mathcal{H}$. The adjoint A^* of A *is the operator defined by*

$$\mathsf{D}(A^*) := \big\{f \in \mathcal{H} \mid \exists f^* \in \mathcal{H} \text{ with } \langle f^*, g\rangle = \langle f, Ag\rangle \text{ for all } g \in \mathsf{D}(A)\big\}$$

and $A^*f := f^*$ *for all* $f \in \mathsf{D}(A^*)$.

Let us note that the density of $\mathsf{D}(A)$ is necessary to ensure that A^* is well-defined. Indeed, if f_1^*, f_2^* satisfy for all $g \in \mathsf{D}(A)$

$$\langle f_1^*, g\rangle = \langle f, Ag\rangle = \langle f_2^*, g\rangle,$$

then $\langle f_1^* - f_2^*, g\rangle = 0$ for all $g \in \mathsf{D}(A)$, and this equality implies $f_1^* = f_2^*$ only if $\mathsf{D}(A)$ is dense in $\mathcal{H}$. Note also that once $\big(A^*, \mathsf{D}(A^*)\big)$ is defined, one has

$$\langle A^*f, g\rangle = \langle f, Ag\rangle \qquad \forall f \in \mathsf{D}(A^*) \text{ and } \forall g \in \mathsf{D}(A).$$

Exercise 1.4.8. *Show that if* $\big(A, \mathsf{D}(A)\big)$ *is closable, then* $\mathsf{D}(A^*)$ *is dense in* $\mathcal{H}$.

Some relations between A and its adjoint A^* are gathered in the following lemma.

Lemma 1.4.9. *Let* $\big(A, \mathsf{D}(A)\big)$ *be a densely defined linear operator on* $\mathcal{H}$. *Then*

(i) $\big(A^*, \mathsf{D}(A^*)\big)$ *is closed,*

(ii) One has $\mathsf{Ker}(A^*) = \mathsf{Ran}(A)^\perp$,

(iii) If $\big(B, \mathsf{D}(B)\big)$ *is an extension of* $\big(A, \mathsf{D}(A)\big)$, *then* $\big(A^*, \mathsf{D}(A^*)\big)$ *is an extension of* $\big(B^*, \mathsf{D}(B^*)\big)$.

Proof. a) Consider $\{f_n\}_{n\in\mathbb{N}} \subset \mathsf{D}(A^*)$ such that $\mathrm{s-}\lim_{n\to\infty} f_n = f \in \mathcal{H}$ and such that $\mathrm{s-}\lim_{n\to\infty} A^*f_n = h \in \mathcal{H}$. Then for each $g \in \mathsf{D}(A)$ one has

$$\langle f, Ag\rangle = \lim_{n\to\infty}\langle f_n, Ag\rangle = \lim_{n\to\infty}\langle A^*f_n, g\rangle = \langle h, g\rangle.$$

Hence $f \in \mathsf{D}(A^*)$ and $A^*f = h$, which proves that A^* is closed.

b) Let $f \in \mathsf{Ker}(A^*)$, *i.e.* $f \in \mathsf{D}(A^*)$ and $A^*f = 0$. Then, for all $g \in \mathsf{D}(A)$, one has

$$0 = \langle A^*f, g\rangle = \langle f, Ag\rangle$$

meaning that $f \in \mathsf{Ran}(A)^\perp$. Conversely, if $f \in \mathsf{Ran}(A)^\perp$, then for all $g \in \mathsf{D}(A)$ one has

$$\langle f, Ag\rangle = 0 = \langle 0, g\rangle$$

meaning that $f \in \mathsf{D}(A^*)$ and $A^* f = 0$, by the definition of the adjoint of A.

c) Consider $f \in \mathsf{D}(B^*)$ and observe that $\langle B^* f, g\rangle = \langle f, Bg\rangle$ for any $g \in \mathsf{D}(B)$. Since $\big(B, \mathsf{D}(B)\big)$ is an extension of $\big(A, \mathsf{D}(A)\big)$, one infers that $\langle B^* f, g\rangle = \langle f, Ag\rangle$ for any $g \in \mathsf{D}(A)$. Now, this equality means that $f \in \mathsf{D}(A^*)$ and that $A^* f = B^* f$, or more explicitly that A^* is defined on the domain of B^* and coincide with this operator on this domain. This means precisely that $\big(A^*, \mathsf{D}(A^*)\big)$ is an extension of $\big(B^*, \mathsf{D}(B^*)\big)$. □

Let us finally introduce the analogue of the bounded self-adjoint operators but in the unbounded setting. These operators play a key role in quantum mechanics and their study is very well developed.

Definition 1.4.10. *A densely defined linear operator* $\big(A, \mathsf{D}(A)\big)$ *is* self-adjoint *if* $\mathsf{D}(A^*) = \mathsf{D}(A)$ *and* $A^* f = Af$ *for all* $f \in \mathsf{D}(A)$.

Note that as a consequence of Lemma 1.4.9.(i) a self-adjoint operator is always closed. Recall also that in the bounded case, a self-adjoint operator was characterized by the equality

$$\langle Af, g\rangle = \langle f, Ag\rangle \tag{1.18}$$

for any $f, g \in \mathcal{H}$. In the unbounded case, such an equality still holds if $f, g \in \mathsf{D}(A)$. However, let us emphasize that (1.18) does not completely characterize a self-adjoint operator. In fact, a densely defined operator $\big(A, \mathsf{D}(A)\big)$ satisfying (1.18) is called *a symmetric operator*, and self-adjoint operators are special instances of symmetric operators (but not all symmetric operators are self-adjoint). For a symmetric operator the adjoint operator $\big(A^*, \mathsf{D}(A^*)\big)$ is an extension of $\big(A, \mathsf{D}(A)\big)$, but the equality of these two operators holds only if $\big(A, \mathsf{D}(A)\big)$ is self-adjoint. Note also that for any symmetric operator the scalar $\langle f, Af\rangle$ is real for any $f \in \mathsf{D}(A)$.

Exercise 1.4.11. *Show that a symmetric operator is always closable.*

Let us add one more definition related to self-adjoint operators.

Definition 1.4.12. *A symmetric operator* $\big(A, \mathsf{D}(A)\big)$ *is* essentially self-adjoint *if its closure* $\big(\overline{A}, \mathsf{D}(\overline{A})\big)$ *is self-adjoint. In this case* $\mathsf{D}(A)$ *is called* a core *for* $\overline{A}$.

A following *fundamental criterion for self-adjointness* is important in this context, and its proof can be found in [Amr, Prop. 3.3].

Proposition 1.4.13. *Let* $\big(A, \mathsf{D}(A)\big)$ *be a symmetric operator in a Hilbert space* $\mathcal{H}$. *Then*

(i) $\big(A, \mathsf{D}(A)\big)$ *is self-adjoint if and only if* $\mathsf{Ran}(A + i) = \mathcal{H}$ *and* $\mathsf{Ran}(A - i) = \mathcal{H}$,

(ii) $\big(A, \mathsf{D}(A)\big)$ *is essentially self-adjoint if and only if* $\mathsf{Ran}(A + i)$ *and* $\mathsf{Ran}(A - i)$ *are dense in* $\mathcal{H}$.

For completeness, let us recall the definitions of a spectral family and a spectral measure, and mention one version of the spectral theorem for self-adjoint operators. We do not provide more explanations here and refer to [Amr, Chap. 4] for a thorough introduction to this important result of spectral theory. Later on, it will be useful to have these definitions and this statement at hand.

Definition 1.4.14. A spectral family*, or* a resolution of the identity*, is a family $\{E_\lambda\}_{\lambda\in\mathbb{R}}$ of orthogonal projections in $\mathcal{H}$ satisfying:*

(i) The family is non-decreasing, i.e. $E_\lambda E_\mu = E_{\min\{\lambda,\mu\}}$,

(ii) The family is strongly right continuous, i.e. $E_\lambda = E_{\lambda+0} = s-\lim_{\varepsilon\searrow 0} E_{\lambda+\varepsilon}$,

(iii) $s-\lim_{\lambda\to-\infty} E_\lambda = \mathbf{0}$ *and* $s-\lim_{\lambda\to\infty} E_\lambda = \mathbf{1}$,

Given such a family, one first defines

$$E\big((a,b]\big) := E_b - E_a, \qquad a,b\in\mathbb{R}, \tag{1.19}$$

and extends this definition to all sets $V\in\mathcal{A}_B$, where $\mathcal{A}_B$ denotes the set of Borel sets on $\mathbb{R}$. Thus one ends up with the notion of a *spectral measure*, which consists in a projection-valued map $E:\mathcal{A}_B\to\mathscr{P}(\mathcal{H})$ which satisfies $E(\emptyset)=\mathbf{0}$, $E(\mathbb{R})=\mathbf{1}$, $E(V_1)E(V_2)=E(V_1\cap V_2)$ for any Borel sets V_1,V_2.

Theorem 1.4.15 (Spectral Theorem)**.** *With any self-adjoint operator $\big(A,\mathsf{D}(A)\big)$ on a Hilbert space $\mathcal{H}$ one can associate a unique spectral family $\{E_\lambda\}$, called* the spectral family of A*, such that $\mathsf{D}(A)=\mathsf{D}_{\mathrm{id}}$ with*

$$\mathsf{D}_\varphi := \Big\{f\in\mathcal{H}\mid \int_{-\infty}^{\infty}|\varphi(\lambda)|^2\,\langle E(\mathrm{d}\lambda)f,f\rangle<\infty\Big\}.$$

and $A=\int_{-\infty}^{\infty}\lambda\,E(\mathrm{d}\lambda)$. Conversely any spectral family or any spectral measure defines a self-adjoint operator in $\mathcal{H}$ by the previous formulas.

1.5 Resolvent and spectrum

We come now to the important notion of the spectrum of an operator. As already mentioned in the previous section we shall often speak about a linear operator A, its domain $\mathsf{D}(A)$ being implicitly taken into account. Recall also that the notion of a closed linear operator has been introduced in Definition 1.4.6.

The notion of the inverse of a bounded linear operator has already been introduced in Definition 1.2.7. By analogy we say that any linear operator A is *invertible* if $\mathsf{Ker}(A)=\{0\}$. In this case, the inverse A^{-1} gives a bijection from $\mathsf{Ran}(A)$ onto $\mathsf{D}(A)$. More precisely $\mathsf{D}(A^{-1})=\mathsf{Ran}(A)$ and $\mathsf{Ran}(A^{-1})=\mathsf{D}(A)$. It can then be checked that if A is closed and invertible, then A^{-1} is also closed. Note also if A is closed and if $\mathsf{Ran}(A)=\mathcal{H}$ then $A^{-1}\in\mathscr{B}(\mathcal{H})$. In fact, the boundedness of A^{-1} is a consequence of the closed graph theorem and one says in this case that A is *boundedly invertible* or *invertible in $\mathscr{B}(\mathcal{H})$*.

Definition 1.5.1. *For a closed linear operator A its* resolvent set $\rho(A)$ *is defined by*

$$\begin{aligned}\rho(A) :=& \big\{z \in \mathbb{C} \mid (A - z) \text{ is invertible in } \mathscr{B}(\mathcal{H})\big\} \\ =& \big\{z \in \mathbb{C} \mid \mathsf{Ker}(A - z) = \{0\} \text{ and } \mathsf{Ran}(A - z) = \mathcal{H}\big\}.\end{aligned}$$

For $z \in \rho(A)$ the operator $(A-z)^{-1} \in \mathscr{B}(\mathcal{H})$ is called the resolvent of A at the point z. The spectrum $\sigma(A)$ of A *is defined as the complement of $\rho(A)$ in $\mathbb{C}$,* i.e.

$$\sigma(A) := \mathbb{C} \setminus \rho(A). \tag{1.20}$$

The following statement summarized several properties of the resolvent set and of the resolvent of a closed linear operator.

Proposition 1.5.2. *Let A be a closed linear operator on a Hilbert space $\mathcal{H}$. Then*

(i) The resolvent set $\rho(A)$ is an open subset of $\mathbb{C}$,

(ii) If $z_1, z_2 \in \rho(A)$ then the first resolvent equation *holds, namely*

$$(A - z_1)^{-1} - (A - z_2)^{-1} = (z_1 - z_2)(A - z_1)^{-1}(A - z_2)^{-1} \tag{1.21}$$

(iii) If $z_1, z_2 \in \rho(A)$ then the operators $(A - z_1)^{-1}$ and $(A - z_2)^{-1}$ commute,

(iv) In each connected component of $\rho(A)$ the map $z \mapsto (A - z)^{-1}$ is holomorphic.

As a consequence of the previous proposition, the spectrum of a closed linear operator is always closed. In particular, $z \in \sigma(A)$ if $A-z$ is not invertible or if $\mathsf{Ran}(A-z) \neq \mathcal{H}$. The first situation corresponds to the definition of an eigenvalue:

Definition 1.5.3. *For a closed linear operator A, a value $z \in \mathbb{C}$ is* an eigenvalue of A *if there exists $f \in \mathsf{D}(A)$, $f \neq 0$, such that $Af = zf$. In such a case, the element f is called* an eigenfunction of A associated with the eigenvalue z. *The dimension of the vector space generated by all eigenfunctions associated with an eigenvalue z is called* the geometric multiplicity of z. *The set of all eigenvalues of A is denoted by $\sigma_p(A)$, and is often called* the point spectrum *of A.*

Let us still provide two properties of the spectrum of an operator in the special cases of a bounded operator or of a self-adjoint operator.

Exercise 1.5.4. *By using the Neumann series, show that for any $B \in \mathscr{B}(\mathcal{H})$ its spectrum is contained in the ball in the complex plane of center 0 and of radius $\|B\|$.*

Lemma 1.5.5. *Let A be a self-adjoint operator in $\mathcal{H}$.*

(i) Any eigenvalue of A is real,

(ii) More generally, the spectrum of A is real, i.e. $\sigma(A) \subset \mathbb{R}$,

(iii) Eigenvectors associated with different eigenvalues are orthogonal to one another.

Proof. a) Assume that there exists $z \in \mathbb{C}$ and $f \in \mathsf{D}(A)$, $f \neq 0$ such that $Af = zf$. Then one has

$$z\|f\|^2 = \langle f, zf\rangle = \langle f, Af\rangle = \langle Af, f\rangle = \langle zf, f\rangle = \overline{z}\|f\|^2.$$

Since $\|f\| \neq 0$, one deduces that $z \in \mathbb{R}$.

b) Let us consider $z = \lambda + i\varepsilon$ with $\lambda, \varepsilon \in \mathbb{R}$ and $\varepsilon \neq 0$, and show that $z \in \rho(A)$. Indeed, for any $f \in \mathsf{D}(A)$ one has

$$\begin{aligned}\|(A-z)f\|^2 &= \|(A-\lambda)f - i\varepsilon f\|^2 \\ &= \big\langle (A-\lambda)f - i\varepsilon f, (A-\lambda)f - i\varepsilon f\big\rangle \\ &= \|(A-\lambda)f\|^2 + \varepsilon^2\|f\|^2.\end{aligned}$$

It follows that $\|(A-z)f\| \geq |\varepsilon|\|f\|$, and thus $A - z$ is invertible.

Now, for any for any $g \in \mathsf{Ran}(A-z)$ let us observe that

$$\|g\| = \big\|(A-z)(A-z)^{-1}g\big\| \geq |\varepsilon|\big\|(A-z)^{-1}g\big\|.$$

Equivalently, it means for all $g \in \mathsf{Ran}(A-z)$, one has

$$\big\|(A-z)^{-1}g\big\| \leq \frac{1}{|\varepsilon|}\|g\|. \tag{1.22}$$

Let us finally observe that $\mathsf{Ran}(A-z)$ is dense in $\mathcal{H}$. Indeed, by Lemma 1.4.9 one has

$$\mathsf{Ran}(A-z)^\perp = \mathsf{Ker}\big((A-z)^*\big) = \mathsf{Ker}(A^* - \overline{z}) = \mathsf{Ker}(A - \overline{z}) = \{0\}$$

since all eigenvalues of A are real. Thus, the operator $(A-z)^{-1}$ is defined on the dense domain $\mathsf{Ran}(A-z)$ and satisfies the estimate (1.22). As explained just before the Exercise 1.4.4, it means that $(A-z)^{-1}$ continuously extends to an element of $\mathscr{B}(\mathcal{H})$, and therefore $z \in \rho(A)$.

c) Assume that $Af = \lambda f$ and that $Ag = \mu g$ with $\lambda, \mu \in \mathbb{R}$ and $\lambda \neq \mu$, and $f, g \in \mathsf{D}(A)$, with $f \neq 0$ and $g \neq 0$. Then

$$\lambda\langle f, g\rangle = \langle Af, g\rangle = \langle f, Ag\rangle = \mu\langle f, g\rangle,$$

which implies that $\langle f, g\rangle = 0$, or in other words that f and g are orthogonal. □

1.6 Positivity and polar decomposition

The notion of positive operators can be introduced either in a Hilbert space setting or in a C^*-algebraic setting. The next definition is based on the former framework, and its analog in the latter framework will be mentioned subsequently.

Definition 1.6.1. *A densely defined linear operator* $\big(A, \mathsf{D}(A)\big)$ *in* $\mathcal{H}$ *is positive if*

$$\langle f, Af\rangle \geq 0 \qquad \text{for any } f \in \mathsf{D}(A). \tag{1.23}$$

Clearly, such an operator is symmetric, see the paragraph following Definition 1.4.10. If A is bounded one also infers that A is self-adjoint, but this might not be the case if A is unbounded. It is then a natural question to check whether there exists some self-adjoint extensions of A which are still positive. We shall not go further in this direction and stick to the self-adjoint case. More precisely, a self-adjoint operator $\big(A, \mathsf{D}(A)\big)$ is positive if (1.23) holds.

For positive self-adjoint operators, the following consequences of the spectral theorem are very useful, see Theorem 1.4.15.

Proposition 1.6.2. *For any positive and self-adjoint operator* $\big(A, \mathsf{D}(A)\big)$ *in* $\mathcal{H}$ *the following properties hold:*

(i) $\sigma(A) \subset [0, \infty)$,

(ii) There exists a unique self-adjoint and positive operator $\big(B, \mathsf{D}(B)\big)$ *such that* $A = B^2$ *on* $\mathsf{D}(A)$. *The operator* B *is called* the positive square root *of* A *and is denoted by* $A^{1/2}$

Since in a purely C^*-algebraic the scalar product in (1.23) does not exist (note that this statement is not really correct because of the GNS representation) one usually says that a bounded operator A is positive if $A = A^*$ and $\sigma(A) \subset [0, \infty)$. However, this definition coincides with the one mentioned above as long as one considers bounded operators only.

Remark 1.6.3. *If* A *is an arbitrary element of* $\mathscr{B}(\mathcal{H})$, *observe that* A^*A *and* AA^* *are positive operators. Indeed, self-adjointness follows easily from Exercise 1.2.4 while positivity is obtained by the equalities*

$$\langle f, A^*Af\rangle = \langle Af, Af\rangle = \|Af\|^2 \geq 0$$

and similarly for AA^*. *In fact, the set* $\{A^*A \mid A \in \mathscr{B}(\mathcal{H})\}$ *is equal to the set of all positive operators in* $\mathscr{B}(\mathcal{H})$.

Let us add one statement which contains several properties of bounded positive operators. It can be stated in a purely C^*-algebraic framework, but we present it for simplicity for $\mathscr{B}(\mathcal{H})$ only. Note that if $A \in \mathscr{B}(\mathcal{H})$ we often denote its positivity by writing $A \geq 0$. Now, if A_1, A_2 are bounded and self-adjoint operators, one writes $A_1 \geq A_2$ if $A_1 - A_2 \geq 0$. We shall also use the notation $\mathscr{B}(\mathcal{H})_+$ for the set of positive elements of $\mathscr{B}(\mathcal{H})$.

Proposition 1.6.4. *(i) The sum of two positive elements of* $\mathscr{B}(\mathcal{H})$ *is a positive element of* $\mathscr{B}(\mathcal{H})$, *(ii)* *The set* $\mathscr{B}(\mathcal{H})_+$ *is equal to* $\{A^*A \mid A \in \mathscr{B}(\mathcal{H})\}$,

(iii) *If* A, B *are self-adjoint elements of* $\mathscr{B}(\mathcal{H})$ *and if* $C \in \mathscr{B}(\mathcal{H})$, *then* $A \geq B \Rightarrow C^*AC \geq C^*BC$,

(iv) *If* $A \geq B \geq 0$, *then* $A^{1/2} \geq B^{1/2}$,

(v) *If* $A \geq B \geq 0$, *then* $\|A\| \geq \|B\|$,

(vi) *If* A, B *are positive and invertible elements of* $\mathscr{B}(\mathcal{H})$, *then* $A \geq B \Rightarrow B^{-1} \geq A^{-1} \geq 0$,

(vii) *For any* $A \in \mathscr{B}(\mathcal{H})$ *there exist* $A_1, A_2, A_3, A_4 \in \mathscr{B}(\mathcal{H})_+$ *such that*

$$A = A_1 - A_2 + iA_3 - iA_4.$$

Proof. See Lemma 2.2.3, Theorem 2.2.5 and Theorem 2.2.6 of [Mur]. □

We finally state and prove a very useful result for arbitrary element of $B \in \mathscr{B}(\mathcal{H})$. For that purpose we first introduce

$$|B| := (B^*B)^{1/2}. \tag{1.24}$$

Theorem 1.6.5 (Polar decomposition)**.** *For any* $B \in \mathscr{B}(\mathcal{H})$ *there exists a unique partial isometry* $W \in \mathscr{B}(\mathcal{H})$ *such that*

$$W|B| = B \qquad \text{and} \qquad \mathsf{Ker}(W) = \mathsf{Ker}(B). \tag{1.25}$$

In addition, $W^*B = |B|$.

Proof. For any $f \in \mathcal{H}$ one has

$$\big\||B|f\big\|^2 = \langle |B|f, |B|f\rangle = \langle f, |B|^2 f\rangle = \langle f, B^*Bf\rangle = \|Bf\|^2,$$

which means that the map

$$W_0 : |B|\mathcal{H} \ni |B|f \mapsto Bf \in \mathcal{H}$$

is well-defined, isometric, and also linear. It can then be uniquely extended to a linear isometric map from the closure $\overline{|B|\mathcal{H}}$ to $\mathcal{H}$. This extension is still denoted by W_0. We can thus define the operator $W \in \mathscr{B}(\mathcal{H})$ by $W = W_0$ on $\overline{|B|\mathcal{H}}$ and $W = \mathbf{0}$ on its orthocomplement. It then follows that $W|B| = B$, and W is isometric on $\mathsf{Ker}(W)^\perp$ since $\mathsf{Ker}(W) = \overline{|B|\mathcal{H}}^\perp$. Thus, W is a partial isometry and $\mathsf{Ker}(W) = \mathsf{Ker}(|B|)$. Now, since for any $f, g \in \mathcal{H}$ one has

$$\langle W^*Bf, |B|g\rangle = \langle Bf, Bg\rangle = \langle f, B^*Bg\rangle = \langle |B|f, |B|g\rangle,$$

one deduces that $\langle W^*Bf, h\rangle = \langle |B|f, h\rangle$ for any $h \in \overline{|B|\mathcal{H}}$, and then for any $h \in \mathcal{H}$. Thus $W^*B = |B|$, and since $\mathsf{Ker}(W^*) = \mathsf{Ran}(W)^\perp = \mathsf{Ran}(B)^\perp$, one infers that $\mathsf{Ker}(B) = \mathsf{Ker}(|B|) = \mathsf{Ker}(W)$.

For the uniqueness, suppose that there exists another partial isometry $W' \in \mathscr{B}(\mathcal{H})$ such that $W'|B| = B$ and $\mathsf{Ker}(W') = \mathsf{Ker}(B)$. Then W' is equal to W on $\overline{|B|\mathcal{H}}$ and on $\overline{|B|\mathcal{H}}^\perp$ both operators are equal to $\mathbf{0}$. As a consequence, $W' = W$. □