Exercise 1 *Consider the spiral in* $\mathbb{R}^3$ *defined by the function*

$$f : \mathbb{R} \ni t \mapsto \big(\cos(2t), \sin(2t), 3t\big) \in \mathbb{R}^3.$$

Compute the length of this curve between $t = 1$ *and* $t = 3$.

Exercise 2 *For a function* $f : \mathbb{R}^n \to \mathbb{R}^d$*, give the definition of "f is differentiable at $X \in \mathbb{R}^n$". What is the derivative of f at X ?*

Exercise 3 (Cylindrical coordinates) *Consider the map* $\Phi : [0, \infty) \times [0, 2\pi) \times \mathbb{R} \to \mathbb{R}^3$ *with*

$$\Phi(r, \theta, z) := \big(r\cos(\theta), r\sin(\theta), z\big).$$

Compute the Jacobian matrix corresponding to this function, and the corresponding Jacobian determinant. Is the Jacobian matrix symmetric ?

Exercise 4 *Consider the function* $f : \mathbb{R}^2 \to \mathbb{R}$ *defined by* $f(x, y) = x - y^2$.

(i) Sketch the k-level sets of this function for $k = -2, -1, 0, 1, 2$,

(ii) Compute the gradient ∇f of f, and represent the vector field $(x, y) \mapsto [\nabla f](x, y)$ on the drawing of question (i), i.e. at any $(x, y) \in \mathbb{R}^2$ draw an arrow corresponding to the vector $[\nabla f](x, y)$,

(iii) What do you observe when you compare the k-level sets and the gradient.

Exercise 5 *The aim of this exercise is to determine some local maximal value of the function* $f : \mathbb{R}^2 \to \mathbb{R}$ *defined by*

$$f(x, y) = 3xy - 3x^2 - y^3$$

on the domain $\Omega := [-1, 1] \times [-1, 1]$*. For that purpose,*

(1) Determine the critical point(s) of f in Ω,

(2) Compute the Hessian matrix of f at the critical point(s),

(3) Determine the sign of the eigenvalues of the Hessian matrix at the critical point(s) (for that purpose, you can use the indication mentioned in the reminder below) and discuss if each critical point is a local minimum, a local maximum or a saddle point,

(4) Determine the restriction of f on the edge of the square Ω defined by $y = -1$, and determine the local extrema(s) of f on this edge. Compare what you have found with the result obtained in (3).

Reminder: For a matrix $\mathcal{A} := \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ recall that $\mathsf{Det}(\mathcal{A}) := ad - bc$ is *the determinant* of $\mathcal{A}$ and that $\mathsf{Tr}(\mathcal{A}) := a + d$ denotes *the trace* of $\mathcal{A}$. If λ_1 and λ_2 denote the eigenvalues of a symmetric matrix $\mathcal{A}$, then one has $\lambda_1\lambda_2 = \mathsf{Det}(\mathcal{A})$ and $\lambda_1 + \lambda_2 = \mathsf{Tr}(\mathcal{A})$.

Total 17 pts

Midterm

Exercise 1

2 pts

One has $f(t) = (\cos(2t), \sin(2t), 3t)$

$$\Rightarrow \quad f'(t) = (-2\sin(2t), 2\cos(2t), 3)$$

$$\Rightarrow \quad \|f'(t)\| = \left(4\sin^2(2t) + 4\cos^2(2t) + 9\right)^{1/2}$$

$$= \sqrt{13}.$$ 1

Thus, the length of the curve between $t=1$ and $t=3$ is given by $L = \int_1^3 \|f'(t)\| \, dt = \sqrt{13}\int_1^3 dt = \underline{\underline{2\sqrt{13}}}.$

1

Exercise 2

3 pts

f is differentiable at X if the partial derivatives of f at X exist, namely if $\frac{\partial f_j}{\partial x_k}$ exists 1 for $j = 1, \dots, d$ and for $k = 1, \dots, n$, and if

$$f(X+H) = f(X) + Df(X)H + \|H\| g(H)$$

1

for all H with $\|H\|$ small enough and for $g : \mathbb{R}^n \to \mathbb{R}^d$ satisfying $g(H) \longrightarrow 0$ as $H \longrightarrow 0$.

Note that $f'(X) \equiv Df(X)$ is the $d \times n$ matrix given by the partial derivatives, namely $(Df(X))_{jk} = \frac{\partial f_j}{\partial x_k}(X)$. The derivative is then a $d \times n$ matrix.

1

Exercise 3 3 pts

$\phi(r,\theta,z) = (r\cos(\theta), r\sin(\theta), z)$, and therefore

$$D\phi(r,\theta,z) = \begin{pmatrix} \frac{\partial \phi_1}{\partial r} & \frac{\partial \phi_1}{\partial \theta} & \frac{\partial \phi_1}{\partial z} \\ \frac{\partial \phi_2}{\partial r} & \frac{\partial \phi_2}{\partial \theta} & \frac{\partial \phi_2}{\partial z} \\ \frac{\partial \phi_3}{\partial r} & \frac{\partial \phi_3}{\partial \theta} & \frac{\partial \phi_3}{\partial z} \end{pmatrix}(r,\theta,z)$$

$$= \begin{pmatrix} \cos(\theta) & -r\sin(\theta) & 0 \\ \sin(\theta) & r\cos(\theta) & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad 1\tfrac{1}{2} \longleftarrow \text{Jacobian matrix}$$

($\frac{1}{2}$ for correct equalities)

Then $\text{Det}\left(D\phi(r,\theta,z)\right) = r\cos^2(\theta) + r\sin^2(\theta) = \underline{r}$. 1

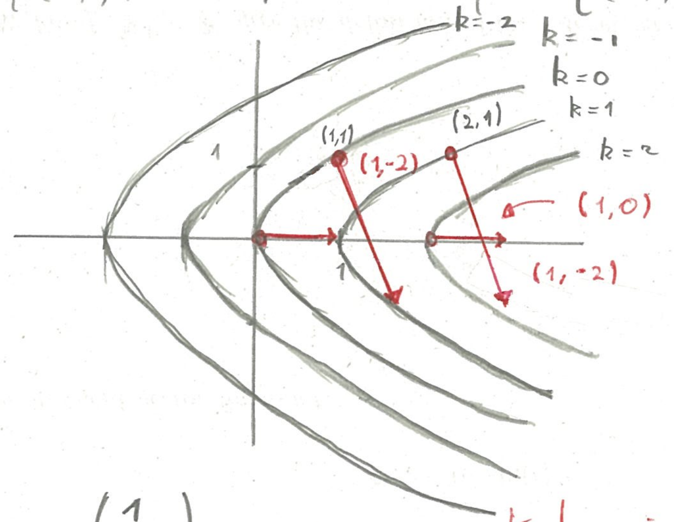
Jacobian determinant.

The Jacobian matrix is not symmetric and has no reason to be symmetric. $\frac{1}{2}$

Exercise 4 3 pts

i) $L_k = \{(x,y)\in\mathbb{R}^2 \mid x - y^2 = k\} = \{(x,y)\in\mathbb{R}^2 \mid y = \pm\sqrt{x-k}\}$

1



ii) $\nabla f(x,y) = \begin{pmatrix} 1 \\ -2y \end{pmatrix}$ represented in red. 1

iii) We observe (and it has been proved in Homework 4, Exercise 3, that the gradient is always perpendicular to the k-level sets. 1

Exercise 5   6 pts

1) Critical points :

$$\nabla f(x,y) = \begin{pmatrix} 3y - 6x \\ 3x - 3y^2 \end{pmatrix}.$$

Thus, (x,y) is a critical point if it satisfies

$$\begin{cases} 3y - 6x = 0 \\ 3x - 3y^2 = 0 \end{cases} \iff \begin{cases} y = 2x \\ x = 4x^2 \end{cases} \iff \begin{cases} x = 0 \\ y = 0 \end{cases} \text{ or } \begin{cases} x = \frac{1}{4} \\ y = \frac{1}{2} \end{cases}.$$

Critical points : $\underline{(0,0)}$ 1 and $\underline{(\frac{1}{4}, \frac{1}{2})}$. Both are in Ω. 1

2) $\partial_x^2 f(x,y) = -6$, $\partial_y^2 f(x,y) = -6y$

$\partial_x \partial_y f(x,y) = 3$

$$\Rightarrow \quad H_f(0,0) = \underline{\begin{pmatrix} -6 & 3 \\ 3 & 0 \end{pmatrix}} \; \tfrac{1}{2} \quad \text{and} \quad H_f(\tfrac{1}{4}, \tfrac{1}{2}) = \underline{\begin{pmatrix} -6 & 3 \\ 3 & -3 \end{pmatrix}}. \; \tfrac{1}{2}$$

3) $\mathrm{Det}\begin{pmatrix} -6 & 3 \\ 3 & 0 \end{pmatrix} = -9$ and $\mathrm{Tr}\begin{pmatrix} -6 & 3 \\ 3 & 0 \end{pmatrix} = -6$

$\Rightarrow$ The eigenvalues λ_1 and λ_2 have a different sign $\Rightarrow$ $(0,0)$ is a <u>saddle point</u>. $\frac{1}{2}$

$\mathrm{Det}\begin{pmatrix} -6 & 3 \\ 3 & -3 \end{pmatrix} = 9$ and $\mathrm{Tr}\begin{pmatrix} -6 & 3 \\ 3 & -3 \end{pmatrix} = -9$.

$\Rightarrow$ The eigenvalues have the same sign, and are negative. Thus $(\frac{1}{4}, \frac{1}{2})$ is a <u>local maximum</u>. $\frac{1}{2}$

One has $f(\frac{1}{4}, \frac{1}{2}) = 3\,\frac{1}{8} - 3\,\frac{1}{16} - \frac{1}{8} = \underline{\underline{\frac{1}{16}}}$.

4) On the edge defined by $y = -1$ one has

$g(x) := f(x,-1) = -3x - 3x^2 + 1$, for $x \in [-1, 1]$. $\frac{1}{2}$

Then $g'(x) = -3 - 6x = 0 \iff x = -\frac{1}{2}$

and the point $x = -\frac{1}{2}$ is the global maximum of

the function $x \longmapsto -3x^2 - 3x + 1$. $\frac{1}{2}$

One then observes that

$$g(-\tfrac{1}{2}) = f(-\tfrac{1}{2}, -1) = \frac{3}{2} - \frac{3}{4} + 1 = \underline{\underline{\frac{7}{4}}} > \frac{1}{16}.$$ $\frac{1}{2}$

Thus the value taken by f at $(-\frac{1}{2}, -1)$ is bigger than the value at $(\frac{1}{4}, \frac{1}{2})$ but the point $(-\frac{1}{2}, -1)$ is not a critical point but only a local maximum of f on the domain Ω, and more precisely on the frontier of Ω.

$\frac{1}{2}$