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7.2 Spectral measure

Def: A spectral family (or a resolution of the identity) is a family $\{E_\lambda\}_{\lambda \in \mathbb{R}} \subset \mathcal{P}(\mathcal{H})$ s.t.:

($\mathcal{P}(\mathcal{H})$: set of all ortho projection in $\mathcal{H}$)

1, $E_\lambda E_\mu = E_{\min\{\lambda,\mu\}}$ $\Rightarrow$ family is commuting

2, $E_\lambda = E_{\lambda+0}$ $\Rightarrow \lim\limits_{\varepsilon \to 0^+} \| E_{\lambda+\varepsilon} f - E_\lambda f \| = 0$.

3, $\operatorname*{s-lim}\limits_{\lambda \to -\infty} E_\lambda = \mathbb{0}$, $\operatorname*{s-lim}\limits_{\lambda\to\infty} E_\infty = \mathbb{1}$

$\lim\limits_{\lambda \to -\infty} \| E_\lambda f \| = 0 \quad \forall f$ $\qquad \lim\limits_{\lambda \to \infty} \| E_\lambda f - f \| = 0$

Def: The support of the family

$$\operatorname{Supp}\{E_\lambda\} = \{\mu \in \mathbb{R} \mid E_{\mu+\varepsilon} - E_{\mu-\varepsilon} \neq 0 \ \forall \varepsilon > 0\}$$

We set $E((a,b]) = E_b - E_a$

$\Rightarrow$ We define a function from $\mathcal{A}_B$ (the set of all Borel set) to $\mathcal{P}(\mathcal{H})$

This function is called the <u>spectral measure</u> associated with $\{E_\lambda\}$

Properties:

$E(\emptyset) = \mathbb{0}$

$E(\mathbb{R}) = \mathbb{1}$

$E(\cup I_k) = \sum\limits_{k=1}^{N} E(I_k)$, $I_i \cap I_j = \emptyset$

For any ~~fix~~ $f \in \mathcal{H}$

$$F_f(\lambda) = \langle f, E_\lambda f \rangle = \langle E_\lambda f, E_\lambda f \rangle = \| E_\lambda f \|^2$$

all the properties of F of last week.

$m_f(v) := \langle f, E(v) f\rangle = \|E(v) f\|^2 \quad \forall v \in \mathcal{A}_B$

m_f is a bounded Borel measure.

Consider $\int_a^b \varphi(\lambda) E(d\lambda) = \lim \sum_{j=0}^{N} \varphi(y_j) E([x_j, x_{j+1}])$

finner partition of $[a,b]$

for $\varphi \in C([a,b])$

<u>Thm</u>: This limit exists for $\varphi \in C[a,b]$ (in the strong sense) and we denote the limit by $\int_a^b \varphi(\lambda) E(d\lambda) \in \mathcal{B}(\mathcal{H})$

<u>Proposition</u>: let $\{E_\lambda\}$ be a spectral (or $\Leftrightarrow$ $E(\cdot)$ a spectral measure)

1) $\|\int \varphi(\lambda) E(d\lambda)\| \leq \sup_{x \in [a,b] \cap \operatorname{supp}\{E_\lambda\}} |\varphi(x)|$.

2) $\left(\int \varphi(\lambda) E(d\lambda)\right)^* = \int \overline{\varphi(\lambda)} E(d\lambda)$ $\Rightarrow$ φ is real, $\int \varphi(\lambda) E d\lambda$ is selfadjoint

3) $\|\int \varphi(\lambda) E(d\lambda) f\| = \int \varphi(\lambda) m_f(d\lambda)$.

4) if $\varphi \in C([a,b])$

$$\left(\int_a^b \varphi(x) E(d\lambda)\right)\left(\int_a^b \psi(x) E(d\lambda)\right) = \int_a^b \varphi(\lambda)\psi(\lambda) E(d\lambda).$$

<u>Rem</u>: How can one extend this definition $\int_a^b \varphi(\lambda) E(d\lambda)$?

1) if $\varphi \in C_b(\mathbb{R}) \Rightarrow$ OK by 3)

bounded

2) if φ is not bounded but continuous, but m_f has compact support, OK.

3) if φ is not bounded but continuous and if $\int_{\mathbb{R}} |\varphi(\lambda)| m_f(d\lambda) < \infty$, then $\int \varphi(\lambda) E(d\lambda) f$ is OK for some f.

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More precisely, we set:

$$D_\varphi = \{ f \in \mathcal{H} \mid \int_{\mathbb{R}} |\varphi(\lambda)| \, m_f(d\lambda) < \infty \} \subset \mathcal{H}$$

and we set $(D_\varphi, \int \varphi(\lambda) E(d\lambda))$ is a linear operator with domain D_φ.

<u>Rem</u>: D_φ is dense in $\mathcal{H}$ $\left(\Leftarrow \operatorname{s-lim}_{\lambda \to \infty} E_\lambda f = f \ \forall f \in \mathcal{H} \right)$

<u>Def</u>: For any $\{E_\lambda\}_{\lambda \in \mathbb{R}}$ (spectral family) and for $\varphi = id$, the operator $(D_{id}, \int_{\mathbb{R}} \lambda E(d\lambda))$ is called the self-adjoint op. associated with $\{E_\lambda\}$.

<u>Spectral theorem</u>

For any self-adjoint op. $(D(A), A)$ there exists a unique spectral family $\{E_\lambda\}$ (or spectral measure) s.t
$D(A) = D_{id}$ and $A = \int \lambda E(d\lambda)$

<u>Rem</u>: starting from $(D(A), A)$, we set $\int \varphi(\lambda) E(d\lambda) =: \varphi(A)$

In particular, in $\mathcal{H} = L^2(\mathbb{R})$, and $(D(X), X)$, then
$\varphi(X) = \varphi(X)$
op of multiplication by

With this notation:

$\varphi(A)$ is self-adjoint iff φ is real-value
$\varphi(A)$ is bounded "iff" $\varphi \in C_b(\mathbb{R})$
iff $\varphi \in L^\infty(\mathbb{R})$
$\varphi(A)$ is unitary iff $|\varphi(x)| = 1 \ \forall x$

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For any $t \in \mathbb{R}$, consider $x \mapsto e^{itx}$

We can define e^{itA}

$$e^{isA} e^{itA} = \left(e^{is\cdot} e^{it\cdot}\right)(A) = e^{i(s+t)A}$$

$\Rightarrow \{e^{itA}\}_{t \in \mathbb{R}}$ is a unitary group which is "strongly continuous".

<u>Stone's theorem</u>

There is a bijective relation between self-adj op. and strongly continuous unitary group $\{U_t\}$ $\left(\lim_{t \to 0} U_t f = f\right)$

One has $D(A) = \left\{ f \in \mathcal{H} \mid \exists\, s\text{-}\lim_{t \to 0} \frac{(U_t - 1)f}{t} \right\}$

$$Af = \lim_{t \to 0} -\frac{i}{t} (U_t - 1) f$$