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Proposition: If $(D(A), A)$ is self-adjoint, then
1, $\sigma(A) \subset \mathbb{R}$
2, if $\lambda, \mu \in \sigma_p(A)$, $\lambda \neq \mu$ and if $Af = \lambda f$ and $Ag = \mu g$ then $f \perp g$.

Proof: 2,
$$\lambda \langle f, g\rangle = \langle \lambda f, g\rangle = \langle Af, g\rangle = \langle f, Ag\rangle = \langle f, \mu g\rangle = \mu \langle f, g\rangle$$
$$\Leftrightarrow \langle f, g\rangle = 0 \Leftrightarrow f \perp g.$$

1, Set $z = \lambda + i\varepsilon$ for $\lambda, \varepsilon \in \mathbb{R}$, $\varepsilon \neq 0$. and let us show that $z \in \rho(A)$

Consider: $f \in D(A)$.
$$\|(A-z)f\|^2 = \|(A-\lambda)f + i\varepsilon f\|^2 \underset{\text{because } A \text{ is self-adjoint}}{=} \|(A-\lambda)f\|^2 + \varepsilon^2\|f\|^2 \geq \varepsilon^2\|f\|^2$$
$$\Rightarrow \operatorname{Ker}(A-z) = \{0\}.$$

Consider: $g \in \operatorname{Ran}(A-z) \Rightarrow g = (A-z)(A-z)^{-1}g$.
$$\|g\|^2 = \|(A-z)(A-z)^{-1}g\|^2 \geq \varepsilon^2\|(A-z)^{-1}g\|^2$$
$$\Rightarrow \|(A-z)^{-1}g\|^2 \leq \frac{1}{\varepsilon^2}\|g\|^2$$

In addition $\operatorname{Ran}(A-z)$ is dense.
Indeed: $\operatorname{Ran}(A-z)^{\perp} = \operatorname{Ker}(A-z)^* = \operatorname{Ker}(A-\bar{z}) \overset{\text{because all eigenvalue are real (see below)}}{=} \{0\}$

$\Rightarrow (A-z)^{-1}: \operatorname{Ran}(A-z) \to D(A-z)$ is densely defined and bounded

$\Rightarrow (A-z)^{-1}$ extends continuously to a bounded operator
$\Rightarrow (A-z)^{-1} \in \mathcal{B}(\mathcal{H})$. $\Rightarrow z \in \rho(A)$.

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3) $\sigma_p(A) \subset \mathbb{R}$ (eigenvalues are real)

Let $z \in \mathbb{C}$ s.t $Af = zf$, then

$$z\|f\|^2 = z\langle f, f\rangle = \langle f, zf\rangle = \langle f, Af\rangle = \langle Af, f\rangle$$
$$= \langle zf, f\rangle = \bar{z}\langle f, f\rangle$$
$$\Rightarrow z = \bar{z} \Rightarrow z \in \mathbb{R}.$$

6) Perturbation theory

Let $(D(A), A)$ be self-adjoint and $B \in \mathcal{B}(\mathcal{H})$ s.t $B = B^*$

Lemma: $(D(A), A+B)$ is self-adjoint

Def: An operator $(D(B), B)$ is A-bonded if $D(A) \subset D(B)$ and $\exists\, \alpha, \beta \in \mathbb{R}^+$ s.t

$$\|Bf\| \leqslant \alpha\|Af\| + \beta\|f\| \quad \forall f \in D(A)$$

The infimum of α is called the A-bonded of B.

Remark. If $B \in \mathcal{B}(\mathcal{H})$, then $\alpha = 0$ but it is also possible to have A-bonded equal to 0 with $B \notin \mathcal{B}(\mathcal{H})$

Thm: (Kato-Rellich)

Let $(D(A), A)$ be self-adjoint and $(D(B), B)$ be A-bonded with A-bonded < 1 and symmetric. Then

1) $(D(A), A+B)$ is self-adjoint
2) B is also $(A+B)$-bonded
3) If A is semi-bounded, then $A+B$ is semi-bounded

Def: A self-adjoint operator A is positive $(A \geqslant 0)$ if $\langle f, Af\rangle \geqslant 0 \quad \forall f \in D(A)$

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.) A is lower semi-bounded if $\exists\, c \in \mathbb{R}$ s.t $A + c \geq 0$

.) A is upper semi-bounded if $A - c \leq 0$ for some $c \geq 0$

These definition are equivalent to:

. $\sigma(A) \subset [0, \infty)$

. $\sigma(A) \subset [c, \infty)$ for some $c \in \mathbb{R}$.

. $\sigma(A) \subset (-\infty, c]$ for some $c \in \mathbb{R}$

2nd resolvent equation: for $z \in \rho(A) \cap \rho(A+B)$

then: $(A-z)^{-1} - (A+B-z)^{-1} = (A-z)^{-1} B (A+B-z)^{-1}$

$= (A+B-z)^{-1} B (A-z)^{-1}$.

formal Proof: $(A-z)^{-1} B (A+B-z)^{-1}$

$= (A-z)^{-1} (z - A + B + A - z) (A+B-z)^{-1}$

$= (A-z)^{-1} (A+B-z)(A+B-z)^{-1} - (A-z)^{-1}(A-z)(A+B-z)^{-1}$

$= (A-z)^{-1} - (A+B-z)^{-1}$

Examples,

1, $L^2(\mathbb{R})$, the operator $(D(X), X)$

with $D(X) = \{ f \in L^2(\mathbb{R}) \mid \int_{\mathbb{R}} x^2 |f(x)|^2 dx < \infty$

$[Xf](x) = x f(x)$

$\mathrm{Ker}(X) = \{0\}$

$\Rightarrow (X-z)^{-1} : \mathrm{Ran}(X-z) \longrightarrow D(X-z)$

For any $z \in \mathbb{C}$ with $\mathrm{Im}(z) \neq 0$

$x \longmapsto (x-z)^{-1} = (x - \lambda - i\varepsilon)^{-1} = \dfrac{x - \lambda + i\varepsilon}{(x-\lambda)^2 + \varepsilon^2}$

is a bounded function $\Rightarrow$ the operator $(X-z)^{-1}$ is bounded

2, Let $\mathcal{H} = L^2(\mathbb{R}^d)$

$\varphi : \mathbb{R}^d \to \mathbb{C}$, Lebesgue measurable.

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We set $\Pi_\varphi = \varphi(X)$ for the operator $[\Pi_\varphi f](x) = \varphi(x) f(x)$, called a multiplication operator
and $D(\varphi(X)) = \{ f \in \mathcal{H} \mid \int_{\mathbb{R}^d} |\varphi(x) f(x)|^2 dx < \infty \}$.

Lemma: $\varphi(X) \in B(\mathcal{H}) \Leftrightarrow \varphi \in L^\infty(\mathbb{R}^d)$

Remark. Assume $\varphi \in L^\infty(\mathbb{R}^d)$

- $\varphi(X)$ is self-adjoint $\Leftrightarrow$ φ is a real function
- $\varphi(X)$ is projection $\Leftrightarrow$ $\varphi(x) \in \{0, 1\}$.
- $\varphi(x)$ is a unitary operator $\Leftrightarrow$ $|\varphi(x)| = 1$
- $\varphi(x)$ is a partial isometry $\Leftrightarrow$ $|\varphi(x)| \in \{0; 1\}$
- $\varphi(x)$ is isometry $\Leftrightarrow$ φ is unitary

Lemma: Let $\varphi : \mathbb{R}^d \to \mathbb{R}$, Lebesgue measurable, then $(D(\varphi(X)), \varphi(X))$ is self-adjoint.