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Hilbert spaces and linear operations.

I / Hilbert spaces.

Def : A Hilbert space is complex vector space with a strictly positive inner product $\langle \cdot , \cdot \rangle$, complete for the norm $\|\cdot\| := \sqrt{\langle \cdot , \cdot \rangle}$ and seperable ($\Leftrightarrow$ with a countable basis)

$$\forall f, g \in \mathcal{H} \quad \langle f, g \rangle = \overline{\langle g, f \rangle} \quad \text{(conjugate)}$$

$$\langle \alpha f + \beta g, h \rangle = \alpha \langle f, h \rangle + \beta \langle g, h \rangle$$

Observe : $\langle f, \alpha g \rangle = \bar{\alpha} \langle f, g \rangle$

$$\langle f, f \rangle = \|f\|^2 = 0 \text{ iff } f = 0$$

Def : A sequence $\{f_n\} \subset$ Vector space with a norm, is Cauchy if $\forall \varepsilon > 0$, $\exists N$ large enough s.t $\|f_n - f_m\| \leq \varepsilon$ $\forall n, m \geq N$

Def : $\mathcal{H}$ is complete iff each Cauchy sequence converges in $\mathcal{H}$ $\Leftrightarrow \exists f_\infty \in \mathcal{H}$ s.t $\lim_{n \to \infty} \|f_n - f_\infty\| = 0$

Ex : $\mathbb{C}^n = \{(a_1, a_2, \dots, a_n) \mid a_j \in \mathbb{C}\}$

For $a, b \in \mathbb{C}^n$, $\langle a, b \rangle = \sum a_j \overline{b_j}$

Ex : $\ell^2(\mathbb{Z}) = \{(\dots a_{-2}, a_{-1}, a_0, a_1, a_2, \dots) \mid a_j \in \mathbb{C}, \sum_{j \in \mathbb{Z}} |a_j|^2 < \infty\}$

For $a, b \in \ell^2(\mathbb{Z})$, $\langle a, b \rangle = \sum_{j \in \mathbb{Z}} a_j \overline{b_j}$

Ex : $L^2(\mathbb{R}^d)$

$f, g \in L^2(\mathbb{R}^d)$, $\langle f, g \rangle = \int_{\mathbb{R}^d} f(x) \overline{g(x)} \, dx$

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Properties

- $|\langle f, g\rangle| \leqslant \|f\| \, \|g\|$ (Schwartz inequality)
- $\|f+g\| \leqslant \|f\| + \|g\|$ (triangular inequality)
- $\|f+g\|^2 \leqslant 2\|f\|^2 + 2\|g\|^2$
- $\big| \|f\| - \|g\| \big| \leqslant \|f-g\|$

Topologies on $\mathcal{H}$

- A sequence $\{f_n\} \subset \mathcal{H}$ converges strongly to $f_\infty \in \mathcal{H}$ if $\lim_{n\to\infty} \|f_n - f_\infty\| = 0$

- It converges weakly to $f_\infty \in \mathcal{H}$ if for any $g \in \mathcal{H}$, one has $\lim_{n\to\infty} \langle g, f_n - f_\infty\rangle = 0$

- If a sequence converges strongly then it converges weakly.

Lemma

- $s\text{-}\lim_{n\to\infty} f_n = f_\infty \Leftrightarrow w\text{-}\lim_{n\to\infty} f_n = f_\infty$ and $\lim_{n\to\infty} \|f_n\| = \|f_\infty\|$

- A subspace M of $\mathcal{H}$ is a linear subset of $\mathcal{H}$ (stable for addition and multiply by scalar). If M is closed (= complete) then M is a Hilbert space for the induced scalar product.

Examples: 1) Take $f_1, f_2, \dots, f_n \in \mathcal{H}$
and $M = \text{span}(f_1, f_2, \dots, f_n)$ is a complete subspace

2) $\mathcal{H} = L^2(\mathbb{R})$
$M = L^2(\mathbb{R}^+)$ is a closed subspace of $\mathcal{H}$

Def: If M is a subset of $\mathcal{H}$ and set $M^\perp = \{f \in \mathcal{H} \mid \langle f, g\rangle = 0$ for any $g \in M\}$

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$M^\perp$ is called orthogcomplement.

Thm: (Projection thm)

Let M be a closed subspace of $\mathcal{H}$. Then for any $f \in \mathcal{H}$, $\exists$ $f_M \in M$ and $M^\perp$ s.t: $f = f_M + f_{M^\perp}$.

Def: Set $\mathcal{H}^* = \{\varphi : \mathcal{H} \longrightarrow \mathbb{C}$ s.t φ is linear and bounded$\}$ (dual space), i.e $\varphi(\alpha f + g) := \alpha\varphi(f) + \varphi(g)$ and $\exists$ c s.t $|\varphi(g)| \leq c\ \|g\|$ $\forall g \in \mathcal{H}$.

Example: set $\varphi(f) = \langle f, g\rangle$ for any fixed $g \in \mathcal{H}$. It means, any $g \in \mathcal{H}$ defines an element of $\mathcal{H}^*$ by $\varphi(\cdot) = \langle \cdot, g\rangle$

Riesz lemma

Any element of $\mathcal{H}^*$ is of this form, it means for any $\varphi \in \mathcal{H}^*$, $\exists g \in \mathcal{H}$ s.t $\varphi(\cdot) = \langle \cdot, g\rangle$

Also denoted $\mathcal{H}^* \simeq \mathcal{H}$. One also has:

$|\varphi(g)| \leq \|\varphi\|\ \|g\|$ (always possible to find a g satisfied with the equality, ie: $\|\varphi\| = \|g\|$) where we set:

$$\|\varphi\| = \sup_{f \in \mathcal{H}, f \neq 0} \frac{|\varphi(f)|}{\|f\|} \quad \text{(norm of the functional)}$$

II / Bounded linear operators

Def: A map $B : \mathcal{H} \longrightarrow \mathcal{H}$ is a bounded linear operator if $B(f + \alpha g) = Bf + \alpha Bg$ and $\exists$ $c \geq 0$ s.t $\|Bf\| \leq c\|f\|$ $\forall f \in \mathcal{H}$.

Then we set: $$\|B\| := \sup_{\substack{f \in \mathcal{H} \\ f \neq 0}} \frac{\|Bf\|}{\|f\|}$$

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$$= \sup_{\substack{f \in M_1, \|f\|=1 \\ g \in M_2, \|g\|=1}} |\langle Bf, g\rangle| \text{ with } M_1, M_2 \text{ dense subset of } \mathcal{H}.$$

Def: M is dense in $\mathcal{H}$ if $\forall \varepsilon > 0$ and any $f \in \mathcal{H}$, $\exists g \in M$ s.t $\|f-g\| < \varepsilon$. (Ex, $C_c(\mathbb{R})$ is dense in $L^2(\mathbb{R})$)
i.e, $\forall f \in \mathcal{H}$, $\exists \{g_n\} \subset M$ s.t $\lim_{n\to\infty} \|g_n - f\| = 0$

The set of all bounded and linear operators is denoted by $B(\mathcal{H})$

$B(\mathcal{H})$ is endowed with an involution: taking the adjoint, it means, to any $B \in \mathcal{H}$, there exists $B^* \in B(\mathcal{H})$ which satisfies: $\langle f, Bg\rangle = \langle B^* f, g\rangle$.

Properties

- $\|B^*\| = \|B\|$
- $(B^*)^* = B$
- $(AB)^* = B^* A^*$
- $\|B^* B\| = \|B\|^2$
- For a matrix $\alpha \in M_n(\mathbb{C})$: $(\alpha^*)_{ij} = \overline{\alpha_{ji}}$
- If $\mathcal{H} = \mathbb{C}^n$, then $B(\mathcal{H}) = M_n(\mathbb{C})$

Topologies on $B(\mathcal{H})$

Consider $\{B_n\} \subset B(\mathcal{H})$ and $B_\infty \in B(\mathcal{H})$

- $\{B_n\}$ converges uniformly to B_∞ if
$$\lim_{n\to\infty} \|B_n - B_\infty\| = 0$$

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- $\{B_n\}$ converges strongly to B_∞ if $\forall g \in \mathcal{H}$
  $\lim_{n\to\infty} \| (B_n - B_\infty) g \| = 0$
- $\{B_n\}$ converges weakly to B_∞ if $\forall f, g \in \mathcal{H}$
  $\lim_{n\to\infty} \langle f, (B_n - B_\infty) g \rangle = 0$
- u-lim $\Rightarrow$ s-lim $\Rightarrow$ w-lim

Remark: $B(\mathcal{H})$ is complete for these 3 topologies