Scattering theory and topological quantities

1) Scattering theory

Early stage of quantum mechanics

- $\mathcal{H}$ a Hilbert space (complex vector space with scalar product)

For example : $\ell^2(\mathbb{Z}^d) = \{ a = (a_n)_{n \in \mathbb{Z}^d} \mid \sum_{n \in \mathbb{Z}^d} |a_n|^2 < \infty \}$

$$\text{with } \langle a, b \rangle = \sum_{n \in \mathbb{Z}^d} \bar{a}_n b_n$$

Or $L^2(\mathbb{R}^n) = \{ f : \mathbb{R}^n \to \mathbb{C} \mid \int |f(x)|^2 dx < \infty \}$

$$\text{with } \langle f, g \rangle = \int_{\mathbb{R}^n} \bar{f}(x)\, g(x)\, dx$$

The scalar product induces a norm : $\|f\| = \langle f, f \rangle^{1/2}$ (way to measure a distance

- $H = (D(H), H)$ a self-adjoint operator in $\mathcal{H}$.

$\Rightarrow \langle f, Hg \rangle = \langle Hf, g \rangle \quad \forall f, g \in D(H)$, but not enough. $\rightsquigarrow$ real spectrum

For example : $H_0 = -\Delta = -\sum_{j=1}^{n} \partial_{x_j}^2$ in $L^2(\mathbb{R}^n)$

$$H = -\Delta + V(X)$$

↑ multiplication operator

with $V : \mathbb{R}^n \to \mathbb{R}$ with bounded support.

H describes a quantum system, also called the Hamiltonian operator.

- <u>Schrödinger evolution</u> is defined by the equation (with $\hbar = 1$, $m = \frac{1}{2}$ ...)

$$\begin{cases} i\frac{\partial}{\partial t}\psi(t) = H\psi(t) \\ \psi(0) = \psi_0 \quad \text{initial condition} \end{cases}$$

Solution : $\psi(t) = e^{-itH}\psi_0$

$\{e^{-itH}\}_{t\in\mathbb{R}}$ is called the <u>evolution group</u>.

$\left(e^{-i(t+s)H} = e^{-itH}e^{-isH} \text{ and } e^{-i0H} = 1\right)$.

- <u>Natural question</u> How can one study $e^{-itH}\psi_0$ for large $|t|$?

<u>Answer</u> : by comparison with a simpler evolution group. More precisely, for a given $\psi_0 \in \mathcal{H}$, can one find $\psi_\pm \in \mathcal{H}$ and a simpler self-adjoint operator H_0 s.t.

$$\| e^{-itH}\psi_0 - e^{-itH_0}\psi_\pm \| \xrightarrow{t\to\pm\infty} 0$$

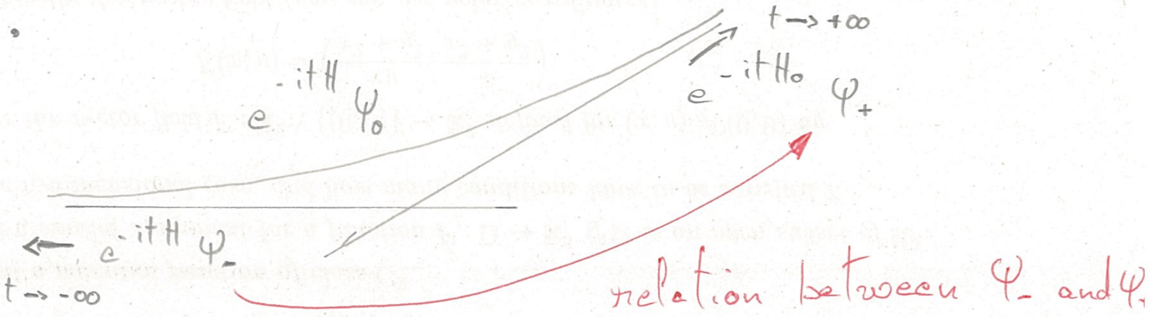
$$\Leftrightarrow \| \psi_0 - e^{itH}e^{-itH_0}\psi_\pm \| \xrightarrow{t\to\pm\infty} 0 \quad ?$$

Define $W_\pm \psi_\pm := \lim_{t\to\pm\infty} e^{itH}e^{-itH_0}\psi_\pm$

if these limits exist.

called wave operators.

Thus, if $\varphi_0 = W_\pm \varphi_\pm$ for some $\varphi_\pm \in \mathcal{H}$, then $e^{-itH}\varphi_0$ can be described asymptotically by $e^{-itH_0}\varphi_\pm$.



$$W_-\varphi_- = \varphi_0 = W_+\varphi_+ \quad (*)$$

$$\Rightarrow \underbrace{W_+^* W_-}_{:= S}\varphi_- = W_+^* W_+ \varphi_+ = \varphi_+$$

scattering operator ↑ if $W_+^* W_+ = 1$

$\rightsquigarrow$ $S\varphi_- = \varphi_+$ describes a scattering process.

In summary, our assumptions are:

H1) H, H_0 self-adj. op. in $\mathcal{H}$, H_0 absolutely continuous — no eigenvalues — meaning of simple

H2) $W_\pm = s\text{-}\lim_{t\to\pm\infty} e^{itH} e^{-itH_0}$ exist

H3) $\mathrm{Ran}\, W_+ = \mathrm{Ran}\, W_-$ (for $(*)$)
$\Leftrightarrow W_+\mathcal{H} = W_-\mathcal{H}$

Then, automatically,

i) $W_\pm^* W_\pm = 1$

ii) S is a unitary operator ($S^{-1} = S^*$)

iii) S commutes with H_0 ($[S, H_0] = 0$)

$\rightsquigarrow$ they can be diagonalized simultaneously

$$S \simeq \{S(\lambda)\}_{\lambda \in \sigma(H_0)}$$ ← generalization of the set of eigenvalues

scattering matrix at energy λ

: 2) Levinson's theorem (since 1949)

↳ relation of the form

regularization (for integrability)

$$\frac{1}{2\pi}\int_{\sigma(H_0)} \left\{ \mathrm{tr}\left\{ i\, S(\lambda)^* S'(\lambda) \right\} - c(\lambda) \right\} d\lambda$$

expectation value ?

$$= \#\{\text{eigenvalues of } H\} + \delta$$

correction (due to 0-energy effect).

Remarks :
- many authors for many models
- purely analytic
- regularization + correction depend on model
- beside H1 - H3 additional assumptions necessary

⇝ Topological version :

$$\mathrm{Wind}_r\,(q(W_-)) = \#\{\text{eigenvalues of } H\}$$

↑ regularized

very important role played by W_-

$$\equiv \#\,\sigma_p(H)$$

Remarks :
- both sides are stable under small perturbations ⇝ topological equality
- δ is on the other side of the equality
- regularization automatic
- W_- more important than S.

↑ "see" the bound states

does not see the bound states

Lemma If $\varphi \in D(H) \subset \mathcal{H}$ satisfies $H\varphi = \lambda\varphi$, then $\varphi \perp W_\pm \mathcal{H}$.

Proof $|\langle \varphi, W_\pm \varphi_\pm \rangle|$

$$= |\lim_{t\to\pm\infty} \langle \varphi, e^{itH} e^{-itH_0} \varphi_\pm \rangle|$$

$$= \lim_{t\to\infty} |\langle e^{-i\lambda t}\varphi, e^{-itH_0} \varphi_\pm \rangle|$$

$$= \lim_{t\to\pm\infty} |\langle \varphi, \underbrace{e^{-itH_0}}_{\text{it spreads}} \varphi_\pm \rangle| = 0$$

property of H_0 a.c.

Assumption $H4$: $\mathrm{Ran}(W_\pm) = \mathcal{H}_p(H)^\perp$

space generated by eigenvectors.

and $\dim \mathcal{H}_p(H) < \infty$

$\Leftrightarrow$ $\#\{\text{eigenvalues of } H\} < \infty$. often proved

temporary

3) Mathematical background

Main idea : equivalence classes of objects

$\hookrightarrow$ properties which are common for a family of object

Recall that $B(\mathcal{H}) = \{$ bounded linear operators on $\mathcal{H}\}$
and $K(\mathcal{H}) = \{$ compact operators on $\mathcal{H}\}$ limit of matrices

These are examples of C^*-algebras :
(vector space + algebra with a norm and an involution (the adjoint). Examples : $\mathbb{C}$, $B(\mathcal{H})$, $K(\mathcal{H})$

Let $\mathcal{E}$ be a C^*-algebra.

$K_0(\mathcal{E}) \simeq \{P \in \mathcal{E} \mid P = P^2 = P^*\} / \sim$ homotopy

$K_1(\mathcal{E}) \simeq \{U \in \mathcal{E} \mid U^* = U^{-1}\} / \sim$ homotopy

equivalence classes

K-groups for C^*-algebras

If $\mathcal{J} \subset \mathcal{E}$ are C^*-algebras and $\mathcal{J}$ is an ideal in $\mathcal{E}$ (i.e. $je \in \mathcal{J}$ $\forall j \in \mathcal{J}$ and $e \in \mathcal{E}$)

then $\mathcal{E}/\mathcal{J} = \{[e+j] \mid j \in \mathcal{J}\}$ is again a C^*-algebra

We write $q : \mathcal{E} \to \mathcal{E}/\mathcal{J}$ the quotient map

again, considering $\mathcal{E}/\mathcal{J}$ is considering equivalence classes

Thm Topological Levinson's theorem

Assume H1 - H4 and let $\mathcal{K}(\mathcal{H}) \subset \mathcal{E} \subset \mathcal{B}(\mathcal{H})$ s.t. 1) $\mathcal{E}$ is a C^*-algebra, 2) $W_- \in \mathcal{E}$ and 3) $\mathcal{E}/\mathcal{K}(\mathcal{H}) \simeq C(\mathbb{S}^1, \mathcal{K}(\mathcal{H}'))$

continuous functions on a circle with values in $\mathcal{K}(\mathcal{H}')$

Then $\mathrm{Wind}_{\tau}[q(W_-)] = \mathrm{Tr}(P_p) = \#\sigma_p(H)$.

Some explanation

because values in $\mathcal{K}(\mathcal{H}')$

Remark: Main ingredients are:

$W_- \rightsquigarrow \mathcal{E}$

good understanding

usually information not available yet.

4) Aplications

Validity checked on several models.

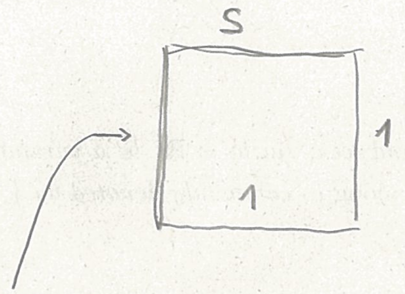
Example 1 : Potential scattering in $\mathbb{R}^3$

$\mathcal{H} = L^2(\mathbb{R}^3)$, $H_0 = -\Delta$, $H = -\Delta + V$

with $V : \mathbb{R}^3 \longrightarrow \mathbb{R}$ s.t. $|V(x)| \leq c\,(1+|x|)^{-7-\varepsilon}$.

Then $\mathcal{E} = \dots$, and

$$\mathcal{E}/\mathcal{K}(\mathcal{H}) \cong C\big(\square, \mathcal{K}(L^2(\mathbb{S}^2))\big) \cong C(\mathbb{S}, \mathcal{K}(L^2(\mathbb{S}^2)))$$



$$x \longmapsto 1 + \tfrac{1}{2}\left(1 + \tanh(x) - i\,\frac{1}{\cosh(x)}\right)(S(0) - 1) \equiv R$$

and $\mathrm{Wind}_n\big((R, S, 1, 1)\big) = \#\,\sigma_p(H)$

↑ related to 0-energy through $S(0)$.

Example 2 : Ahranov - Bohm operator

... Not today.

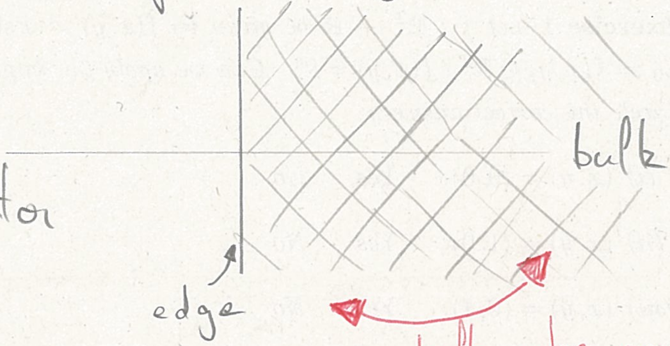
⇝ link with adiabatic process

5) Relation with surface states

for the time being

Consider H describing a perfect crystal located in a half space

Let H_0 be the free operator (no crystal)



bulk edge correspondence.

⚠ No scattering

Then for suitable $\varphi_\pm \in \mathcal{H}$,

$$W_\pm \varphi_\pm := s\text{-}\lim_{t \to \pm\infty} e^{itH} e^{-itH_0} \varphi_\pm$$

⚠ Before, $\operatorname{Ran}(W_\pm) = \mathcal{H}_p(H)^\perp$

Now, $\mathcal{H}_p(H) = \{0\}$ but $\operatorname{Ran}(W_\pm) = \mathcal{H}_{ss}(H)^\perp$

surface states

states which stay close to the boundary

$$\hookrightarrow \lim_{R \to \infty} \sup_{t \in \mathbb{R}} \| \chi_{\mathbb{R}^{n-1} \times [R,\infty)}(X)\, e^{-itH} \varphi \| = 0$$

<u>Expectation 1</u> : $\operatorname{Wind}_n(q(W_-)) = \operatorname{Tr}(P_{ss}(H))$

not the previous one, extension of them!

<u>Expectation 2</u> : If not a perfect crystal, then $\mathcal{H}_p(H) \neq \{0\}$, detect them in addition to $\mathcal{H}_{ss}(H)$ ...

2 different scales