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For any $B \in \mathcal{B}(\mathcal{H})$, $\operatorname{Ran}(B) := \{g \in \mathcal{H} \mid g = Bf \text{ for some } f \in \mathcal{H}\} = B\mathcal{H}$.

<u>Def</u>: $B \in \mathcal{B}(\mathcal{H})$ is invertible if $\ker(B) = \{0\}$
$\ker(B) = \{f \in \mathcal{H} \mid Bf = 0\}$

Then $B^{-1} : \operatorname{Ran}(B) \to \mathcal{H}$ and satisfies: $B^{-1}Bf = f \ \forall f \in \mathcal{H}$
$BB^{-1}g = g \ \forall g \in \operatorname{Ran}(B)$

If $\operatorname{Ran}(B) = \mathcal{H}$, then $B^{-1} : \mathcal{H} \to \mathcal{H}$ is a bounded operator

In this case we say that B is boundedly invertible, or invertible in $\mathcal{B}(\mathcal{H})$.

<u>Notation</u>: $\mathbb{0}, \mathbb{1} \in \mathcal{B}(\mathcal{H})$
$\mathbb{0}f = 0 \ \forall f \in \mathcal{H}$, $\mathbb{1}f = f \ \forall f \in \mathcal{H}$

<u>Lemma</u>: Neumann series
If $B \in \mathcal{B}(\mathcal{H})$ with $\|B\| < 1$ then $(\mathbb{1} - B)$ is boundedly invertible and $(\mathbb{1} - B)^{-1} = \sum_{n=0}^{\infty} B^n$ (which is norm convergent)

<u>III</u> / Special classes in $\mathcal{B}(\mathcal{H})$

Let $B \in \mathcal{B}(\mathcal{H})$

- B is self-adjoint if $B^* = B$ ($\Leftrightarrow \langle f, Bg \rangle = \langle Bf, g \rangle$)
- B is normal if $B^*B = BB^*$
- B is a projection if $B^2 = B$, and an orthogonal projection if $B^2 = B = B^*$
- B is a unitary if $B^{-1} = B^*$ ($\Leftrightarrow B^*B = \mathbb{1}$ and $B^*B = \mathbb{1}$)
- B is an isometry if $B^*B = \mathbb{1}$ ($\|Bf\|^2 = \langle Bf, Bf \rangle = \langle B^*Bf, f \rangle = \langle f, f \rangle = \|f\|$)
- B is a partial isometry if B^*B is an orthogonal projection

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(called initial set projection) $\Rightarrow BB^*$ is also an orthogonal project
(called final set projection).

About compact operators:

Consider $\{f_j, g_j\}_{j=1}^{N} \subset \mathcal{H} \ni f$

We define $A := \sum_{j=1}^{N} |f_j\rangle\langle g_j|$ with $|f_j\rangle\langle g_j| f = \underbrace{\langle g_j, f\rangle}_{\in \mathbb{C}} \underbrace{f_j}_{\in \mathcal{H}}$

A is called a finite rank operator. The set of all finite rank operators is denoted by $\mathcal{F}(\mathcal{H}) \subset \mathcal{B}(\mathcal{H})$

Rem: $\operatorname{Ran}(A) = \operatorname{span}(f_1, \dots, f_n)$ (at most of dim n)
and $\ker(A)$ "=" $\operatorname{span}(g_1, \dots, g_n)^{\perp}$ if $f_j \neq 0 \ \forall j$.

Def: An operator $B \in \mathcal{B}(\mathcal{H})$ is compact if $\forall \varepsilon > 0$, $\exists A \in \mathcal{F}(\mathcal{H})$ s.t. $\|B - A\| < \varepsilon$.

Compact operators are very close to matrices. The set of compact operators is denoted by $\mathcal{K}(\mathcal{H}) \subset \mathcal{B}(\mathcal{H})$ ($\mathcal{F}(\mathcal{H}) \subset \mathcal{K}(\mathcal{H})$)

Properties

- If $B \in \mathcal{K}(\mathcal{H})$, then $B^* \in \mathcal{K}(\mathcal{H})$
- $\mathcal{K}(\mathcal{H})$ is norm closed, a subalgebra of $\mathcal{B}(\mathcal{H})$.
- $\mathcal{K}(\mathcal{H})$ is an ideal ($\Leftrightarrow$ if $B \in \mathcal{K}(\mathcal{H})$, $A \in \mathcal{K}(\mathcal{H})$ then $AB, BA \in \mathcal{K}(\mathcal{H})$)
- If $\{f_n\} \subset \mathcal{H}$ with $f_n \xrightarrow[n\to\infty]{w} f_\infty$ and if $B \in \mathcal{K}(\mathcal{H})$, then $Bf_n \xrightarrow[n\to\infty]{s} Bf_\infty$
- If $(A_n) \subset \mathcal{B}(\mathcal{H})$ with $A_n \xrightarrow[n\to\infty]{s} A_\infty$ and if $B \in \mathcal{K}($

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then $A_n B \xrightarrow[n\to\infty]{\text{Uniformly}} A_\infty B$ and $B A_n^* \xrightarrow[n\to\infty]{V} B A_\infty^*$

(A compact operator improves convergence).

IV. Unbounded operators.

<u>Def</u>: A linear operator is a pair $(\underbrace{D(A)}_{\text{domain of operator } A}, A)$ with

$D(A) \subset \mathcal{H}$ and $A : D(A) \longrightarrow \mathcal{H}$ is linear map

$A(f+\alpha g) = Af + \alpha Ag \quad \forall f, g \in D(A),\ \alpha \in \mathbb{C}$

(In particular, any $B \in \mathcal{B}(\mathcal{H})$ is a linear operator with $D(B) = \mathcal{H}$)

$(D(A), A)$ is densely defined if $D(A)$ is dense in $\mathcal{H}$

⚠ What about $A+B$? only defined on $D(A) \cap D(B)$

and AB? defined on $\{f \in D(B) \mid Bf \in D(A)\}$.

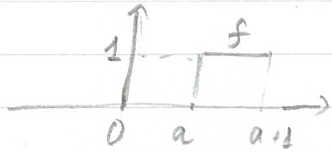
$(D(A), A)$ is <u>unbounded</u> if there is no $C>0$ s.t $\|Af\| = C\|f\|$ $\forall f \in D(A)$. For unbounded operators, we can not write $\|A\|$!

<u>Ex</u>: $\mathcal{H} = L^2(\mathbb{R})$, $(D(X), X)$ with $f \in D(X) = \{f \in \mathcal{H} \mid \int x^2 |f(x)|^2 dx < \infty\}$ and $[Xf](x) = x f(x)$ (X is called multiplication by variable operator).

<u>Then</u> $(D(X), X)$ is not bounded.

<u>Proof</u>: Choose f as a square function.

$f_a(x) = \begin{cases} 1 \text{ if } x \in [a, a+1] \\ 0, \text{ otherwise} \end{cases}$



$$\|f_a\|^2 = \int_{\mathbb{R}} |f_a(x)|^2 dx = \int_a^{a+1} 1\, dx = 1$$

$$\|Xf_a\|^2 = \int_{\mathbb{R}} x^2 |f_a(x)|^2 dx = \int_a^{a+1} x^2 \cdot 1\, dx = \frac{(a+1)^3 - a^3}{3}$$

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$\Rightarrow$ there is no $c>0$ s.t $\|X f_a\| = c\|f_a\|$ $\forall a$

$\Rightarrow$ Unbounded.

Observe that we could have defined $(C_c(\mathbb{R}), X)$, $(S(\mathbb{R}), x)$
The same expression (here X) can have different domain

<u>Example</u>: Consider $\mathcal{H} = L^2(\mathbb{R})$, $(\underbrace{S(\mathbb{R})}_{\text{Schwartz space}}, \overset{=D}{-i\partial x})$

with $(Df)(x) = -i f'(x)$ for any $f \in S(\mathbb{R})$.

Observe that if $\mathcal{F} : L^2(\mathbb{R}) \to L^2(\mathbb{R})$ denotes the Fourier transform (it is a unitary operator) then $\mathcal{F}\, S(\mathbb{R}) = S(\mathbb{R})$

Then: $\underbrace{\mathcal{F} D \mathcal{F}^{-1}}_{X \text{ (multiplication operator)}} \underbrace{\mathcal{F} f}_{\hat{f}}$, it means that $[\mathcal{F} D f](\overset{\in \mathbb{R}}{k}) = k \hat{f}(k)$

We can also define: $(\overset{\text{First Sobolev space on } \mathbb{R}}{\mathcal{H}^1(\mathbb{R})}, D) = \{ f \in \mathcal{H} \mid -i\partial_x f \in L^2(\mathbb{R}) \}$
$= \mathcal{F} D(X)$