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Def: If $(D(A), A)$ and $(D(B), B)$ satisfy $D(A) \subset D(B)$ and $Af = Bf$ $\forall f \in D(A)$, then $(D(B), B)$ is an extension of $(D(A), A)$, or $(D(A), A)$ is a restriction of $(D(B), B)$

Def: An operator $(D(A), A)$ is closed if for any sequence $\{f_n\} \subset D(A)$ satisfying $\lim_{n \to \infty} f_n = f \in \mathcal{H}$ and such that $\{Af_n\}$ is a Cauchy sequence in $\mathcal{H}$, then one has $f \in D(A)$ and $s\text{-}\lim_{n \to \infty} Af_n = Af$.

Def: Let $(D(A), A)$ be an unbounded operator and $D(A^*) = \{f \in \mathcal{H} \mid \exists f^* \in \mathcal{H}$ which satisfies $\langle f^*, g \rangle = \langle f, Ag \rangle \ \forall g \in D(A)\}$.
Then we set $A^* f := f^*$
The pair $(D(A^*), A^*)$ is called the adjoint of $(D(A), A)$

Remark: The equality $\langle f, Ag \rangle = \langle A^* f, g \rangle$ $\forall g \in D(A), f \in D(A^*)$ holds.

Lemma: Let $(D(A), A)$ be an unbounded operator. then:
1, $(D(A^*), A^*)$ is closed
2, $\operatorname{Ker}(A^*) = \operatorname{Ran}(A)^\perp$
3, If $(D(B), B)$ is an extension of $(D(A), A)$ then $(D(A^*), A^*)$ is an extension of $(D(B^*), B^*)$

Def: An operator $(D(A), A)$ is self-adjoint if
$(D(A), A) = (D(A^*), A^*)$

⚠ If $(D(A), A)$ is self-adjoint, then $\langle f, Ag \rangle = \langle Af, g \rangle$ $\forall f, g \in D(A)$. But this equality is not the definition of self-adjoint. It only says that $(D(A), A) \subset (D(A^*), A^*)$. An operator
(since $(A^*)^* \neq A$)

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satisfying this equality is called symmetric.

5, Resolvent and spectrum.

Def: The operator $(D(A), A)$ is invertible if $\ker(A) = \{0\}$
Then $A^{-1}: Ran(A) \longrightarrow D(A)$ such that $AA^{-1}f = f$
$\forall f \in Ran(A)$ and $A^{-1}Ag = g$ $\forall g \in D(A)$

Remark: If $(D(A), A)$ is closed and invertible then $(D(A^{-1}), A^{-1})$ is also closed. In addition $Ran(A) = \mathcal{H}$ then A^{-1} is defined on all $\mathcal{H}$

Examples:

1, $\mathcal{H} = L^2(\mathbb{R})$, $(\mathcal{H}_1, X)$
Is $(\mathcal{H}_1, X)$ invertible ?
Check: $\ker(X)$
$f \in \ker(X)$ if $Xf = 0$
$\Leftrightarrow [Xf](x) = x f(x) = 0 \ \forall x$
$\Leftrightarrow f = 0$
$\Rightarrow (\mathcal{H}_1, X)$ is invertible
Its inverse is $(D(X^{-1}), X^{-1})$
$D(X^{-1}) = \{ f \in \mathcal{H} \mid \int \left|\frac{1}{x} f(x)\right|^2 dx < \infty \}$

2, $\mathcal{H} = L^2(\mathbb{R})$, $(\mathcal{H}, \sin(X))$
with $[\sin(X) f] = \sin(x) f(x)$ (op of multiplication by the sine function)
$(\mathcal{H}, \sin(X))$ is invertible with inverse $\left(D\left(\frac{1}{\sin X}\right), \frac{1}{\sin X}\right)$

3, If we start with $\left(D\left(\frac{1}{\sin(X)}\right), \frac{1}{\sin(X)}\right)$, this operator is invertible, with inverse $(\mathcal{H}, \sin(X))$ which is bounded

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Def: let $(D(A), A)$ be a closed op. The resolvent set $\rho(A)$ is defined by $\rho(A) := \{ z \in \mathbb{C} \mid (A-z)^{-1} \in B(\mathcal{H}) \}$

$= \{ z \in \mathbb{C} \mid (A-z)^{-1}$ is invertible and $(A-z)^{-1}$ is bounded $\}$

$= \{ z \in \mathbb{C} \mid \ker(A-z) = \{0\}$ and $\mathrm{Ran}(A-z) = \mathcal{H} \}$

with $(A-z)f = Af - zf$.

For any $z \in \rho(A)$, the op $(A-z)^{-1}$ is called the resolvent of A at z

Also the spectrum of A is given by $\sigma(A) := \mathbb{C} \setminus \rho(A)$
$= \{ z \in \mathbb{C} \mid$ either $\ker(A-z) \neq 0$ or $\mathrm{Ran}(A-z) \neq \mathcal{H} \}$.

Properties:

1, $\rho(A)$ is always open $\Rightarrow \sigma(A)$ is a closed set. with the Neumann series.

2, If $z_1, z_2 \in \rho(A)$, then
$$(A-z_1)^{-1} - (A-z_2)^{-1} = (z_1 - z_2)(A-z_1)^{-1}(A-z_2)^{-1}$$

3, $(A-z_1)^{-1}$ and $(A-z_2)^{-1}$ is commute.

Def. let $(D(A), A)$ be a closed op, assume that there exist $z \in \mathbb{C}$ and $f \in D(A)$ s.t $Af = zf$. Then z is called an eigenvalue and f a corresponding eigenfunction.

$\sigma_p(A) = \{$ all eigenvalues of $A \}$.

Remark: $\sigma_p(A) \subset \sigma(A)$.

Def: The dimension of the vector space generated by all eigenfunctions associated with the eigenvalue z is called the multiplicity of z.

Lemma: If $A \in B(\mathcal{H})$, then $\sigma(A) \subset \underbrace{B(0, \|A\|)}_{\text{(ball)}} \subset \mathbb{C}$

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Proof : $(A-z)^{-1} = -z^{-1}(1-\frac{A}{z})^{-1}$

If $|z| > \|A\|$ then $\|\frac{A}{z}\| < 1$. By the Neumann series, $(1-\frac{A}{z})$ is invertible.

Proposition: Let $(D(A), A)$ be self-adjoint.

1, $\sigma(A) \subset \mathbb{R}$

2, $\sigma_p(A) \subset \mathbb{R}$

3, If $\lambda_1, \lambda_2 \in \sigma_p(A)$ with $\lambda_1 \neq \lambda_2$ and if $Af_1 = \lambda_1 f_1$ and $Af_2 = \lambda_2 f_2$ with $f_1 \neq 0$, $f_2 \neq 0$ then $f_1 \perp f_2$ ($\Leftrightarrow \langle f_1, f_2 \rangle = 0$)