Report

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1 Excercise and Extension in Chapter 2

1.1 Exercise 2.3.3

Theorem 1.1 (Theorem 2.3.2). *Let $a_1 \geq a_2 \geq \cdots \geq a_n \geq 0$ and $b \in \mathbb{C}^n$ with*

$$\sum_{i=1}^{k} b_i^* \leq \sum_{i=1}^{k} a_i$$

for $k \in \{1, ..., n\}$. Then there exist $a^{(1)}, a^{(2)}, ..., a^{(m)} \in \mathbb{C}^n$ with $(a^{(l)})^* = a$ and $\{\lambda_l\} \subset [0, 1]$ satisfying $\sum_{l=1}^{m} \lambda_l = 1$ such that

$$b = \sum_{l=1}^{m} \lambda_l a^{(l)}.$$

In addition, if Φ is a real valued function on $[0, \infty)^n$ and if the function $\phi : \mathbb{C}^n \to \mathbb{R}_+$, defined by $\phi(c) := \Phi(c_1^*, ..., c_n^*)$, is convex on $\mathbb{C}^n$, then

$$\phi(b) \leq \phi(a).$$

Proof. Let H be the convex hull of the set of points $x \in \mathbb{C}^n$ such that $x^* = a = (a_1, ..., a_n)$. We only need to show that b lies in H. Let $f \in (\mathbb{C}^n)^*$, linear functional on $\mathbb{C}^n$. Since H is bounded we write

$$\beta = \max_{c \in H} Re(f(c)).$$

The result follows if we prove that $|f(b)| \leq \beta$ by the geometric version of Hahn-Banach theorem on $\mathbb{C}^n$ (a point outside the convex hull can be separated by a hyperplane), e.g. [4]. Now since $(\mathbb{C}^n)^* = \mathbb{C}^n$, there exists $\alpha \in \mathbb{C}^n$ such that for each $c \in \mathbb{C}^n$, $f(c) = \sum_{j=1}^{n} \alpha_j c_j$. Let $f^*(c) := \sum_{j=1}^{n} \alpha_j^* c_j$. Then by (2.13), $|f(b)| = |\sum_{j=1}^{n} \alpha_j b_j| \leq \sum_{j=1}^{n} \alpha_j^* b_j^* = f^*(b^*)$. For each $i \in \{1, ..., n\}$, there exists $j \in \{1, .., n\}$ such that $\alpha_j^* = |\alpha_i|$. Let us define $a_i' := a_j e^{-i\arg(a_i)}$ for each i, then $(a')^* = (a_1', ..., a_n')^* = a$ and $f(a') = f^*(a)$. Thus $f^*(a) \leq \beta$ and

$$\begin{aligned} f^*(b^*) &= \alpha_n^* \sum_{j=1}^{n} b_j^* + (\alpha_{n-1}^* - \alpha_n^*) \sum_{j=1}^{n-1} b^* + \cdots + (\alpha_1^* - \alpha_2^*) b_1^* \\ &\leq \alpha_n^* \sum_{j=1}^{n} a_j + \cdots + (\alpha_1^* - \alpha_2^*) a_1 = f^*(a). \end{aligned}$$

Therefore $|f(b)| < \beta$ by the previous inequality. The last inequality of Theorem 1.1 is directly obtained by

$$\phi(b) = \phi\left(\sum_{l=1}^{m} \lambda_l a^{(l)}\right) \leq \sum_{l=1}^{m} \lambda_l \phi(a^{(l)}) = \sum_{l=1}^{m} \lambda_l \phi(a) = \phi(a).$$

□

1.2 Exercise 2.3.5

Corollary 1.2 (Corollary 2.3.4). *Let $a, b \in \mathbb{R}^n$ be positive and ordered, and suppose that*

$$\prod_{j=1}^{k} b_j \leq \prod_{j=1}^{k} a_j \quad \textit{for any } \; k \in \{1, ..., n\}.$$

Then, for any continuous, monotone increasing function $g : [0,\infty) \to \mathbb{R}$ with $t \mapsto g(e^t)$ convex, we have that

$$\sum_{j=1}^{k} g(b_j) \le \sum_{j=1}^{k} g(a_j) \quad \text{for any} \ \ k \in \{1, ..., n\}.$$

In particular, (2.14) *can be obtained by taking $g(x) = x$.*

Proof. We can assume that all a_j are not-zero (otherwise the result easily follows). We also assume that all b_j are not-zero, otherwise we can take the value ϵ small enough for the zero term and take the limit as $\epsilon \to 0$. Set $\tilde{a}_j := \gamma a_j$ and $\tilde{b}_j := \gamma b_j$ for γ large enough so that $\tilde{a}_j, \tilde{b}_j \ge 1$ for all j.

Let $\tilde{g}(x) := g(\gamma^{-1}x)$. It is true that $\tilde{g}$ is continuous and increasing. We now show that the map $t \mapsto \tilde{g}(e^t)$ is convex. Indeed, for $a, b \in [0, \infty)$ and $s \in [0,1]$ we have

$$\begin{aligned}
\tilde{g}(e^{as+(1-s)b}) &= g(\gamma^{-1}e^{as+(1-s)b)}) \\
&= g(e^{(a+\ln\gamma^{-1})s+(1-s)(b+\ln\gamma^{-1}))}) \\
&\le sg(e^{(a+\ln\gamma^{-1})}) + (1-s)g(e^{(b+\ln\gamma^{-1}))}) \\
&= sg(\gamma^{-1}e^a) + (1-s)g(\gamma^{-1}e^b) = s\tilde{g}(e^a) + (1-s)\tilde{g}(e^b).
\end{aligned}$$

Next we define $\bar{c}_j := \ln\tilde{c}_j$ for every j $(c := a, b)$, $f(x) := \tilde{g}(e^x)$, $x \in \mathbb{R}_+$ (f is convex and increasing) and $\Phi(x) := \sum_{j=1}^{n} f(x_j)$ for every $x \in [0,\infty)^n$. Then the hypotheses of Theorem 1.1 are satisfied. Therefore for $\bar{a} = (\bar{a}_1, .., \bar{a}_n)$ and $\bar{b} = (\bar{b}_1, .., \bar{b}_n)$ we get

$$\Phi(\bar{b}^*) \le \Phi(\bar{a}^*) \ \Leftrightarrow \ \sum_{j=1}^{n} g(b_j) \le \sum_{j=1}^{n} g(a_j).$$

Last, if we only consider the first $k \le n$ terms, the result follows.

□

1.3 Exercise 2.3.13

Theorem 1.3 (Theorem 2.3.11)**.** *Let Φ be a symmetric norm on c_c, then*

(a) If $a \in J_\Phi$ and $\lim_{j\to\infty} a_j = 0$, then $\Phi(a) = \Phi(a^*)$,

(b) If $a, b \in J_\Phi$ with $\lim_{j\to\infty} a_j = 0$ and $\lim_{j\to\infty} b_j = 0$, and if $\sum_{j=1}^{n} b_j^* \le \sum_{j=1}^{n} a_j^*$ for any $n \in \mathbb{N}$, then $\Phi(b) \le \Phi(a)$.

(c) If $\Phi((1, 0, 0, ...)) = c$, then $c\|a\|_\infty \le \Phi(a) \le c\|a\|_1$ for any $a \in J_\Phi$,

(d) Both J_Φ and $J_\Phi^{(0)}$ are Banach spaces,

(e) If α is a substochastic matrix and $a \in J_\Phi$, resp. $a \in J_\Phi^{(0)}$, then αa is in J_Φ, resp. $\alpha a \in J_\Phi^{(0)}$, and $\Phi(\alpha a) \le \Phi(a)$,

(f) If Φ is inequivalent to $\|\cdot\|_\infty$, then J_Φ consists only of sequences which vanish at infinity,

(g) If $J_\Phi = J_\Psi$, then Φ and Ψ are equivalent norms.

Proof. (a.) ($\star$) For finite sequence a, b, assertion (b) follows from Theorem 2.3.2.

Since $a \in J_\Phi$, $\lim_{n\to\infty} \Phi((a_1, a_2, ..., a_n, 0, ..))$ exists and is finite. Let $n \in \mathbb{N}$, there exists $m > n$ such that $a_1^*, ..., a_n^*$ are among $|a_1|, ..., |a_m|$. Then by ($\star$) one gets

$$\Phi((a_1^*, ..., a_n^*, 0, ..)) \le \Phi((a_1, ..., a_m, 0, ..)).$$

By taking the limit as $n, m \to \infty$, one obtains $\Phi(a^*) \le \Phi(a)$. Moreover, for each $n' \in \mathbb{N}$, ($\star$) implies $\Phi((a_1^*, ..., a_{n'}^*, 0, ..)) \ge \Phi((a_1, ..., a_{n'}, 0, ..))$. Hence $\Phi(a^*) \ge \Phi(a)$ and it implies $\Phi(a^*) = \Phi(a)$.

(b.) Since $a, b \in J_\Phi$, then $a^*, b^* \in J_\Phi$. For $n, m \in \mathbb{N}$ with $n < m$, we have $\Phi((b_1^*, ..., b_n^*, 0, ..)) \le \Phi((a_1^*, ..., a_m^*, 0, ..))$. Therefore $\Phi(b^*) \le \Phi(a^*)$. Finally by (a), $\Phi(b) \le \Phi(a)$.

(c.) We have $\Phi((0, .., 0, a_n, 0, ..)) \le \Phi(a) \le \sum_{n=1}^{\infty} \Phi((0, .., 0, a_n, 0, ..))$. The first inequality is obtained by (b) and second one by the triangle inequality. Hence,

$$c\|a\|_\infty \le \Phi(a) \le c\|a\|_1.$$

The last inequality is true if $a \in \ell_1$.

(d.) It is clear that Φ is a norm in J_Φ and $J_\Phi^{(0)}$. We need only consider J_Φ since $J_\Phi^{(0)}$ is closed in J_Φ. Let $\{a^{(m)}\}_{m\ge1}$ be a Cauchy sequence in J_Φ. By assertion (c), we obtain that $\{a_n^{(m)}\}_{m\ge1}$ is a Cauchy sequence in $\mathbb{C}$. Then there exists a sequence $a = (a_n)$ such that $a_n^{(m)} \to a_n$ for each n. Next by applying (b) we get

$$\begin{aligned} \Phi((a_1 - a_1^{(m)}, .., a_N - a_N^{(m)}, 0, .)) &= \lim_{n\to\infty} \Phi((a_1^{(n)} - a_1^{(m)}, .., a_N^{(n)} - a_N^{(m)}, 0, .)) \\ &\le \lim_{n\to\infty} \Phi(a^{(n)} - a^{(m)}). \end{aligned}$$

Taking the limit as $N \to \infty$ we discover that $\Phi(a - a^{(m)}) \to 0$ as $m \to \infty$. Therefore $a \in J_\Phi$ and J_Φ is complete.

(e.) First define $b := \alpha a$. Since both J_Φ and $J_\Phi^{(0)}$ are Banach spaces, we only prove for $J_\Phi^{(0)}$ (a similar argument will apply to J_Φ). Now let $b^N := (b_1, ..., b_N, 0, ..)$ for $N \in \mathbb{N}$. By Proposition 2.3.7 and assertion (b), we obtain $\Phi(b^N) \le \Phi(a^*)$. Since $J_\Phi^{(0)}$ is the closure of c_c, we only need to show that $\alpha a \in J_\Phi^{(0)}$ for $a \in c_c$. Indeed if $a \in c_c$, $a = (a_1, ..., a_m, 0, ..)$, only m first columns of α count for αa. Given $\epsilon > 0$, we can choose $p \in \mathbb{N}$ so that $\sum_{i>p} |\alpha_{ij}| < \epsilon$ for $j = 1, .., m$. Then

$$\Phi((0, 0, ..., 0, (\alpha a)_{p+1}, ..)) \le \epsilon\Phi(a) \quad \text{by (b)}.$$

Therefore $\Phi(\alpha a) \le \Phi(((\alpha a)_1, .., (\alpha a)_p, 0, ..)) + \epsilon\Phi(a) < \infty$ and so $\alpha a \in J_\Phi^0$.

(f.) Suppose $a \in J_\Phi$ and $a_n \nrightarrow 0$. Then there exists a subsequence $\bar{a} = (\bar{a}_j)$ of a such that $\bar{a}_j \to d \ne 0$ for some $d \in \mathbb{C}$ and $|\bar{a}_j - d| \le \frac{1}{2^j}$. Let us write $\mathbf{1} := (1, 1, ...)$. It is clear that $\bar{a} \in J_\Phi$ and we will show that $\bar{a} - d\mathbf{1} \in J_\Phi$. Indeed by assertion (b) and (c), we obtain $\Phi(\bar{a} - d\mathbf{1}) \le \Phi((\frac{1}{2}, .., \frac{1}{2^j}, ...)) \le \Phi((1, 0, ...)) < \infty$. Therefore $|d|\Phi(\mathbf{1}) \le \Phi(\bar{a} - d\mathbf{1}) + \Phi(\bar{a}) < \infty$ and it implies $\|a\|_\infty \Phi(\mathbf{1}) \ge \Phi(a)$. This inequality and (c) show that Φ is equivalent to $\|\cdot\|_\infty$.

(g.) By the first inequality of assertion (c), one can shows that the identity map from J_Φ to J_Ψ is closed. Then the closed graph theorem implies that Φ and Ψ are equivalent. □

As usual, let us define Φ' on c_c as the conjugate norm of Φ by setting

$$\Phi'(b) := \sup\left\{ \big|\sum_j a_j b_j\big| \mid a \in c_c, \Phi(a) \leq 1 \right\}$$

for any $b \in c_c$. One can show that the conjugate norm of Φ' is Φ, ($\Phi'' = \Phi$).

Theorem 1.4 (Theorem 2.3.12)**.** *Let Φ be a symmetric norm on c_c. Then*

(a) $\sum_j |a_j b_j| \leq \Phi(a)\Phi(b)$,

(b) $\big(J_\Phi^{(0)}\big)^* = J_{\Phi'}$ $\big(\big(J_{\Phi'}^{(0)}\big)^* = J_\Phi\big)$ *in the sense that any continuous linear functional on* $J_\Phi^{(0)}$ *has the form* $a \mapsto \sum_j a_j b_j$ *for some* $b \in J_{\Phi'}$,

(c) $J_\Phi^{(0)}$, *resp.* J_Φ, *is reflexive if and only if both* Φ *and* Φ' *are regular.*

Proof. (a.) Let $a, b \in c_c$. WLOG assume $a \neq 0$. Let N be the biggest integer such that a_N and b_N are not zero, then by (2.13) and theorem 2.3.11 (b),

$$\begin{aligned} \sum_{i=1}^N |a_i b_i| &\leq \sum_{i=1}^N a_i^* b_i^* \\ &\leq \Phi(a^*) \sum_{i=1}^N \frac{a_i^*}{\Phi(a^*)} b_i^* \\ &\leq \Phi(a^*)\Phi'(b^*) = \Phi(a)\Phi'(b). \end{aligned}$$

(b.) By assertion (a), every $b \in J_{\Phi'}$ defines a linear functional on $J_\Phi^{(0)}$. For each m, there is $a \in c_c$ so that $\Phi(a) = 1$ and $|\sum a_n b_n| = \Phi'(b_1, ..., b_m, 0, ..)$. It implies that $\Phi'(b)$ is the norm of b as a linear functional. Thus, we only need to show that any $f \in (J_\Phi^{(0)})^*$ is of the requisite form. Since $(\mathbb{C}^n)^* = \mathbb{C}^n$, there exists a sequence $\{b_n\}_{n\geq 1}$ with $f(a) = \sum_n a_n b_n$ for all $a \in c_c$. Moreover, by definition of Φ', $\Phi'(b_1, .., b_n, 0, ..) = \sup\{f(a) \mid a = (a_1, .., a_n, 0, ..), \Phi(a) = 1\} \leq \|f\|$. Therefore $b \in J_{\Phi'}$. Since $b \in J_{\Phi'}$ and $f = b$ on c_c, $f = b$ on $J_\Phi^{(0)}$.

(c.) ($\Leftarrow$) If Φ and Φ' are regular then (b) implies $(J_\Phi^{(0)})^{**} = J_\Phi^{(0)}$.

($\Rightarrow$) If Φ is not regular, $J_\Phi^{(0)} \subset J_\Phi \subseteq (J_\Phi^{(0)})^{**}$, so J_Φ^0 is not reflexive. If Φ' is not regular, $(J_\Phi^{(0)})^{**} = (J_{\Phi'})^*$ is strictly bigger than $J_\Phi = (J_{\Phi'}^{(0)})^*$ by the corollary of Hahn-Banach theorem. Therefore $J_\Phi^{(0)}$ is not reflexive. Finally, if $J_\Phi^{(0)}$ is not reflexive, then $J_\Phi = (J_{\Phi'}^{(0)})^*$ is not reflexive.

□

1.4 Extension 2.5.8

Definition 1. A Banach space X is called *uniformly convex* if and only if for all $0 < \epsilon < 2$, the modulus convexity,

$$\delta(\epsilon) \equiv \inf\left\{1 - \left\|\frac{x+y}{2}\right\| \mid \|x\| = \|y\| = 1, \|x - y\| \geq \epsilon\right\}$$

is strictly positive.

Theorem 1.5. *Let X be a uniformly convex Banach space and suppose that x_n converges to x weakly and that $\|x_n\| \to \|x\|$. Then $\|x_n - x\| \to 0$ (x_n strongly converges to x).*

Proof. We only consider the case $x \neq 0$. It suffices to show that $\|y_n - y\| \to 0$ as $n \to \infty$ for $y_n := \frac{x_n}{\|x_n\|}$ and $y := \frac{x}{\|x\|}$ (since $\lim_{n\to\infty} \|y_n - y\|$ implies $\lim_{n\to\infty} \|x_n - x\|$). We claim that $\|\frac{y_n+y}{2}\| \to 1$. It implies $\|y_n - y\| \to 0$ as the consequence of the assumption. Indeed by the Hahn-Banach theorem, there exists $g \in X^*$ such that $\|g\| = g(y) = 1$. Then we obtain $g(\frac{y_n+y}{2}) \to g(y) = 1$ since y_n converges weakly to y and

$$1 \geq \left\| \frac{y_n + y}{2} \right\| \geq \left| g\left(\frac{y_n + y}{2}\right) \right| \to 1.$$

The squeeze theorem implies $\|\frac{y_n+y}{2}\| \to 1$. □

Here we state some known spaces which have the unifom convexity property.

Example 1.

(1.) Every Hilbert space is uniformly convex space.

(2.) Every closed subspace of a uniformly convex Banach space is uniformly convex.

(3.) L^p space $(1 < p < \infty)$ are uniformly convex.

Theorem 1.6. *For $1 \leq p < \infty$, $\mathcal{J}_p$ is a linear norm space. Moreover $\mathcal{J}_p$ is complete and we have the following inequality :*

(i) For $2 \leq p < \infty$, $\|A + B\|_p^p + \|A - B\|_p^p \leq 2^{p-1}(\|A\|_p^p + \|B\|_p^p)$,

(ii) For $1 \leq p \leq 2$, $\|A + B\|_p^{p'} + \|A - B\|_p^{p'} \leq 2(\|A\|_p^p + \|B\|_p^p)^{\frac{p'}{p}}$,

for $A, B \in \mathcal{J}_p$ and $\frac{1}{p} + \frac{1}{p'} = 1$.

By the theorem 1.8, we see that $\mathcal{J}_p$ is a Banach space.

Theorem 1.7. *For $1 < p < \infty$, $\mathcal{J}_p$ is uniformly convex.*

Proof. Let $0 < \epsilon < 2$. For $\|A\|_p = \|B\|_p = 1$ and $\|A - B\|_p \geq \epsilon$,

(i) $(1 < p \leq 2)$ Applying Theorem 1.8 (ii) we obtain,

$$\|A + B\|_p^{p'} + \leq 2^{p'} - \|A - B\|_p^{p'} \leq 2^{p'} - \epsilon^{p'}.$$

Therefore $2\delta(\epsilon) \geq 2 - (2^{p'} - \epsilon^{p'})^{\frac{1}{p'}} > 0$.

(ii) $(2 < p < \infty)$ Applying Theorem 1.8 (i) we obtain $2\delta(\epsilon) \geq 2 - (2^p - \epsilon^p)^{\frac{1}{p}} > 0$.

Hence $\mathcal{J}_p$ is uniformly convex. □

Corollary 1.8 (Theorem 2.5.7). *Let $p \in [1, \infty)$ and $\{A_n\} \subset \mathscr{I}_p$ and $A \in \mathscr{I}_p$. If $w - \lim_{n\to\infty} A_n = A$ and $\|A_n\| \to \|A\|$, then $\|A_n - A\| \to 0$.*

The Corollary above is direct consequence of Theorem 1.8 and Theorem 1.6 for $p \in (1, \infty)$.

References

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