Chapter 1

Geometric setting

In this Chapter we recall some basic notions on points or vectors in $\mathbb{R}^n$. The norm of a vector and the scalar product between two vectors are also introduced.

1.1 The Euclidean space $\mathbb{R}^n$

We set $\mathbb{N} := \{1, 2, 3, \dots\}$ for the set of *natural numbers*, also called *positive integers*, and let $\mathbb{R}$ be the set of all real numbers.

Definition 1. *One sets*

$$\mathbb{R}^n = \big\{(x_1, x_2, \dots, x_n) \mid x_j \in \mathbb{R} \text{ for all } j \in \{1, 2, \dots, n\}\big\}^1.$$

Alternatively, an element of $\mathbb{R}^n$*, also called* a n-tuple *or* a vector*, is a collection of n numbers* $(x_1, x_2, \dots, x_n)$ *with* $x_j \in \mathbb{R}$ *for any* $j \in \{1, 2, \dots, n\}$*. The number n is called* the dimension of $\mathbb{R}^n$.

In the sequel, we shall often write $X \in \mathbb{R}^n$ for the vector $X = (x_1, x_2, \dots, x_n)$. With this notation, the values $x_1, x_2, \dots, x_n$ are called *the components* or *the coordinates* of X. Note that one often writes (x, y) for elements of $\mathbb{R}^2$ and (x, y, z) for elements of $\mathbb{R}^3$. However this notation is not really convenient in higher dimensions.

The set $\mathbb{R}^n$ can be endowed with two operations, *the addition* and *the multiplication by a scalar.*

Definition 2. *For any* $X, Y \in \mathbb{R}^n$ *with* $X = (x_1, x_2, \dots, x_n)$ *and* $Y = (y_1, y_2, \dots, y_n)$ *and for any* $\lambda \in \mathbb{R}$ *one defines the addition of* X *and* Y *by*

$$X + Y := (x_1 + y_1, x_2 + y_2, \dots, x_n + y_n) \in \mathbb{R}^n$$

and the multiplication of X *by the scalar* λ *by*

$$\lambda X := (\lambda x_1, \lambda x_2, \dots, \lambda x_n) \in \mathbb{R}^n.$$

[1] The vertical line $\mid$ has to be read "such that".

Examples 3. *(i)* $(1,3)+(2,4)=(3,7)\in\mathbb{R}^2$,

(ii) $(1,2,3,4,5)+(5,4,3,2,1)=(6,6,6,6,6)\in\mathbb{R}^5$,

(iii) $3(1,2)=(3,6)\in\mathbb{R}^2$,

(iv) $\pi(0,0,1)=(0,0,\pi)\in\mathbb{R}^3$.

One usually sets

$$\mathbf{0}=(0,0,\dots,0)\in\mathbb{R}^n$$

and this element satisfies $X+\mathbf{0}=\mathbf{0}+X=X$ for any $X\in\mathbb{R}^n$. If $X=(x_1,x_2,\dots,x_n)$ one also writes $-X$ for the element $-1X=(-x_1,-x_2,\dots,-x_n)$. Then, by an abuse of notation, one writes $X-Y$ for $X+(-Y)$ if $X,Y\in\mathbb{R}^n$, and obviously one has $X-X=\mathbf{0}$. Note that $X+Y$ is defined if and only if X and Y belong to $\mathbb{R}^n$, but has no meaning if $X\in\mathbb{R}^n$ and $Y\in\mathbb{R}^m$ with $n\neq m$.

Properties 4. *If $X,Y,Z\in\mathbb{R}^n$ and $\lambda,\mu\in\mathbb{R}$ then one has*

(i) $X+Y=Y+X$, (commutativity)

(ii) $(X+Y)+Z=X+(Y+Z)$, (associativity)

(iii) $\lambda(X+Y)=\lambda X+\lambda Y$, (distributivity)

(iv) $(\lambda+\mu)X=\lambda X+\mu X$,

(v) $(\lambda\mu)X=\lambda(\mu X)$.

1.2 Scalar product and norm

Definition 5. *For any $X,Y\in\mathbb{R}^n$ with $X=(x_1,x_2,\dots,x_n)$ and $Y=(y_1,y_2,\dots,y_n)$ one sets*

$$X\cdot Y:=x_1y_1+x_2y_2+\cdots+x_ny_n=\sum_{j=1}^n x_jy_j$$

and calls this number the scalar product *between X and Y.*

For example, if $X=(1,2)$ and $Y=(3,4)$, then $X\cdot Y=1{\cdot}3+2{\cdot}4=3+8=11$, but if $X=(1,3)$ and $Y=(6,-2)$, then $X\cdot Y=6-6=0$. Be aware that the previous notation is slightly misleading since the dot $\cdot$ between X and Y corresponds to the scalar product while the dot between numbers just corresponds to the usual multiplication of numbers.

Properties 6. *For any $X,Y,Z\in\mathbb{R}^n$ and $\lambda\in\mathbb{R}$ one has*

(i) $X\cdot Y=Y\cdot X$,

(ii) $X \cdot (Y + Z) = X \cdot Y + X \cdot Z$,

(iii) $(\lambda X) \cdot Y = X \cdot (\lambda Y) = \lambda(X \cdot Y)$,

(iv) $X \cdot X \geq 0$, *and* $X \cdot X = 0$ *if and only if* $X = \mathbf{0}$.

Definition 7. *The* Euclidean norm *or simply* norm *of a vector* $X \in \mathbb{R}^n$ *is defined by*

$$\|X\| := \sqrt{X^2} \equiv \sqrt{X \cdot X} = \sqrt{x_1^2 + x_2^2 + \cdots + x_n^2}.$$

The positive number $\|X\|$ *is also referred to as the* magnitude *of* X. *A vector of norm* 1 *is called a* unit vector.

Example 8. *If* $X = (-1, 2, 3) \in \mathbb{R}^3$, *then* $X \cdot X = (-1)^2 + 2^2 + 3^2 = 14$ *and therefore* $\|X\| = \sqrt{14}$.

Remark 9. *If* $n = 2$ *or* $n = 3$ *this norm corresponds to our geometric intuition.*

Properties 10. *For any* $X \in \mathbb{R}^n$ *and* $\lambda \in \mathbb{R}$ *one has*

(i) $\|X\| = 0$ *if and only if* $X = \mathbf{0}$,

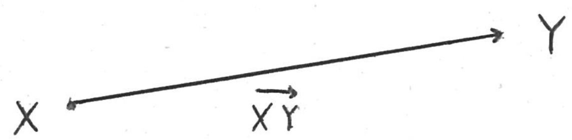
(ii) $\|\lambda X\| = |\lambda| \, \|X\|$,

(iii) $\|-X\| = \|X\|$.

Note that the third point is a special case of the second point.

1.3 Vectors and located vectors

Definition 11. *For any* $X, Y \in \mathbb{R}^n$ *we set* $\overrightarrow{XY}$ *for the arrow starting at* X *and ending at* Y, *and call it the* located vector $\overrightarrow{XY}$.



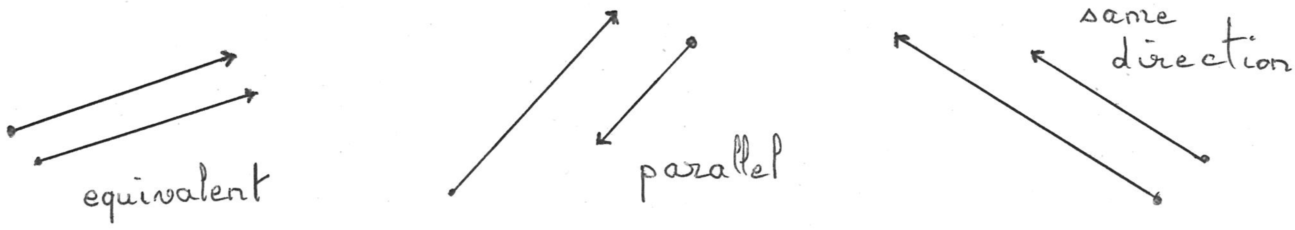
With this definition and for any $X \in \mathbb{R}^n$ the located vector $\overrightarrow{\mathbf{0}X}$ corresponds to the arrow starting at $\mathbf{0}$ and ending at X. This located vector is simply called *a vector* and is often identified with the element X of $\mathbb{R}^n$. This identification should not lead to any confusion in the sequel.

Definition 12. *Let* $A, B, C, D \in \mathbb{R}^n$ *and consider the located vectors* $\overrightarrow{AB}$ *and* $\overrightarrow{CD}$.

(i) $\overrightarrow{AB}$ *is* equivalent to $\overrightarrow{CD}$ *if* $B - A = D - C$,

(ii) $\overrightarrow{AB}$ *is* parallel to $\overrightarrow{CD}$ *if* $B - A = \lambda(D - C)$ *for some* $\lambda \in \mathbb{R}$,

(iii) $\overrightarrow{AB}$ *and* $\overrightarrow{CD}$ have the same direction *if* $B - A = \lambda(D - C)$ *for some* $\lambda > 0$.



Note that the located vector $\overrightarrow{XY}$ is always equivalent to the located vector $\overrightarrow{\mathbf{0}(Y-X)}$ which is located at *the origin* $\mathbf{0}$. This fact follows from the equality

$$(Y - X) - \mathbf{0} = (Y - X) = Y - X.$$

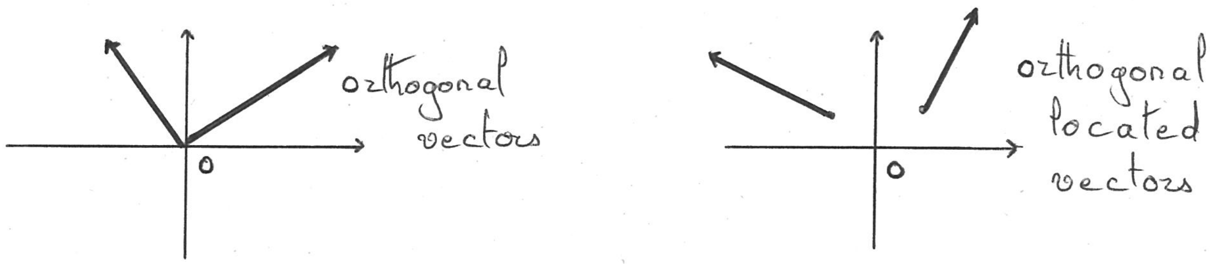
Definition 13. *The* norm *of a located vector* $\vec{XY}$ *is defined by* $\|\overrightarrow{XY}\| := \|Y - X\|$.

Note that this definition corresponds to our intuition in dimension $n = 2$ or $n = 3$ for the length of an arrow. It should also be observed that two equivalent located vectors have the same norm.

Definition 14. *Let* $A, B, C, D \in \mathbb{R}^n$.

(i) The vectors $\overrightarrow{\mathbf{0}A}$ *and* $\overrightarrow{\mathbf{0}B}$ *are* orthogonal *(or* perpendicular*) if* $A \cdot B = 0$. *In this case one writes* $\overrightarrow{\mathbf{0}A} \perp \overrightarrow{\mathbf{0}B}$ *or simply* $A \perp B$,

(ii) The located vectors $\overrightarrow{AB}$ *and* $\overrightarrow{CD}$ *are* orthogonal *if* $\overrightarrow{\mathbf{0}(B-A)}$ *and* $\overrightarrow{\mathbf{0}(D-C)}$ *are orthogonal, namely if* $(B - A) \cdot (D - C) = 0$. *In this case one writes* $\overrightarrow{AB} \perp \overrightarrow{CD}$.



Let us emphasize again that this notion corresponds to our intuition in dimension $n = 2$ or $n = 3$.

Example 15. *In* $\mathbb{R}^n$ *let us set* $E_1 = (1, 0, \dots, 0)$, $E_2 = (0, 1, 0, \dots, 0)$, $\dots$, $E_n = (0, \dots, 0, 1)$ *the* n *different vectors obtained by assigning a* 1 *at the coordinate* j *of* E_j *and* 0 *for all its other coordinates. Then, one easily checks that*

$$E_j \cdot E_k = 0 \text{ whenever } j \neq k \quad \text{and} \quad E_j \cdot E_j = 1 \text{ for any } j \in \{1, 2, \dots, n\}.$$

These n *vectors are said to be* mutually orthogonal.

Theorem 16 (General Pythagoras theorem). *For any $X, Y \in \mathbb{R}^n$ one has*

$$\|X+Y\|^2 = \|X\|^2 + \|Y\|^2 \quad \Longleftrightarrow \quad X \cdot Y = 0.$$

In other words, the equality on the left-hand side holds if and only if X and Y are orthogonal.

Let us add some other useful properties:

Lemma 17. *For any $X, Y \in \mathbb{R}^n$:*

(i) $|X \cdot Y| \leq \|X\| \, \|Y\|$,

(ii) $\|X+Y\| \leq \|X\| + \|Y\|$,

(iii) Let us recall from plane geometry that if one considers the triangle with vertices the points $\mathbf{0}$, X and Y, then the angle θ at the vertex $\mathbf{0}$ satisfies

$$\cos(\theta) = \frac{X \cdot Y}{\|X\| \, \|Y\|}.$$

1.4 Lines and hyperplanes

Let us consider $P, N \in \mathbb{R}^n$ with $N \neq \mathbf{0}$.

The set

$$L_{P,N} := \big\{P + tN \mid t \in \mathbb{R}\big\}$$

defines the line in $\mathbb{R}^n$ passing through P and having the direction parallel to $\overrightarrow{\mathbf{0}N}$.

The set

$$H_{P,N} := \big\{X \in \mathbb{R}^n \mid X \cdot N = P \cdot N\big\}$$

defines a *hyperplane* passing through P and normal to $\overrightarrow{\mathbf{0}N}$. Note that if $P = (p_1, p_2, \dots, p_n)$ and if $N = (n_1, n_2, \dots, n_n)$, then

$$H_{P,N} = \Big\{(x_1, x_2, \dots, x_n) \in \mathbb{R}^n \mid n_1x_1 + n_2x_2 + \dots + n_nx_n = \sum_{j=1}^{n} p_jn_j\Big\}.$$