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Distribution theory

1, Test functions and distributions.

<u>Aim</u>: Give a meaning to "$\int_{\mathbb{R}} f(x)\,\delta_0(x)\,dx = f(0)$"

and explain who is δ_0

<u>Answer</u>: δ_0 is a linear continuous functional on the set of smooth functions with compact support.

<u>Reminder</u>:

Consider $f : \mathbb{R}^n \to \mathbb{C}$ (or $\mathbb{R}$)

Then $\partial_j f(x) \equiv (D_j f)(x) \equiv (\nabla_j f)(x) := \lim_{\varepsilon \to 0} \frac{f(x+\varepsilon E_j) - f(x)}{\varepsilon}$

for any $j \in \{1; \dots, n\}$

, with $E_j = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{pmatrix} \leftarrow$ position j

and $\partial_j f : \mathbb{R}^n \to \mathbb{C}$ is the j^{th} partial derivative of f

Then for any $m \in \mathbb{N}$ one can define:

$$\partial_j^m f = \underbrace{\partial_j \partial_j \dots \partial_j}_{m \text{ times}} f$$

Also for $\alpha \in \mathbb{N}^n$, one can define:

$$\partial^\alpha f := \partial_1^{\alpha_1} \partial_2^{\alpha_2} \dots \partial_n^{\alpha_n} f \quad (\alpha \text{ is called multi-index})$$

<u>Def</u>: A function $f : \mathbb{R}^n \to \mathbb{C}$ is smooth (or C^∞ or belongs to $C^\infty(\mathbb{R}^n)$) if $\partial^\alpha f$ exists and it is continuous for any $\alpha \in \mathbb{N}^n$

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<u>Remark</u> : For $f \in C^\infty(\mathbb{R}^n)$, $\partial_j \partial_k f = \partial_k \partial_j f$ for any $j, k \in \{1, \dots, n\}$

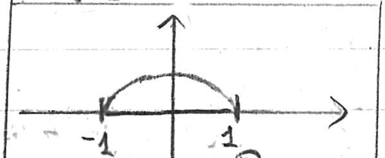
<u>Def</u> : <u>For any $f : \mathbb{R}^n \to \mathbb{C}$</u>, the support of f is $\overline{\{X \in \mathbb{R}^n \mid f(x) \neq 0\}}$ (close set). It is denoted by "$\mathrm{supp}(f)$"

<u>Def</u> : $f \in C^\infty(\mathbb{R}^n)$ is a <u>test function</u> if its support is compact ($\Leftrightarrow$ bounded)
The set of all test function is denoted by $D(\mathbb{R}^n)$

<u>Example</u> : For $X \in \mathbb{R}^n$, i.e $X = (x_1, x_2, \dots, x_n)$
one sets $\|x\| = \sqrt{x_1^2 + x_2^2 + \dots + x_n^2}$ (Euclidean norm on $\mathbb{R}^n$)
and defines $f : \mathbb{R}^n \to \mathbb{R}$

$$f(x) := \begin{cases} \exp\left(-\dfrac{1}{1 - \|x\|}\right) & \text{if } \|X\| < 1 \\ 0 & \text{if } \|X\| \geq 1 \end{cases}$$

$\mathrm{supp}(f) = \overline{B(0,1)}$



Picture of f in $\mathbb{R}$

<u>Rem</u> : $D(\mathbb{R}^n)$ is a vector space, and an algbra ($\Leftrightarrow$ stable for multiplication). Also $D(\mathbb{R}^n)$ is an ideal in $C^\infty(\mathbb{R}^n)$ if $f \in D(\mathbb{R}^n)$ and $g \in C^\infty(\mathbb{R}^n)$ then $fg \in D(\mathbb{R}^n)$

<u>Def</u> (Convergence in $D(\mathbb{R}^n)$)
A sequence $\{f_i\}_{i \in \mathbb{N}} \subset D(\mathbb{R}^n)$ converges to $f_\infty \in D(\mathbb{R}^n)$ if :

1) $\forall \alpha \in \mathbb{N}^n$, $\sup\limits_{X \in \mathbb{R}^n} |\partial^\alpha f_i(X) - \partial^\alpha f_\infty(X)| \xrightarrow{i \to \infty} 0$

$$\equiv \|\partial^\alpha f_i - \partial^\alpha f_\infty\|_\infty \xrightarrow{i \to \infty} 0$$

2) There exists $R > 0$ s.t $\mathrm{supp}(f_i) \subset B(0, R)$

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Def. A distribution T on $\mathbb{R}^n$ is a continuous linear functional on $D(\mathbb{R}^n)$

i.e $T : D(\mathbb{R}^n) \longrightarrow \mathbb{C}$ ($\to$ function on $D(\mathbb{R}^n)$)

s.t $T(f_1 + \alpha f_2) = T(f_1) + \alpha T(f_2)$ ($\to$ linear)

for any $f_1, f_2 \in D(\mathbb{R}^n)$, $\alpha \in \mathbb{C}$

and if $f_i \xrightarrow{i\to\infty} f_\infty$ in $D(\mathbb{R}^n)$ than $T(f_i) \xrightarrow{i\to\infty} T(f_\infty)$ in $\mathbb{C}$

($\to$ continuous)

The set of all distribution is denoted by $D'(\mathbb{R}^n)$, it is a vector space.

One often writes $\langle T, f \rangle$ for $T(f)$

⚠ it is linear in both argument.

Example.

1, Consider $h \in L^1_{loc}(\mathbb{R}^n) = \{h : \mathbb{R}^n \to \mathbb{C} \,/\, \int_{B(Y,1)} |h(x)|dx < \infty$ for any $Y \in \mathbb{R}^n\}$
and set T_h for the distribution defined by $T_h = \int h(x) f(x) dx$ ($< \infty$)

2, For any $Y \in \mathbb{R}^n$, we set $\delta_Y(f) := f(Y)$

i.e $\delta_Y : D(\mathbb{R}^n) \longrightarrow \mathbb{C}$ is a distribution

In particular $\delta_0(f) = f(0)$, we often writes $\delta_0 = \int f(x)\delta_0(x)dx$

3, For any $\alpha \in \mathbb{N}^n$, we set

$$T_Y^\alpha(f) := (\partial^\alpha f)(Y)$$

T_Y^α is a distribution, for any $\alpha \in \mathbb{N}^n$, $Y \in \mathbb{R}^n$

Proposition $T \in D'(\mathbb{R}^n)$ iff : $T : D(\mathbb{R}^n) \to \mathbb{C}$ is linear and if for any $Y \in \mathbb{R}^n$, $R > 0$ there exists $C > 0$ and $m \in \mathbb{N}$ s.t : $|T(f)| \leq C \sum_{|\alpha| \leq m} \|\partial^\alpha f\|_\infty$ for all $f \in D(\mathbb{R}^n)$,

$\operatorname{supp} f \subset \overline{B(Y, R)}$ with $|\alpha| = \alpha_1 + \alpha_2 + \dots + \alpha_n$

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2, Derivatives of distribution

Recall that $\int_a^b f'(x)g(x)dx = f(x)g(x)\Big|_a^b - \int_a^b f(x)g'(x)dx$

If $a = -\infty$, $b = \infty$ and $f, g \in D(\mathbb{R})$ then:

$$\int_{-\infty}^{\infty} f'(x)\, g(x)\, dx = -\int_{-\infty}^{\infty} f(x)\, g'(x)\, dx$$

Similarly on $\mathbb{R}^n$, if $f, g \in D(\mathbb{R}^n)$:

$$\int_{\mathbb{R}^n} (\partial_j f)(x) g(x)\, dx = -\int_{\mathbb{R}^n} f(x) (\partial_j g)(x)\, dx$$

and $\int_{\mathbb{R}^n} (\partial^\alpha f)(x) g(x)\, dx = (-1)^{|\alpha|} \int_{\mathbb{R}^n} f(x) (\partial^\alpha g)(x)\, dx$

Def: For any $T \in D'(\mathbb{R}^n)$, and any $f \in D(\mathbb{R}^n)$ and $\alpha \in \mathbb{N}^n$, one sets:

$$(\partial^\alpha T)(f) := (-1)^{|\alpha|}\, T(\partial^\alpha f)$$

One can write: $\langle \partial^\alpha T, f \rangle = \langle T, (-1)^{|\alpha|} \partial^\alpha f \rangle$

Lemma: $\partial^\alpha T \in D'(\mathbb{R}^n)$

Lemma: If $h \in C^\infty(\mathbb{R}^n)$, then $\partial^\alpha T_h = T_{\partial^\alpha h}$

Example: Let $H : \mathbb{R} \to \mathbb{R}$, $f \in D(\mathbb{R}^n)$

$$H(x) = \begin{cases} 0 & \text{if } x < 0 \\ 1 & \text{if } x \geq 0 \end{cases}$$

Clearly $H \in L^1_{loc}(\mathbb{R})$, $T_H \in D'(\mathbb{R})$

$$\begin{aligned} \partial T_H(f) &= -T_H(f') \\ &= -\int_{-\infty}^{\infty} H(x) f'(x)\, dx \\ &= -\int_0^{\infty} f'(x)\, dx = -f(x)\Big|_{x=0}^{x=\infty} = f(0) = \delta_0(f) \end{aligned}$$