No.

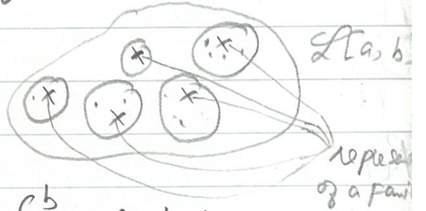
Date

Def: The set of all $\mathcal{L}$. integrable functions on $[a,b]$ is denoted by $\mathcal{L}([a,b])$

Thm: (Lebesgue dominated convergence theorem)
Let $\{f_n\}$ be a sequence of $\mathcal{L}$.m function on $[a,b]$ s.t $\lim_{n\to\infty} f_n(x) = f(x)$ for a.e $x \in [a,b]$.

Suppose $\exists\, g \in \mathcal{L}([a,b])$ with $|f_n| \le g$.
Then $f \in \mathcal{L}([a,b])$ and $\lim_{n\to\infty} \int_a^b f_n(x)\,dx = \int_a^b f(x)\,dx$,
g is called dominating function.

4, L^p- spaces
For any $f, g \in \mathcal{L}([a,b])$, we write $f \sim g$ if $f = g$ a.e on $[a,b]$
Then $\sim$ is an equivalence relation ($f\sim g \Rightarrow g \sim f$; $f \sim f$; $f\sim g$ & $g \sim h$ $\Rightarrow f \sim h$)



Def: $L^1([a,b]) := \mathcal{L}([a,b])/\sim$

For any $f \in L^1([a,b])$, we set $\|f\|_1 := \int_a^b |f(x)|\,dx < \infty$

Proposition: • $f \to \|f\|_1$ defines a norm on $L^1([a,b])$
1, $\|\lambda f\| = |\lambda|\, \|f\|$
2, $\|f+g\| \le \|f\| + \|g\|$
3, $\|f\| \geq 0$ with equality iff $f = 0$
• $(L^1([a,b]), \|\cdot\|_1)$ is a Banach space : $\lim_{n\to\infty} f_n = f_\infty \in L^1([a,b])$

Def: •, For any $p \geq 1$ we set
$L^p = \{f \text{ measurable on } [a,b] \mid |f|^p \text{ is } \mathcal{L}\text{. integrable}\}/\sim$
•, We set $\|f\|_p := \left(\int_a^b |f(x)|^p dx\right)^{1/p}$

No.

Date

Thm: $(L^p([a,b]), \|.\|_p)$ is a Banach space.

Thm: Holder inequality
If $f \in L^p([a,b])$, $g \in L^q([a,b])$ with $p \geq 1$, $\frac{1}{p} + \frac{1}{q} = 1$.
Then $fg \in L^1([a,b])$ and $\|fg\|_1 \leq \|f\|_p \cdot \|g\|_q$

Rem: For the case $p = \infty$: $L^\infty([a,b]) = \mathcal{L}^\infty([a,b]) / \sim$
$\|f\|_\infty = \operatorname{ess} \sup_{x \in [a,b]} |f(x)|$:

$\overset{\text{Holder ineq.}}{\Longrightarrow} \int_a^b |f(x)\, g(x)|\, dx \leq \sup_{x \in [a,b]} f(x) \int_a^b |g(x)|\, dx$

5, Fourier series

$f \in L^2([-\pi,\pi])$, $n \in \mathbb{N}$:

$$\begin{cases} a_0 = a_0(f) := \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x)\, dx \\ a_n = a_n(f) := \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos(nx)\, dx \\ b_n = b_n(f) := \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin(nx)\, dx \end{cases}$$

, are called Fourier coeff.

$$S_n(x) := a_0 + \sum_{j=1}^{n} a_j \cos(jx) + b_j \sin(jx)$$

One has: $S_n \in L^2([a,b])$

$$a_j(S_n) = \begin{cases} a_j(f) & \text{if } j \leq n \\ 0 & \text{if } j > n \end{cases}$$

$$b_j(S_n) = \begin{cases} b_j(f) & \text{if } j \leq n \\ 0 & \text{if } j > n \end{cases}$$

One has: $\lim_{n \to \infty} \|f - S_n\|_{L^2([-\pi,\pi])} = 0 \Leftrightarrow \lim_{n \to \infty} \left(\int_{-\pi}^{\pi} |f(x) - S_n(x)|^2 dx \right)^{1/2} = 0$

If f is slightly more regular (continuity, differentiability) then one has pointwise convergence or uniform convergence
$(\lim_{n \to \infty} S_n(x) = f(x))$ $\quad$ $(\lim_{n \to \infty} \sup_{x \in [-\pi,\pi]} |f(x) - S_n(x)| = 0)$