Trace

m17021a@math.nagoya-u.ac.jp
321701139 M1 SONG SEUNGHOAN.

Def $S, H, D_n : l_\infty \to l_\infty$

① $S((a_1, \cdots)) = (a_2, a_3, \cdots)$

② $H((a_1, \cdots)) = \left(a_1, \frac{a_1+a_2}{2}, \frac{a_1+a_2+a_3}{3}, \cdots\right)$

③ $D_n((a_1, \cdots)) = (\underbrace{a_1, a_1, \cdots, a_1}_{n}, \underbrace{a_2, \cdots, a_2}_{n}, \cdots)$

Extension 3.1.2>

(i) $D_n S = S^n D_n$ for any $n \in \mathbb{N}$.

(ii) $(HS - SH)(a) \in c_0$ for any $a \in l_\infty$.

(iii) $(HD_n - D_n H)(a) \in c_0$ for any $a \in l_\infty$.

<Notation> $\{f(i)\}_i := (f(1), f(2), \cdots)$.

<proof> For $\{a_i\}_i = (a_1, a_2, \cdots)$, determine each coordinate of $S(\{a_i\}_i)$, $D_n(\{a_i\}_i)$, $H(\{a_i\}_i)$

$S(\{a_i\}_i) = \{a_{i+1}\}_i$,

$H(\{a_i\}_i) = \left\{\frac{\sum_{k=1}^{i} a_k}{i}\right\}_i$,

$D_n(\{a_i\}_i) = \{a_{\lfloor \frac{i-1}{n} \rfloor + 1}\}_i$

(i) $D_n S = S^n D_n \quad \forall n \in \mathbb{N}$.

Compare $D_n S(\{a_i\}_i)$ and $S^n D_n(\{a_i\}_i)$.

$\{b_i\}_i = D_n S(\{a_i\}_i) = D_n(\{a_{i+1}\}_i) = D_n(\{a'_i\}_i) = \{a'_{\lfloor \frac{i-1}{n} \rfloor + 1}\}_i = \{a_{\lfloor \frac{i-1}{n} \rfloor + 2}\}_i$

$\left(\{a'_i\}_i \overset{\text{def}}{=} \{a_{i+1}\}_i\right)$

$\{c_i\}_i = S^n D_n(\{a_i\}_i) = S^n(\{a''_i\}_i) = \{a''_{i+n}\}_i = \{a_{\lfloor \frac{(i+n)-1}{n} \rfloor + 1}\}_i = \{a_{\lfloor \frac{i-1}{n} \rfloor + 2}\}_i$

$\left(\begin{aligned}\{a''_i\}_i &\overset{\text{def}}{=} \{D_n\{a_i\}\}_i \\ &= \{a_{\lfloor \frac{i-1}{n} \rfloor + 1}\}_i\end{aligned}\right)$

Thus, $D_n S(\{a_i\}_i) = \{b_i\}_i = \{c_i\}_i = S^n D_n(\{a_i\}_i)$, $\forall \{a_i\}_i \in l_\infty$ //

(ii) $(HS-SH)(a) \in c_0, \quad \forall a \in l_\infty.$

ⓐ $\{b_i\}_i = HS(\{a_i\}_i) = \left(a_2, \frac{a_2+a_3}{2}, \frac{a_2+a_3+a_4}{3}, \dots\right) = \left\{\frac{\sum_{k=1}^{i} a_{k+1}}{i}\right\}_i$

$\{c_i\}_i = SH(\{a_i\}_i) = \left(\frac{a_1+a_2}{2}, \frac{a_1+a_2+a_3}{3}, \dots\right) = \left\{\frac{\sum_{k=1}^{i+1} a_k}{i+1}\right\}_i$

ⓑ $(HS-SH)(\{a_i\}_i) = \{b_i - c_i\}_i$

$$= \left\{\frac{\sum_{k=1}^{i} a_{k+1}}{i} - \frac{\sum_{k=1}^{i+1} a_k}{i+1}\right\}_i$$

$$= \left\{\frac{1}{i(i+1)}\left(\sum_{k=1}^{i}\left((i+1)-i\right)a_{k+1}\right) + \frac{-a_1}{i+1}\right\}_i$$

$$= \left\{\frac{\sum_{k=1}^{i}(a_{k+1}-a_1)}{i(i+1)}\right\}_i$$

ⓒ $\{a_i\}_i \in l_\infty \Rightarrow -\infty < -\sup_l |a_l| \leq a_k \leq \sup_l |a_l| < \infty$

ⓓ By ⓑ and ⓒ,

$$0 = \lim_{i\to\infty} \frac{-2i \sup_l |a_l|}{i(i+1)} \leq \lim_{i\to\infty} \frac{\sum_{k=1}^{i}(a_{k+1}-a_1)}{i(i+1)} \leq \lim_{i\to\infty} \frac{2i \sup_l |a_l|}{i(i+1)} = 0.$$

Thus, $\lim_{i\to\infty} \frac{\sum_{k=1}^{i}(a_{k+1}-a_1)}{i(i+1)} = 0$

and therefore $(HS-SH)(a) \in c_0, \quad \forall a \in l_\infty.$

(iii) $(HD_n - D_nH)(a) \in c_0$, $\forall a \in \ell_\infty$

ⓐ $\{b_i\} = HD_n(\{a_i\}_i) = H(\{a'_i\}_i) = \left\{\frac{\sum_{k=1}^{i} a'_k}{i}\right\}_i = \left\{\frac{\sum_{k=1}^{i} a_{\lfloor\frac{k-1}{n}\rfloor+1}}{i}\right\}_i$

$$\left(\begin{aligned}\{a'_i\}_i &\overset{\text{def}}{=} D_n(\{a_n\}_i)\\ &= \{a_{\lfloor\frac{i-1}{n}\rfloor+1}\}\end{aligned}\right)$$

$\{c_i\} = D_nH(\{a_i\}_i) = D_n(\{a''_i\}_i) = \{a''_{\lfloor\frac{i-1}{n}\rfloor+1}\}_i = \left\{\frac{\sum_{k=1}^{\lfloor\frac{i-1}{n}\rfloor+1} a_k}{\lfloor\frac{i-1}{n}\rfloor+1}\right\}_i$

$$\left(\begin{aligned}\{a''_i\}_i &\overset{\text{def}}{=} H(\{a_i\}_i)\\ &= \left\{\frac{\sum_{k=1}^{i} a_k}{i}\right\}_i\end{aligned}\right)$$

ⓑ Any natural number $i \in \mathbb{N}$ can be uniquely expressed as $i = tn + m$ $(1 \leq m \leq$ with $m, t \in \mathbb{N}$

$\{b_i\}_i = \left\{\frac{\sum_{k=1}^{i} a_{\lfloor\frac{k-1}{n}\rfloor+1}}{i}\right\}_i = \left\{\frac{n\sum_{l=1}^{t} a_l + m a_{t+1}}{tn+m}\right\}_i$

$\{c_i\}_i = \left\{\frac{\sum_{k=1}^{\lfloor\frac{i-1}{n}\rfloor+1} a_k}{\lfloor\frac{i-1}{n}\rfloor+1}\right\}_i = \left\{\frac{\sum_{k=1}^{t+1} a_k}{t+1}\right\}_i$.

ⓒ $b_i - c_i = \frac{n\sum_{l=1}^{t} a_l + m a_{t+1}}{tn+m} - \frac{\sum_{k=1}^{t+1} a_k}{t+1}$

(When $i = tn+m$,)

$$= \frac{1}{(t+1)(tn+m)}\left(\sum_{k=1}^{t}(tn+n)a_k - \sum_{k=1}^{t}(tn+m)a_k\right) + \frac{m a_{t+1}}{tn+m} - \frac{a_{t+1}}{t+1}$$

$$= \frac{1}{(t+1)(tn+m)}\left(\sum_{k=1}^{t}(n-m)a_k\right) + \frac{m a_{t+1}}{tn+m} - \frac{a_{t+1}}{t+1}$$

ⓓ $\lim_{t\to\infty}\frac{-t(n-m)\sup_l|a_l|}{(t+1)(tn+m)} \leq \lim_{t\to\infty}\frac{\sum_{k=1}^{t}(n-m)a_k}{(t+1)(tn+m)} \leq \lim_{t\to\infty}\frac{t(n-m)\sup_l|a_l|}{(t+1)(tn+m)}$

$\quad\quad\quad \| \quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad \|$

$\quad\quad\quad 0 \quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad 0$

Therefore, $\lim_{t\to\infty}\frac{\sum_{k=1}^{t}(n-m)a_k}{(t+1)(tn+m)} = 0$. and $\underbrace{\lim_{t\to\infty}\left(\frac{m a_{t+1}}{tn+m}\right) = \lim_{t\to\infty}\left(\frac{a_{t+1}}{t+1}\right) = 0}_{(\because\ a_t \text{ is bounded.})}$

ⓔ From ⓒ and ⓓ,

$$\lim_{i\to\infty}(b_i - c_i) = 0.$$

and therefore $(HD_n - D_nH)(a) \in c_0$, $\forall a \in l_\infty$.