No.

Date

Lebesgue theory

1, Reminder on Riemann integral

Let P a partition of $[a, b]$, it means
$P = \{x_0, x_1, \ldots, x_n\}$ with $x_0 = a$, $x_n = b$, $x_j < x_{j+1}$

Let $\mathcal{L}^\infty([a, b])$
$= \{ f : [a, b] \to \mathbb{C} \mid \exists c_f > 0$ with $|f(x)| \leq c_f \ \forall x \in [a, b] \}$

We set $\|f\|_\infty = \sup_{x \in (a,b)} |f(x)|$

We set : $\underset{\substack{\uparrow \\ \text{lower}}}{L}(f \in \mathcal{L}^\infty, P) = \sum_{j=1}^{n} \left(\inf_{x \in [x_{j-1}, x_j)} f(x) \right) . (x_j - x_{j-1})$

$\underset{\substack{\uparrow \\ \text{upper}}}{U}(f \in \mathcal{L}^\infty, P) = \sum_{j=1}^{n} \left(\sup_{x \in [x_{j-1}, x_j)} f(x) \right) . (x_j - x_{j-1})$

The lower integral is $\sup_P L(f, P)$, and upper integral is $\inf_P U(f, P)$

Def : f is Riemann integrable on $[a, b]$ if $\sup_P L(f, P) = \inf_P U(f, P)$

the value is denoted by $\int_a^b f(x)\,dx$

Remark : Set $f(x) = \begin{cases} 1 \text{ if } x \in \mathbb{Q} \\ 0 \text{ if } x \notin \mathbb{Q} \end{cases}$

f is not Riemann integrable on $[0, 1]$

Thm : If $f \in C([a, b])$, then f is Riemann integrable on $[a, b]$

2, Lebesgue measure

Reminder : Topology : $V \subset \mathbb{R}^n$ is open if for any $x \in V$
$\exists r > 0$ s.t $B(x, r) \subset V$

No.

Date

V is is <u>closed</u> if $\mathbb{R}^n \setminus V$ is open.

<u>Ex</u>: $[\ \text{———}\ )$ is not open nor closed in $\mathbb{R}$.

V is compact (in $\mathbb{R}^n$) if V is closed and bounded

<u>Def</u>: A closed box (interval) in $\mathbb{R}^n$ is a set $I = \{(x_1, \dots x_n) \in \mathbb{R}^n \mid a_j \leq x_j \leq b_j$ for some $a_j + b_j \in \mathbb{R}$ all $j = 1, \dots, n\}$

Its volume $v(I) := \prod_{j=1}^{n} (b_j - a_j) > 0$

<u>Idea</u>: cover any $V \subset \mathbb{R}^n$ by closed boxes.

<u>Def</u>: For any $V \subset \mathbb{R}^n$, the set $S := \{I_j\}$ is a covering of V if $V \subset \bigcup_i I_j$. S can be made of a finite or a countable ($\Leftrightarrow i \in \mathbb{N}$) number of closed boxes.

We set $\sigma(S) := \sum_j v(I_j) \in [0, \infty]$.

<u>Def</u>: For any $V \subset \mathbb{R}^n$, the <u>Lebesgue outer measure</u> of V denoted by $m^*(V) = \inf_{S \text{ covering of } V} \sigma(S)$

Then for any $\varepsilon > 0$, $\exists$ S covering of V s.t. $\sigma(S) \leq m^*(V) + \varepsilon$

<u>Exercise</u>: $m^*(I) = v(I)$

<u>Properties</u>:

1, If $V \subset W \subset \mathbb{R}^n$, then $m^*(V) \leq m^*(W)$

2, $m^*(V \cup U) \leq m^*(U) + m^*(V)$

3, $m^*(\bigcup_j V_j) \leq \sum_j m^*(V_j)$

⚠ Even if $V_1 \cap V_2 = \emptyset$, one can have $m^*(V_1 \cup V_2) < m^*(V_1) + m^*(V_2)$

No.

Date

Thm: Let $V \subset \mathbb{R}^n$ with $m^*(V) < \infty$ then for any $\varepsilon > 0$, $\exists$ G open set : $V \subset G$ and $m^*(G) < m^*(V) + \varepsilon$

Also true if $m^*(V) = \infty$

Note that if we consider $V \subset W$ then :

$m^*(W) \leq m^*(V) + m^*(W \setminus V)$

$\Rightarrow m^*(W) - m^*(V) \leq m^*(W \setminus V)$, but no information on $W \setminus V$

Def. A set $V \subset \mathbb{R}^n$ is Lebesgue measurable if $\forall \varepsilon > 0$, $\exists$ open set $G \subset \mathbb{R}^n$ with $V \subset G$ s.t $m^*(G \setminus V) \leq \varepsilon$. Then we set $m(V) = m^*(V)$, called Lebesgue measure.

Remark: On $\mathbb{R} \ni x, y$, we say that $x \sim y$ if $x - y \in \mathbb{Q}$ and write $[x] = \{ y \in \mathbb{R} \mid x - y \in \mathbb{Q} \}$

Consider $V \subset \mathbb{R}$ by choosing for one representation for each class (we need the axiom of choice)

Then V is not Lebesgue measurable.

Proposition:

1, If V is open, then V is Lebesgue measurable ($\mathcal{L}.m$)

2, If V satisfies $m^*(V) = 0 \Rightarrow V$ is $\mathcal{L}.m$

3, If $V = \bigcup_j V_j$ with V_i $\mathcal{L}.m$ then V is $\mathcal{L}.m$ with $m(V) \leq \sum_j m(V_j)$

4, If V is closed, then V is $\mathcal{L}.m$ In particular, a closed box is $\mathcal{L}.m$ with $m(I) = v(I)$

Thm: If $\{V_i\}$ of pair wise disjoint $\mathcal{L}.m$ sets, one has $m(\bigcup V_j) = \sum_i m(V_i)$