Tensor

1, Covariant and contravariant transformation.

Consider $\mathbb{R}^n$ with 2 bases $E = (E_1, E_2, \dots E_n)$ and $F = (F_1, F_2, \dots, F_n)$

Clearly F_j is a linear of $E_1, E_2, \dots, E_n$, namely $\exists!\ \alpha_j^1, \alpha_j^2, \dots \alpha_j^n$ s.t: $F_j = \alpha_j^1 E_1 + \alpha_j^2 E_2 + \dots + \alpha_j^n E_n$.

line

We set $\mathcal{A} := (\alpha_j^i) \in M_{n\times n}(\mathbb{R})$

column

Observe that $\mathcal{A}$ is invertible (otherwise E_j could not be expressed with $F_1, \dots F_n$).

<u>Coordinates vector</u> (components vector)

let $V \in \mathbb{R}^n$, then

$$V = \sum_{i=1}^{n} (V^E)^i E_i$$

$$= \sum_{i=1}^{n} (V^F)^j F_j$$

V^E is called coordinate vector w.r.t E.

One has:

$$V = \sum_{j=1}^{n} (V^F)^j F_j$$

$$= \sum_{j=1}^{n} (V^F)^j \sum_{i=1}^{n} \alpha_j^i E_i = \sum_{i=1}^{n} \left(\sum_{j=1}^{n} \alpha_j^i (V^F)^j \right) E_i$$

$$\Rightarrow (V^E)^i = \sum_{j=1}^{n} \alpha_j^i (V^F)^j \quad \forall i$$

$$\Leftrightarrow V^E = \mathcal{A} V^F$$

$$\Leftrightarrow V^F = \mathcal{A}^{-1} V^E$$

<u>Def</u>: A contravariant "quantity" changes like $\mathcal{A}^{-1}$ for a change of bases defined by $\mathcal{A}$

No.

Date

$\Rightarrow$ The coordinates vector is contravariant.

Linear maps

Let $T : \mathbb{R}^n \to \mathbb{R}^n$ be a linear map and let $V \in \mathbb{R}^n$

Consider $(TV)^F = \mathcal{A}^{-1} (TV)^E$

$\Rightarrow T^{F\to F} V^F = \mathcal{A}^{-1} T^{E\to E} V^E$ ← representation of T w.r.t E

$\Rightarrow T^{F\to F} V^F = \mathcal{A}^{-1} T^{E\to E} \mathcal{A} V^F$

Since V is arbitrary, one gets:

$$T^{F\to F} = \mathcal{A}^{-1} T^{E\to E} \mathcal{A}$$

Observe that $((TV)^E)^i = (T^{E\to E} V^E)^i$
$= \sum_{j=1}^{n} (T^{E\to E})^i_j (V^E)^j$

$\Rightarrow (T^{E\to E})^i_j = (TE_j)^i$

$\Rightarrow TE_j$ defines the j-col of $T^{E\to E}$

Def: A covariant "quantity" changes like $\mathcal{A}$ for a change of bases defined by $\mathcal{A}$.

$\Rightarrow$ A linear map is one time covariant and one time contravariant.

Linear Functional:

let $L : \mathbb{R}^n \to \mathbb{R}$ be a linear map and $V \in \mathbb{R}^n$

$$LV = L\left(\sum_i (V^F)^i F_i\right) = \sum_i (V^F)^i L(F_i)$$

$$= \sum_i (V^F)^i L^F_i$$ ← representation of L w.r.t F

On the other hand: $LV = L \sum_j (V^E)^j E_j$

No.

Date

$$= L \sum_j (\mathcal{A} V^F)^j E_j$$

$$= \sum_j \sum_i \alpha_i^j (V^F)^i L_j^E = \sum_i (V^F)^i \sum_j \alpha_i^j L_j^E$$

$$\Rightarrow L_i^F = \sum_j \alpha_i^j L_j^E$$

If we set $L^F = (L_1^F, \dots L_n^F)$, then: $\underset{(n\times n)}{L^F} = \underset{(n\times n)}{L^E} \underset{(n\times n)}{\mathcal{A}}$

$\Rightarrow$ A linear functional is covariant.

2, Tensor

<u>Def 1</u>: A tensor of type (p,q) (also called a (p,q)-tensor) is a collection of numbers associated with E.

$$(T^E)^{i_1, i_2, \dots, i_p}_{j_1, j_2, \dots, j_q} \text{ with } i_k \in \{1, \dots n\},\ j_k \in \{1, \dots, n\}.$$

which is p times contravariant and q times covariant, which means

$$(T^F)^{i_1 \dots i_p}_{j_1 \dots j_q} = \sum_{i_1' \dots i_p' = 1}^{n} \sum_{j_1' \dots j_q' = 1}^{n} (\mathcal{A}^{-1})^{i_1}_{i_1'} \dots (\mathcal{A}^{-1})^{i_p}_{i_p'} (T^E)^{i_1' \dots i_p'}_{j_1' \dots j_q'} \mathcal{A}^{j_1'}_{j_1} \dots \mathcal{A}^{j_q'}_{j_q}$$

<u>Ex</u>: . The coordinates vector is a $(1,0)$ _ tensor
. A linear map from $\mathbb{R}^n \to \mathbb{R}^n$ is a $(1,1)$ _ tensor
. A linear functional is a $(1,0)$ _ tensor

<u>Def</u>: The dual of $\mathbb{R}^n$, denoted by $\mathbb{R}^{n*}$, consists in all linear functionals from $\mathbb{R}^n$ to $\mathbb{R}$.

<u>Lemma</u>: 1, $\mathbb{R}^{n*}$ is a vector space of dimension n

2, For any basis $E = (E_1, \dots, E_n)$ of $\mathbb{R}^n$, there exists a

No.

Date

unique basis, called dual basis, $\mathcal{E} = (\mathcal{E}^1, \dots \mathcal{E}^n)$ of $\mathbb{R}^{n*}$ s.t $\mathcal{E}^j(E_k) = \delta^j_k = \begin{cases} 1 & \text{if } j = k \\ 0, & \text{otherwise} \end{cases}$.

Observe that if $\mathcal{F}$ is the dual basis of F, then for any $V \in \mathbb{R}^n$:

$$\mathcal{F}^j(V) = \mathcal{F}^j\left(\sum_k (V^F)^k F_k\right) = \sum_k (V^F)^k \mathcal{F}^j(F_k)$$

$$= \sum_k (V^F)^k \delta^j_k = (V^F)^j$$

$$= (\alpha^{-1} V^E)^j = \sum (\alpha^{-1})^j_i (V^E)^i$$

$$= \sum_i (\alpha^{-1})^j_i \mathcal{E}^i(V)$$

$$\Rightarrow \mathcal{F}^j = \sum_i (\alpha^{-1})^j_i \mathcal{E}^i$$

Def 2 A tensor of type (p, q) is a multilinear map T:
$$\underbrace{\mathbb{R}^{n*} \times \mathbb{R}^{n*} \times \dots \times \mathbb{R}^{n*}}_{p} \times \underbrace{\mathbb{R}^n \times \dots \times \mathbb{R}^n}_{q} \longrightarrow \mathbb{R}.$$

Remark: By choosing one basis for $\mathbb{R}^n$, we get q copies of this basis and p copies of the dual basis, and any change of the initial basis gives a quantity which is p times contravariant and q times covariant.