17/4/19

Reminder

- A distribution is a continuous linear function from $D(\mathbb{R}^n) \to \mathbb{C}$
- The set of all distribution is denoted by $D'(\mathbb{R}^n)$
- Derivative of T : $(\partial^\alpha T)(f) = (-1)^{|\alpha|} T(\partial^\alpha f)$, $\alpha \in \mathbb{N}^n$

  If $T = T_h$, $\partial^\alpha T_H = T_{\partial^\alpha h}$

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2) Recall $P_v \frac{1}{x} \in D(\mathbb{R})$ P_v : principal value???

$\boxed{\ln(|x|)' = P_v(\frac{1}{x})}$

$$
\begin{aligned}
(P_v \tfrac{1}{x})(f) &= \lim_{\varepsilon \to 0} \int_{\mathbb{R}\setminus[-\varepsilon,\varepsilon]} \frac{1}{x} f(x)\,dx \\
\underset{D(\mathbb{R})}{\cap} \\
&= \lim_{\varepsilon \to 0} \int_{\mathbb{R}\setminus[-\varepsilon,\varepsilon]} \ln(|x|)' f(x)\,dx \\
&= \lim_{\varepsilon \to 0} - \int \ln(|x|) f'(x)\,dx + O(\varepsilon) \qquad \lim_{\varepsilon\to 0} O(\varepsilon) = 0 \\
&= - \int_{\mathbb{R}} \ln(|x|) f'(x)\,dx \\
&= - T_{\ln(|\cdot|)}(f') = \partial T_{\ln(|\cdot|)}(f)
\end{aligned}
$$

For $n = 3$, consider

$h : \mathbb{R}^3 \to \mathbb{R}$, $h(x) = \frac{1}{\|x\|}$ $\int_{B(0,1)} \frac{1}{\|x\|}\,dx = 4\pi \int_0^1 \frac{1}{r} r^2\,dr < \infty$

$h \in L^1_{loc}(\mathbb{R}^3)$

T_h is a distribution $\begin{cases} T_h(f) = \int h(x) f(x)\,dx \\ \forall h \in L^1_{loc}(\mathbb{R}^n) \end{cases}$

$\Delta T_h = (\partial_1^2 + \partial_2^2 + \partial_3^2) T_h = -4\pi \delta_0$ delta

$\Delta \frac{1}{\|x\|} = -4\pi \delta_0$

3) Other operations on distributions

- Multiplication by a C^∞ - function

  If $T \in D'(\mathbb{R}^n)$, $g \in C^\infty(\mathbb{R}^n)$, $f \in D(\mathbb{R}^n)$

  Then $\underbrace{(gT)}_{\in D'(\mathbb{R}^n)}(f) := T(gf)$

Remark: if $T = T_h$ with $h \in L^1_{loc}(\mathbb{R}^n)$ then $gT_h = T_{gh}$

Ex : 1. $g\delta_0 = g(0)\,\delta_0$

or $n=1$

2. $(g\delta_0)' = \delta(g\delta_0) = g\delta_0' + g'(0)\,\delta_0$

Convergence in $D'(\mathbb{R}^n)$

Def : A sequence $\{T_i\}_{i\in\mathbb{N}} \subset D'(\mathbb{R}^n)$ converges to $T_\infty \in D'(\mathbb{R}^n)$ in the sense of distribution if $T_i(f) \xrightarrow{i\to\infty} T_\infty(f)$ for any $f \in D(\mathbb{R}^n)$

In fact, if $\{T_i(f)\}$ converges for any $f \in D(\mathbb{R}^n)$, then there exists $T_\infty \in D'(\mathbb{R}^n)$ such that $\lim_{i\to\infty} T_i(f) = T_\infty(f)$

Remark : If $\{T_i\}$ converges to T_∞ in the sense of distribution, then $\{\partial^\alpha T_i\}$ converges to $\partial^\alpha T_\infty$ for any $\alpha \in \mathbb{N}^n$

Exercise.

Consider $h_i(x) := \dfrac{\sin(ix)}{x} \in L^1_{loc}(\mathbb{R})$ and $T_{h_i} \xrightarrow{i\to\infty} \pi\delta_0$

$$T_{h_i} = \int_{\mathbb{R}} \frac{\sin(ix)}{x} f(x)\,dx \xrightarrow{i\to\infty} \pi f(0)$$

Convolution of distribution

↑ can be defined for "some" distributions

$$(T_h * T_2)(f) : T_2(h * f)$$

4) Fourier transform, Schwartz function and tempered distribution

For $f \in L^1(\mathbb{R}^n) = \{ f : \mathbb{R}^n \to \mathbb{C} \mid \int_{\mathbb{R}^n} |f(x)|\,dx < \infty \}$

we define
$$(\mathcal{F} f)(x) \equiv \hat{f}(x) = \frac{1}{(2\pi)^{\frac{n}{2}}} \int e^{-iy\cdot x} f(y)\,dy$$

and call it the Fourier transform of f

Properties

1) $\mathcal{F} : L^1(\mathbb{R}^n) \longrightarrow C_0(\mathbb{R}^n)$ is linear

$C_0(\mathbb{R}^n) = \{ f : \mathbb{R}^n \to \mathbb{C} \mid f \text{ is continuous and goes to } 0 \text{ at } \infty \}$

for any $\varepsilon > 0$, $\exists R > 0$ (large) such that $|f(x)| \le \varepsilon \quad \forall x$ with $\|x\| \geq R$

2) $|\hat{f}(x)| \leq \int |f(y)|\,dy \quad \forall x \in \mathbb{R}^n$

3) $\mathcal{F}(f * g) = \hat{f} \cdot \hat{g}$ with $(f * g)(x) = \frac{1}{(2\pi)^{\frac{n}{2}}} \int f(x-y)\, g(y)\, dy$

4) If $f, \partial_j f \in L^1(\mathbb{R}^n)$

then $\widehat{-i\partial_j f}(y) = y_j \hat{f}(y)$

Remark: $L^2(\mathbb{R}^n) = \{ f : \mathbb{R}^n \to \mathbb{C} \mid \int |f(x)|^2 dx < \infty \}$

Proposition

$\mathcal{F}$ extends to a bijective map from $L^2(\mathbb{R}^n)$ to $L^2(\mathbb{R}^n)$,

and $\|f\|_{L^2} = \|\hat{f}\|_{L^2}$

$\mathcal{F}f$ if $f \in L^2(\mathbb{R}^n) \cap L^1(\mathbb{R}^n)$

with $\|f\|_{L^2} = \left(\int_{\mathbb{R}^n} |f(x)|^2 dx \right)^{1/2}$

" $\int e^{-ix\cdot y} f(y)\,dy$ "

Remark

any $f \in D(\mathbb{R}^n) \subset L^2(\mathbb{R}^n) \cap L^1(\mathbb{R}^n)$

It would be natural to define

$(\mathcal{F}T)(f) = T(\underbrace{\mathcal{F}f}_{\notin D(\mathbb{R}^n)})$ for any $f \in D(\mathbb{R}^n)$

$\notin D(\mathbb{R}^n)$ most of the time

not possible

Def: Let $S(\mathbb{R}^n)$ be the subset of $C^\infty(\mathbb{R}^n)$ such that

$$\sup_{x \in \mathbb{R}^n} |\underbrace{x^\beta}_{x_1^{\beta_1} x_2^{\beta_2} \cdots x_n^{\beta_n}} (\partial^\alpha f)(x)| < \infty \quad \forall \alpha, \beta \in \mathbb{N}^n$$

and $f \in S(\mathbb{R}^n)$

It means that $\partial^\alpha f$ has to decrease faster than any polynomials

Ex: $x \longmapsto e^{-\|x\|} \in S(\mathbb{R}^n)$
$\notin D(\mathbb{R}^n)$

Clearly $D(\mathbb{R}^n) \subset S(\mathbb{R}^n) \subset C^\infty(\mathbb{R}^n)$

Properties:

1) $S(\mathbb{R}^n)$ is a vector space

2) $x^\beta \partial^\alpha f \in S(\mathbb{R}^n)$ whenever $f \in S(\mathbb{R}^n)$

3) $\mathcal{F}S(\mathbb{R}^n) = S(\mathbb{R}^n)$

$$(\mathcal{F}^{-1}f)(x) = \frac{1}{(2\pi)^{\frac{n}{2}}} \int e^{ix\cdot y} f(y)\,dy$$

Def (Convergence in $S(\mathbb{R}^n)$)

A sequence $\{f_i\}_{i\in\mathbb{N}} \subset S(\mathbb{R}^n)$ converges $\underset{\checkmark}{\text{in } S(\mathbb{R}^n)}$ to $f_\infty \in S(\mathbb{R}^n)$

if $\sup_{x\in\mathbb{R}^n} | x^\beta \partial^\alpha (f_i - f_\infty)(x) | \xrightarrow{i\to\infty} 0$

Def A tempered distribution is a distribution which is continuous on $S(\mathbb{R}^n)$, i.e

if $f_i \to f_\infty$ in $S(\mathbb{R}^n)$, then

$T(f_i) \xrightarrow{i\to\infty} T(f_\infty)$

We set $S'(\mathbb{R}^n)$ for the set of all tempered distribution

Clearly $S'(\mathbb{R}^n) \subset D'(\mathbb{R}^n)$, and for example

$T_{x\to e^{x^2}} \notin S'(\mathbb{R}^n)$

Indeed $T_{x\to e^{x^2}}(f) = \int e^{x^2} f(x)\,dx \overset{?}{<} \infty$

for any $f \in S(\mathbb{R}^n)$

not true for some $f \in S(\mathbb{R}^n)$

Ex: $f(x) = e^{-\|x\|}$

Good point: if $T \in S'(\mathbb{R}^n)$

$(\mathcal{F}T)(f) := T(\mathcal{F}f)$ for any $f \in S(\mathbb{R}^n)$