Report 2.

Trace

321701139 SONG SEUNGHOAN.

(◦ Let $L^1_c(\mathbb{R}^n)$ be the set of compactly supported $L^1(\mathbb{R}^n)$ functions. and $L^1_{mod}(\mathbb{R}^n)$ be the set of modulated $L^1(\mathbb{R})$ functions.)

Proposition: $L^1_c(\mathbb{R}^n)$ is not dense in $L^1_{mod}(\mathbb{R}^n)$.

(proof) (i) It is sufficient to give a function $f \in L^1_{mod}(\mathbb{R}^n)$ such that is not in the $\|\cdot\|_{L^1_{mod}}$ closure of $L^1_c(\mathbb{R}^n)$.

Show $f := (1+|x|)^{-2n}$ is an instance of above.

(ii) "$f \in L^1_{mod}(\mathbb{R}^n)$".

$\forall t;$ $(1+t)^n \int_{|x|>t} (1+|x|)^{-2n} dx$ (∵ volume of sphere dim $n-1$)

$= (1+t)^n \int_t^\infty (1+|x|)^{-2n} \cdot k|x|^{n-1} d|x|$ (k: positive real constant.)

$\leq k \cdot (1+t)^n \int_t^\infty (1+|x|)^{-2n} \cdot (1+|x|)^{n-1} d|x|$

$= k \cdot (1+t)^n \cdot \int_t^\infty (1+|x|)^{-n-1} d|x|$

$= k \cdot (1+t)^n \frac{1}{-n} [-(1+t)^{-n}]$

$= \frac{k}{n}.$

Therefore, $\|f\|_{L^1_{mod}} = \sup_{t>0} (1+t)^n \int_{|x|>t} (1+|x|)^{-2n} dx < \frac{k}{n} < \infty.$

∴ $f \in L^1_{mod}(\mathbb{R}^n)$.

(iii) "$\exists \tilde{k} > 0$; $\forall g \in L^1_c(\mathbb{R}^n)$; $\|f-g\|_{L^1_{mod}} \geq \tilde{k}$."

Take sufficiently large $t_g > 1$ so that $g(x) = 0$ for $|x| > t_g$.

$(1+t_g)^n \int_{|x|>t_g} |(f-g)(x)| dx$

$= (1+t_g)^n \int_{|x|>t_g} |f(x)| dx$

$= (1+t_g)^n \int_{|x|>t_g} (1+|x|)^{-2n} dx$

$= k(1+t_g)^n \int_{t_g}^\infty (1+|x|)^{-2n} |x|^{n-1} d|x|$

$\geq k\, t_g^n \int_{t_g}^\infty (1+|x|)^{-2n} |x|^{n-1} d|x|$

* only when $t_g \geq 1$. $\geq k\, t_g^n \int_{t_g}^\infty (2|x|)^{-2n} |x|^{n-1} d|x|$

$= \frac{k}{2^{2n}} t_g^n \cdot \frac{1}{n} \cdot t_g^{-n} = \frac{k}{n \cdot 2^{2n}}.$

Therefore,

$\forall g \in L^1_c(\mathbb{R})$;

$$\|f-g\|_{L^1_{mod}} = \sup_{t>0} (1+t)^n \int_{|x|>t} |(f-g)(x)|\, dx$$

$$\geq (1+t_g)^n \int_{|x|>t} |(f-g)(x)|\, dx$$

$$\geq \frac{k}{n \cdot 2^{2n}}.$$

$\therefore$ f is not a limit point of $L^1_c(\mathbb{R}^n)$. Q.E.D.

Reference〉 S. Lord, F. Sukochev, D. Zanin : Sigular traces, theory and applications, 2013.