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Let $\varphi : \mathbb{R} \to \mathbb{C}$ with $\check{\varphi} \in L^1(\mathbb{R})$

Then: $\varphi(A) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \check{\varphi}(t) e^{-itA} dt$

Indeed: $\langle f, \varphi(A) f \rangle = \int \varphi(\lambda) m_f(d\lambda)$ with $m_f(\cdot) = \langle f, E(\cdot) f \rangle$

$$= \int m_f(d\lambda) \frac{1}{\sqrt{2\pi}} \int e^{-it\lambda} \check{\varphi}(t) dt$$

$$= \int dt\, \check{\varphi}(t) \underbrace{\int_{\mathbb{R}} m_f(d\lambda) e^{-it\lambda}}_{\langle f, e^{-itA} f \rangle}$$

$$= \langle f, \int dt\, \check{\varphi}(t) e^{-itA} f \rangle$$

Use polarization identity: $\varphi(A) = \int dt\, \check{\varphi}(t) e^{-itA}$

7.3 Spectral parts of a self-adj op

Let $(D(A), A)$ be self-adjoint, with spectral measure $E(\cdot)$

One has $\operatorname{Ran}(E(\{\mu\})) := \{ f \in \mathcal{H} \mid E(\{\mu\}) f = f \}$

Also one has: $\operatorname{Ran}(E(\{\mu\})) = \{ f \in D(A) \mid Af = \mu f \}$

Indeed one has: $\| (A-\mu) f \|^2 = \int |\lambda - \mu|^2 \langle f, E(d\lambda) f \rangle$

Def: $\mathcal{H}_p(A) := \bigoplus_{\mu} \operatorname{Ran}(E\{\mu\})$ = subspace spanned by all eigenvectors of A = $\{ f \in \mathcal{H} \mid m_f$ has supp on points only$\}$

Note that the set of eigenvalue is countable

$\mathcal{H}_{ac}(A) = \{ f \in \mathcal{H} \mid m_f$ is absolutely continuous w.r.t $\mathcal{L} \}$

$\mathcal{H}_{sc}(A) = \{ f \in \mathcal{H} \mid m_f$ is singular continuous w.r.t $\mathcal{L} \}$

Thm: 1) $\mathcal{H} = \mathcal{H}_p(A) \oplus \mathcal{H}_{ac}(A) \oplus \mathcal{H}_{sc}(A)$

then $A = A_p \oplus A_{ac} \oplus A_{sc}$. Each part is self-adjoint in the respective space

2) $\forall \varphi \in C_b(\mathbb{R})$

$$\varphi(A) = \varphi(A_p) \oplus \varphi(A_{ac}) \oplus \varphi(A_{sc})$$

and $\sigma(A) = \sigma(A_p) \cup \sigma(A_{ac}) \cup \sigma(A_{sc})$

and $\sigma(A_p) = \overline{\sigma_p(A)}$ ← close

We also set $\mathcal{H}_c(A) := \mathcal{H}_{sc}(A) \oplus \mathcal{H}_{ac}(A) \to \sigma_c(A) = \sigma_{ac}(A) \cup \sigma_{sc}(A)$

$\sigma_d(A) = \{\lambda \in \sigma_p(A) \mid \exists \varepsilon > 0 \text{ with } \sigma(A) \cap (\lambda - \varepsilon, \lambda + \varepsilon) = \{\lambda\}\}$

discrete

$\sigma_{ess} = \sigma(A) \setminus \sigma_d(A)$

essential

Thm: let A be self-adjoint op and let B be A_compact then $\sigma_{ess}(A+B) = \sigma_{ess}(A)$

Weyl criterion let A be a self-adjoint op

a) $\lambda \in \sigma(A)$ iff $\exists \{f_n\} \subset D(A)$ with $\|f_n\| = 1$ and $\text{s-}\lim_{n\to\infty} (A-\lambda) f_n = 0$

b) $\lambda \in \sigma_{ess}(A)$ iff $\exists \{f_n\} \subset D(A)$ with $\|f_n\| = 1$, $\text{w-}\lim_{n\to\infty} f_n = 0$ and $\text{s-}\lim_{n\to\infty} (A-\lambda) f_n = 0$

7.4 Spectral representation

let A be a self-adj op in $\mathcal{H}$. Then there exists $U: \mathcal{H} \to L^2(\sigma(A), H(\lambda), d\mu) =: H$

s.t. $UAU^* = X$ (mult by λ) U is unitary.

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<u>Example</u>: 1, Consider $\mathcal{H} = L^2(\mathbb{R}^3)$, $A = X^2 = X_1^2 + X_2^2 + X_3^2$

$\sigma(A) = [0, \infty) = \mathbb{R}_+$.

$\mathcal{H} = L^2(\mathbb{R}_+, L^2(S^2))$

$$\left(\begin{array}{l} [\mathcal{U}f](r, \cdot) \in L^2(S^2) \\ \int |f(x)|^2 dx = \int_0^\infty dr \int_{S^2} |f(r,\omega)|^2 r^2 d\omega \\ \text{we miss "}\sqrt{\ }\text{" such to meet } \mathcal{U} X^2 \mathcal{U}^* = X \end{array} \right)$$

In summary, every self-adjoint op is unitary equivalent to a multiplication op by X in a suitable Hilbert space

$\mathcal{H} = L^2(\sigma(A), \mathcal{H}_\lambda, \mu(d\lambda))$ is called a direct

$= \int_{\sigma(A)}^{\oplus} \mathcal{H}_\lambda \, \mu(d\lambda)$ integral of Hilbert space