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3) Lebesgue integration (on $\mathbb{R}$)

We look for an integral which generalizes Riemann integral

$\Rightarrow$ If f is Riemann integrable $\Rightarrow$ Lebesgue integrable.

Def: $f : [a,b] \to \mathbb{R}$

f is Lebesgue measurable on $[a,b]$ if for any $S \in \mathbb{R}$, the set $\{x \in [a,b] \mid f(x) \geq S\}$ is $\mathcal{L}.m$

Lemma: • The set of $\mathcal{L}.m$ function on $[a,b]$ is a vector space

• The product of 2 $\mathcal{L}.m.f$ is $\mathcal{L}.m$

• The quotient of 2 $\mathcal{L}.m.f$ is $\mathcal{L}.m$, if the denominator is not equal to 0 on $[a,b]$.

Remark: . For Riemann integral, a piecewise cts function on (a,b) with a finite number of jump is OK

. For Lebesgue integral, we can do much better.

Def: Let $f, g : [a,b] \to \mathbb{R}$.

• $f = g$ a.e. (almost everywhere) if $\{x \in [a,b] \mid f(x) \neq g(x)\}$ has Lebesgue measure 0.

• $f \leq g$ a.e. if $\{x \in [a,b] \mid f(x) > g(x)\}$ has Lebesgue measure 0.

Prop: Let f, g $[a,b] \to \mathbb{R}$. If f is $\mathcal{L}.m.$ and $g = f$ a.e. then g is $\mathcal{L}.m$

Additional concept: Let $\{f_n\}_{n\in\mathbb{N}} \subset \mathcal{L}^\infty([a,b])$ and assume that for any $x \in [a,b]$, $\{f_n(x)\}_n$ is bounded. For any $x \in [a,b]$, we define:

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- $f^*(x) := \lim_{n\to\infty} \sup f_n(x) = \lim_{n\to\infty} \underbrace{\sup_{m\geq n} \{f_m(x)\}}$

this sequence is decreasing with n.

- $f_*(x) := \lim_{n\to\infty} \inf f_n(x) = \lim_{n\to\infty} \underbrace{\inf_{m\geq n} \{f_m(x)\}}$

this sequence is increasing with n

It means, we define 2 functions:

$f^*, f_* : [a,b] \to \mathbb{R}$

Clearly: $f^*(x) \geq f_*(x) \quad \forall x \in [a,b]$

with an equality if $\lim_{n\to\infty} f_n(x)$ exists.

<u>Thm</u>: If each f_n is $\mathcal{L}$.m. on $[a,b]$ and $\{f_n(x)\}_n$ is bounded for any $x \in [a,b]$, then f_* and f^* are $\mathcal{L}$.m

<u>Corollary</u>: If $\{f_n\} \subset \mathcal{L}^\infty([a,b])$ and each f_n is $\mathcal{L}$.m.

If there exists $f_\infty : [a,b] \to \mathbb{R}$ s.t. $\lim_{n\to\infty} f_n(x) = f_\infty(x)$

for a.e. $x \in [a\,b]$, then f_∞ is $\mathcal{L}$.m.

<u>Def</u>: A measurable partition P of $[a,b]$ consists in a finite collection $\{E_j\}_{j=1}^N \subset [a,b]$ s.t:

(E_j: subset of $[a,b]$)

- Each E_j is $\mathcal{L}$.m set
- $\bigcup_j E_j = [a,b]$
- $m(E_j \cap E_k) = 0 \quad \forall j \neq k$.

For any $f \in \mathcal{L}^\infty([a,b])$ and for any measurable partition P we set:

$$U[f,P] = \sum_j \left(\sup_{x\in E_j} f(x)\right) m(E_j).$$

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$$L[f,P] = \sum_j \left(\inf_{x\in E_j} f(x)\right) m(E_j).$$

Def: f is Lebesgue integrable if $\sup_{P\in \text{all measurable } P} L[f,P] = \inf_{P} U[f,P] =: \#$

and then we set ${}_a\int_L^b f(x)\,dx = \#$

Exercise: If f is $\mathcal{L}.m$ on $[a,b]$

$$f(x) = \begin{cases} 1 & \text{if } x \in \mathbb{Q} \\ 0 & \text{if } x \in \mathbb{R}\setminus\mathbb{Q} \end{cases}$$

then ${}_a\int_L^b f(x)\,dx = 0$

Thm: If f is Rieman integrable then f is Lebesgue integrable.

Thm: Let $f \in \mathcal{L}^\infty([a,b])$. Then f is Lebesgue integrable iff f is $\mathcal{L}.m$

Remark: • If $f, g \in \mathcal{L}^\infty([a,b])$ and f is $\mathcal{L}.m$ and $f(x) = g(x)$ a.e. then: $\int_L f(x)\,dx = \int_L g(x)\,dx$

• For unbounded function f consider: $f = f_+ - f_-$,
$f_\pm(x) = \begin{cases} f(x) & \text{if } \pm f(x) \geq 0 \\ 0 & \text{otherwise} \end{cases}$. Then for f_+, consider $f_+^N(x) = \begin{cases} f_+(x) & \text{if } f(x) < N \\ N & \text{otherwise} \end{cases}$
then $f_+^N \in \mathcal{L}^\infty([a,b])$.

Def: • f_+ is $\mathcal{L}$ integrable on $[a,b]$ if f_+^N is $\mathcal{L}.m$ and $\lim_{N\to\infty} \int_L f_+^N(x)\,dx$ converges.

• f is $\mathcal{L}.m$ if f_+ and f_- are $\mathcal{L}.m$

Similar construction, for $[a,\infty)$, $(-\infty,b]$ or $(-\infty,\infty)$.