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Remember:

If $\varphi: \mathbb{R}^n \to \mathbb{R}$, measurable then the op $(\underbrace{\varphi(X)}, D(\varphi(X)))$ is self adjoint. (multiplication op in $L^2(\mathbb{R}^n)$)

$$D(\varphi(x)) = \{ f \in \mathcal{H} \mid \int |\varphi(x) f(x)|^2 dx < \infty \}$$

Example: $(X_j, D(X_j))$ position op for variable j $(x_1, \dots x_n)$
position operator.

If $\alpha \in \mathbb{N}^n$, then $X^\alpha = x_1^{\alpha_1} x_2^{\alpha_2} \dots x_n^{\alpha_n}$ also defines a self-adjoint op in $L^2(\mathbb{R}^n)$.

Lemma: If $\varphi: \mathbb{R}^n \to \mathbb{R}$ measurable then $\sigma(\varphi(x)) = \varphi(\mathbb{R}^n)$

$$[\mathcal{F}f](k) = \frac{1}{\sqrt{(2\pi)^n}} \int_{\mathbb{R}^n} e^{-ixk} f(x) dx.$$

$$x_j \mathcal{F}f = \mathcal{F}(-i\partial_j) f$$

$\Rightarrow D_j = P_j = \mathcal{F}^* x_j \mathcal{F}$

$\to$ Similarly, $P^\alpha = P_1^{\alpha_1} \dots P_n^{\alpha_n} = \mathcal{F}^* X^\alpha \mathcal{F}$.

with domain $D(P^\alpha) = \mathcal{F}^* D(X^\alpha) = \{ f \in \mathcal{H} \mid f = \mathcal{F}^* g$ with $g \in D(X^\alpha) \}$; $D(X^\alpha) = \{ f \in L^2(\mathbb{R}^n) \mid \int |X^\alpha f(x)|^2 dx < \infty \}$.

Moregenerally, for $\varphi: \mathbb{R}^n \to \mathbb{R}$, we set:

$\varphi(P) := \mathcal{F}^* \varphi(x) \mathcal{F}$, called a convolution op.

Indeed
$$\begin{aligned} \varphi(P) f &= \mathcal{F}^* \varphi(x) \mathcal{F} f \\ &= \mathcal{F}^* (\varphi \hat{f}) \\ &= \check{\varphi} * f, \text{ with } (\check{\varphi} * f)(x) = \frac{1}{(2\pi)^{n/2}} \int \check{\varphi}(y) f(x-y) dy \end{aligned}$$

Lemma: $\sigma(\varphi(P)) = \varphi(\mathbb{R}^n)$

Proof: $\varphi(P) = \mathcal{F}^* \varphi(x) \mathcal{F}$

unitary transformation $\Rightarrow$ they don't change the spectrum

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7, Spectral theory of self-adjoint operator.

7.1 Stieltjes measure

Consider: $F: \mathbb{R} \to \mathbb{R}$ s.t:

1, $F(\lambda) \geq F(\mu)$ if $\lambda \geq \mu$

2, $F(\lambda) = F(\lambda + 0) = \lim_{\varepsilon \to 0^+} F(\lambda + \varepsilon)$ (cts from the right)

3, $F(-\infty) = 0$, $F(\infty) = \rho < \infty$

Observe that $\lim_{\varepsilon \to 0^+} F(\lambda - \varepsilon)$ exists but can be different from $F(\lambda)$.

Remark: The number of jumps is countable. We then define a measure m_F on $\mathbb{R}$:

$$m_F((a,b]) := F(b) - F(a)$$

Then we extend this measure to all Borel sets of $\mathbb{R}$ (countable union or intersection of open intervals and their complement) by $m_F(V) = \inf \sum_k m_F(\underset{\text{interval } (a_k, b_k]}{I_k})$

For any V Borel set of $\mathbb{R}$

One gets:

$$m_F((a,b]) = F(b) - F(a)$$
$$m_F([a,b]) = F(b) - F(a - 0^+)$$
$$m_F((a,b)) = F(b - 0^+) - F(a)$$
$$m_F(\{a\}) = F(a) - F(a - 0^+)$$

Thus, from F satisfying 1) - 3) we obtain a Borel measure on $\mathbb{R}$ which is bounded, because $m_F(\mathbb{R}) = \rho < \infty$

Reciprocally, if m is a bounded Borel measure, we can define $F(\lambda) = m((-\infty, \lambda])$

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3 types of measures: m a bounded Borel measure on $\mathbb{R}$.

1) m is pure point if for any Borel set V, there exists $\{X_j\}_{j \in \mathbb{N}} \in V$ s.t $m(V) = \sum_j m(\{X_j\})$.

2) m is absolutely continuous (w.r.t Lebesgue measure) if $\exists f : \mathbb{R} \to \mathbb{R}^+$, $\mathcal{L}$.measurable and $m(V) = \int_V f(x)\,dx$

3) m is singular continuous (w.r.t $\mathcal{L}$.m) if $m(\{X\}) = 0 \ \forall x \in \mathbb{R}$. and $\exists$ Borel V with $\int_V dx = 0$ ($\mathcal{L}$.m of V is 0) and s.t $m(\mathbb{R} \setminus V) = 0$

$\uparrow$ Cantor set

Thm. Any bounded Borel measure m satisfies:

$m = m_{pp} + m_{ac} + m_{sc}$, unique decomposition