ON SCHATTEN-VON NEUMANN CLASS PROPERTIES OF PSEUDODIFFERENTIAL OPERATORS. THE CORDES-KATO METHOD

GRUIA ARSU

Communicated by Şerban Strătilă

ABSTRACT. We investigate the Schatten-class properties of pseudodifferential operators with the (revisited) method of Cordes and Kato. As symbol classes we use classes similar to those of Cordes in which the $L^\infty$-conditions are replaced by $L^p$-conditions, $1 \leq p < \infty$.

KEYWORDS: Calderón, Cordes, Kato, Schatten, Schrödinger, Vaillancourt, Weyl calculus, pseudodifferential operator, trace-class.

MSC (2000): Primary 35S05, 43Axx, 46-XX, 47-XX; Secondary 42B15, 42B35.

INTRODUCTION

In two classical papers [10] and [19], H.O. Cordes and T. Kato develop an elegant method to deal with pseudodifferential operators. In [10], H.O. Cordes shows, among others, that if a symbol $a(x, \xi)$ defined on $\mathbb{R}^n \times \mathbb{R}^n$ has bounded derivatives $D^\alpha_x D^\beta_\xi a$ for $|\alpha|, |\beta| \leq [n/2] + 1$, then the associated pseudodifferential operator $a(x, D)$ is $L^2$-bounded. This result, known as Calderón-Vaillancourt theorem, appears for the first time in [7], except that the number of the required derivatives is different. Cordes-Kato method can be used to obtain Calderón-Vaillancourt theorem with a minimal Hölder continuity assumption on the symbol of a pseudodifferential operator. (See [8] and [5] for another approach).

The main purpose of the present paper is to describe an extension of the Cordes-Kato method which can be used to prove Schatten-class properties of pseudodifferential operators. Among others, we prove that $a(x, D) \in B_p(L^2(\mathbb{R}^n))$ if $D^\alpha_x D^\beta_\xi a \in L^p(\mathbb{R}^n \times \mathbb{R}^n)$ for $|\alpha|, |\beta| \leq [n/2] + 1$ and $1 \leq p < \infty$, where for a Hilbert space $\mathcal{H}$, we denote by $B_p(\mathcal{H})$ the Schatten ideal of compact operators on $\mathcal{H}$ whose singular values lie in $l^p$. It is remarkable that this method can be used for $(X, \tau)$-quantization, in particular, for both the Weyl quantization and Kohn-Nirenberg quantization.
For other approaches to the $L^2$-boundedness and Schatten-class properties of pseudodifferential operators we refer to the works [12], [13], [25], [26], [27], [28].

1. Weyl Calculus in the Schrödinger Representation

We find it convenient to present the results in the Schrödinger representation formalism (or Weyl systems formalism).

A symplectic space is a real finite dimensional vector space $\mathcal{S}$ equipped with a real antisymmetric nondegenerate bilinear form $\sigma$.

We define the Fourier measure $d\xi$ as the unique translation invariant, Borel regular measure on $\mathcal{S}$ such that the (symplectic) Fourier transform 
$$(\mathcal{F}_\mathcal{S} a)(\xi) \equiv \hat{a}(\xi) = \int_{\mathcal{S}} e^{-i\sigma(\xi, \eta)} a(\eta) d\eta$$
is involutive (i.e. $\mathcal{F}_\mathcal{S}^2 = 1$) and unitary on $L^2(\mathcal{S})$. We use the same notation $\mathcal{F}_\mathcal{S}$ for the extension to $\mathcal{S}^*(\mathcal{S})$ of this Fourier transform.

We recall some facts in connection with the theory of canonical commutation relations (see [6], [18]). We denote by $\mathcal{M}(\mathcal{S})$ the space of integrable measures on $\mathcal{S}$ equipped with a unital $*$-algebra structure defined by the twisted convolution 
$$(\mu \times \nu)(\xi) = \int_{\mathcal{S}} e^{i\sigma(\xi, \eta)} \mu(\xi - \eta) \nu(\eta) \, d\eta$$
as product, $\mu^*(\xi) = \overline{\mu(-\xi)}$ as involution and $\delta$ the Dirac measure at 0 as unit. These definition must be interpreted in the sense of distributions, i.e. for $f \in C_0(\mathcal{S})$:

$$\int_{\mathcal{S}} f(\xi) \, d(\mu \times \nu)(\xi) = \int \int_{\mathcal{S}} e^{i\sigma(\xi, \eta)} f(\xi + \eta) d\mu(\xi) d\nu(\eta),$$

$$\int_{\mathcal{S}} f(\xi) \, d\mu^*(\xi) = \int_{\mathcal{S}} f(-\xi) d\mu(\xi).$$

Let $\hat{\mathcal{M}}(\mathcal{S})$ be the space of Fourier transforms of measures in $\mathcal{M}(\mathcal{S})$, provided with the product $a \circ b = \mathcal{F}_\mathcal{S}(a \times b)$ and with the usual conjugation $a^* = \overline{a}$ ($= \mathcal{F}_\mathcal{S}(\overline{a})$). $\hat{\mathcal{M}}(\mathcal{S})$ becomes a unital $*$-algebra, with the function identically equal to 1 ($\mathcal{F}_\mathcal{S}(\delta) = 1$) as unit. Observe that $\hat{\mathcal{M}}(\mathcal{S})$ consists of bounded continuous functions on $\mathcal{S}$. The product $a \circ b$ is called composition product.

By a representation of a symplectic space $\mathcal{S}$ on a Hilbert space $\mathcal{H}$ we understand a strongly continuous map $\mathcal{W}$ from $\mathcal{S}$ to the set of unitary operators on $\mathcal{H}$ satisfying:

$$(1.1) \quad \mathcal{W}(\xi) \mathcal{W}(\eta) = e^{i\sigma(\xi, \eta)} \mathcal{W}(\xi + \eta) \quad \text{for all } \xi, \eta \in \mathcal{S}.$$
This implies $\mathcal{W}(0) = 1$, $\mathcal{W}(\xi)^* = \mathcal{W}(-\xi)$ and
\begin{equation}
\mathcal{W}(\xi)\mathcal{W}(\eta) = e^{i\xi\eta}\mathcal{W}(\eta)\mathcal{W}(\xi)\quad \text{for all } \xi, \eta \in \mathcal{S}.
\end{equation}
The couple $(\mathcal{H}, \mathcal{W})$ is also called the Weyl system associated to the symplectic space $\mathcal{S}$.

For integrable Borel measures $\mu$ on $\mathcal{S}$, we denote
\begin{equation}
\mathcal{W}(\mu) = \int_{\mathcal{S}} \mathcal{W}(\xi) \, d\mu(\xi).
\end{equation}
One can easily check then that
$$\mathcal{W}(\mu \times v) = \mathcal{W}(\mu)\mathcal{W}(v) \quad \text{and} \quad \mathcal{W}(\mu^*) = \mathcal{W}(\mu)^*,$$
i.e. $\mathcal{W} : \mathcal{M}(\mathcal{S}) \rightarrow \mathcal{B}(\mathcal{H})$ is a $*$-representation of $\mathcal{M}(\mathcal{S})$ on the Hilbert space $\mathcal{H}$.

Note that for any $\xi \in \mathcal{S}$, $\mathcal{W}(\delta_\xi) = \mathcal{W}(\xi)$, where $\delta_\xi$ is the Dirac measure at $\xi$.

There is a bijective correspondence between faithful $*$-representations $\mathcal{O}p : \widehat{\mathcal{M}}(\mathcal{S}) \rightarrow \mathcal{B}(\mathcal{H})$ and strongly continuous representations $\mathcal{W} : \mathcal{S} \rightarrow \mathcal{U}(\mathcal{H})$. If $\mathcal{O}p$ is irreducible then $\mathcal{W}$ acts irreducibly in $\mathcal{H}$. The correspondence is specified by
\begin{equation}
\mathcal{O}p(a) = a(R) = \mathcal{W}(\hat{a})(\xi) = \int_{\mathcal{S}} \mathcal{W}(\xi) \, d\hat{a}(\xi)
\end{equation}
for $a \in \widehat{\mathcal{M}}(\mathcal{S})$. The application $\mathcal{O}p$ is called Weyl calculus, $\mathcal{O}p(a)$ being the operator associated to the symbol $a$.

Let $\mathcal{S}$ be the dense linear subspace of $\mathcal{H}$ consisting of the $C^\infty$ vectors of the representation $\mathcal{W}$
\begin{equation}
\mathcal{S} = \mathcal{S}(\mathcal{H}, \mathcal{W}) = \{ \varphi \in \mathcal{H} : \mathcal{S} \ni \xi \rightarrow \mathcal{W}(\xi)\varphi \in \mathcal{H} \text{ is a } C^\infty \text{ map} \}.
\end{equation}

For each $\xi \in \mathcal{S}$ the family $\{ \mathcal{W}(t\xi) \}_{t \in \mathbb{R}}$ is a strongly continuous unitary representation of $\mathbb{R}$ in $\mathcal{H}$ which leaves $\mathcal{S}$ invariant. We denote $\sigma(\xi, R)$ the infinitesimal generator of this group, so that
\begin{equation}
\mathcal{W}(t\xi) = e^{it\sigma(\xi, R)}, \quad \text{for all } t \in \mathbb{R}.
\end{equation}
Clearly $\mathcal{S} \subset D(\sigma(\xi, R))$, $\mathcal{S}$ is stable under $\sigma(\xi, R)$, and $\sigma(\xi, R)|\mathcal{S}$ is essentially self-adjoint by Nelson’s lemma (Theorem VIII.11 in [21]). From (1.1) and (1.2) we get
\begin{equation}
\sigma(\xi + \eta, R) = \sigma(\xi, R) + \sigma(\eta, R),
\end{equation}
\begin{equation}
\mathcal{W}(\xi)\sigma(\eta, R)\mathcal{W}(-\xi) = \sigma(\eta, R) + \sigma(\xi, \eta), \quad i[\sigma(\xi, R), \sigma(\eta, R)] = \sigma(\xi, \eta)
\end{equation}
on $\mathcal{S}$ for all $\xi, \eta \in \mathcal{S}$.

Note that $R$ in $\sigma(\xi, R)$ or $a(R)$ is the pair $(Q, P)$, where $Q$ is position and $P$ is momentum observable in physics and the pair $(x, D)$ for users of pseudodifferential calculus as presented, for example, in Chapter XVIII of [15] or in [11].
The space $S$ can be described in terms of the subspaces $D(\sigma(\xi, R))$

$$S = \bigcap_{k \in \mathbb{N}} \bigcap_{\xi_1, \ldots, \xi_k \in \mathcal{G}} D(\sigma(\xi_1, R) \cdots \sigma(\xi_k, R))$$

$$\bigcap_{k \in \mathbb{N}} \bigcap_{\xi_1, \ldots, \xi_k \in \mathcal{B}} D(\sigma(\xi_1, R) \cdots \sigma(\xi_k, R)),$$

where $\mathcal{B}$ is a (symplectic) basis. The topology in $S$ defined by the family of semi-norms 

$$\{ \| \cdot \|_{k, \xi_1, \ldots, \xi_k} \}_{k \in \mathbb{N}, \xi_1, \ldots, \xi_k \in \mathcal{G}}$$

$$\| \varphi \|_{k, \xi_1, \ldots, \xi_k} = \| \sigma(\xi_1, R) \cdots \sigma(\xi_k, R) \varphi \|_H, \quad \varphi \in S,$$

makes $S$ a Fréchet space. We denote by $S^*$ the space of all continuous, antilinear (semilinear) mappings $S \to \mathbb{C}$ equipped with the weak topology $\sigma(S^*, S)$. Since $S \hookrightarrow \mathcal{H}$ continuously and densely, and since $\mathcal{H}$ is always identified with its adjoint $\mathcal{H}^*$, we obtain a scale of dense inclusions

$$S \hookrightarrow \mathcal{H} \hookrightarrow S^*$$

such that, if $\langle \cdot, \cdot \rangle : S \times S^* \to \mathbb{C}$ is the antiduality between $S$ and $S^*$ (antilinear in the first and linear in the second argument), then for $\varphi \in S$ and $u \in \mathcal{H}$, if $u$ is considered as an element of $S^*$, the number $\langle \varphi, u \rangle$ is just the scalar product in $\mathcal{H}$. For this reason we do not distinguish between the the scalar product in $\mathcal{H}$ and the antiduality between $S$ and $S^*$.

For $\xi \in \mathcal{G}$, $\mathcal{W}(\xi)$ and $\mathcal{W}(-\xi)$ are topological isomorphisms of $S$ such that $\mathcal{W}(-\xi)\mathcal{W}(\xi) = \mathcal{W}(\xi)\mathcal{W}(-\xi) = 1_S$. Hence we can extend $\mathcal{W}(\xi)$ to $S^*$ as the adjoint of the mapping of $\mathcal{W}(-\xi)|S$. Then $\mathcal{W}(\xi)$ and $\mathcal{W}(-\xi)$ are isomorphisms of $S^*$ onto itself such that $\mathcal{W}(-\xi) = \mathcal{W}(\xi)^{-1}$.

The first formula of (1.7) and an induction argument give

$$\mathcal{W}(-\xi)\sigma(\eta_1, R) \cdots \sigma(\eta_k, R)\mathcal{W}(\xi) = (\sigma(\eta_1, R) + \sigma(\eta_1, \xi)) \cdots (\sigma(\eta_k, R) + \sigma(\eta_k, \xi))$$

for all $\xi, \eta_1, \ldots, \eta_k \in \mathcal{G}$. Let $\mathcal{B} = \{ \varepsilon_1, \ldots, \varepsilon_n, \varepsilon^1, \ldots, \varepsilon^n \}$ be a symplectic basis, i.e. we have for $j, k = 1, \ldots, n$

$$\sigma(\varepsilon_j, \varepsilon_k) = \sigma(\varepsilon_j^*, \varepsilon_k^*) = \delta_{jk} - \delta_{jk} = 0$$

where $\delta_{jk}$ is the Kronecker delta, equal to 1 when $j = k$ and 0 when $j \neq k$. By making a suitable choice of the vectors $\eta_1, \ldots, \eta_k$ in $\mathcal{B}$, we obtain

$$e^{i(x,p)}Q^\alpha e^{-i(x,p)} = (Q + x)^\alpha, \quad e^{-i(Q,p)} P_\beta e^{i(Q,p)} = (P + p)^.\beta.$$  

Here we used the following notations

$$\mathcal{G} = X + X^*, \quad X = \mathbb{R}\varepsilon_1 + \cdots + \mathbb{R}\varepsilon_n, \quad X^* = \mathbb{R}\varepsilon_1^* + \cdots + \mathbb{R}\varepsilon_n^*,$$

$$\xi = x + p, \quad x = x^1\varepsilon_1 + \cdots + x^n\varepsilon_n, \quad p = p_1\varepsilon^1 + \cdots + p_n\varepsilon_n^*,$$

$$x^1 = \sigma(\varepsilon^1, \xi), \ldots, x^n = \sigma(\varepsilon^n, \xi), \quad p_1 = \sigma(-\varepsilon_1, \xi), \ldots, p_n = \sigma(-\varepsilon_n, \xi),$$

$$\langle x, p \rangle = \sigma(p, x) = x^1p_1 + \cdots + x^np_n,$$

$$x^\alpha = (x^1)^{\alpha_1} \cdots (x^n)^{\alpha_n}, \quad p_\beta = (p_1)^{\beta_1} \cdots (p_n)^{\beta_n}, \quad \alpha, \beta \in \mathbb{N}^n,$$
These formulas together with the binomial formula give

\[ x^\alpha = (Q + x - Q)^\alpha = \sum_{0 \leq \gamma \leq \alpha} (-1)^{|\gamma|} \binom{\alpha}{\gamma} (Q + x)^{\alpha - \gamma} Q^\gamma, \]

\[ p_\beta = (P - (P - p))_\beta = \sum_{0 \leq \gamma \leq \beta} (-1)^{|\gamma|} \binom{\beta}{\gamma} P_{\beta - \gamma} (P - p)_\gamma. \]

Hence on \( S \) we have

\[ x^\alpha p_\beta e^{i(Q,p)} e^{-i(x,p)} = p_\beta e^{i(Q,p)} e^{-i(x,p)} x^\alpha \]

\[ = \sum_{0 \leq \gamma \leq \beta, 0 \leq \delta \leq \alpha} (-1)^{|\gamma|+|\delta|} \binom{\beta}{\gamma} \binom{\alpha}{\delta} P_{\beta - \gamma} Q^{\alpha - \delta} e^{i(Q,p)} e^{-i(x,p)} Q^\delta P_\gamma. \]

We can now prove the following result.

**Lemma 1.1.** If \( \varphi, \psi \in S \), then the map \( \mathcal{S} \ni \xi \rightarrow \langle \psi, \mathcal{W}(\xi) \varphi \rangle_{S,S^*} \in \mathbb{C} \) belongs to \( S(\mathcal{S}) \). Moreover, for each continuous seminorm \( p \) on \( S(\mathcal{S}) \) there are continuous seminorms \( q \) and \( q' \) on \( S \) such that

\[ p(\langle \psi, \mathcal{W}(\cdot) \varphi \rangle_{S,S^*}) \leq q(\psi)q'(\varphi). \]

**Proof.** With the above notations for \( \xi = x + p \) we have

\[ \mathcal{W}(\xi) = \mathcal{W}(x + p) = e^{-\frac{i}{2}c(p,x)} \mathcal{W}(p) \mathcal{W}(x) = e^{-\frac{i}{2}c(p,x)} e^{i(Q,p)} e^{-i(x,p)} \]

so it suffices to study the behaviour of the map \( (x, p) \rightarrow e^{i(Q,p)} e^{-i(x,p)} \) at infinity. We may use the decomposition of \( x^\alpha p_\beta e^{i(Q,p)} e^{-i(x,p)} \) to obtain that

\[ |x^\alpha p_\beta \langle \psi, e^{i(Q,p)} e^{-i(x,p)} \varphi \rangle| \]

\[ \leq \sum_{0 \leq \gamma \leq \beta, 0 \leq \delta \leq \alpha} \binom{\beta}{\gamma} \binom{\alpha}{\delta} \|Q^{\alpha - \delta} p_{\beta - \gamma} \psi, e^{i(Q,p)} e^{-i(x,p)} Q^\delta P_\gamma \varphi\| \]

\[ \leq \sum_{0 \leq \gamma \leq \beta, 0 \leq \delta \leq \alpha} \binom{\beta}{\gamma} \binom{\alpha}{\delta} \|Q^{\alpha - \delta} p_{\beta - \gamma} \psi\| \|Q^\delta P_\gamma \varphi\| < \infty. \]
and duality form, which is a non-degenerate bilinear form. The symplectic space is
\[ \{11\} \]
and pseudodifferential calculus as presented, for example, in Chapter XVIII of \[15\] or
\[ \text{lagrangian if and only if } 2 \dim E \]
both are valid, i.e.
\[ \| \alpha \]
\[ \| \gamma \]

The above integrals make sense if they are taken in the weak sense, i.e. for \( \varphi, \psi \in \mathcal{S} \)
\[ \langle \varphi, \mathcal{W}(\mu) \psi \rangle_{\mathcal{S}, \mathcal{S}^*} = \langle \langle \varphi, \mathcal{W}(\cdot) \psi \rangle_{\mathcal{S}, \mathcal{S}^*}, \mu \rangle_{\mathcal{S}(\mathcal{G}), \mathcal{S}^*(\mathcal{G})} \]
\[ \langle \varphi, \mathcal{O}(a) \psi \rangle_{\mathcal{S}, \mathcal{S}^*} = \langle \langle \varphi, \mathcal{W}(\cdot) \psi \rangle_{\mathcal{S}, \mathcal{S}^*}, a \rangle_{\mathcal{S}(\mathcal{G}), \mathcal{S}^*(\mathcal{G})} \]
Moreover, from Lemma 1.1 one obtains that
\[ |\langle \varphi, \mathcal{W}(\mu) \psi \rangle_{\mathcal{S}, \mathcal{S}^*}| + |\langle \varphi, \mathcal{O}(a) \psi \rangle_{\mathcal{S}, \mathcal{S}^*}| \leq p(\langle \varphi, \mathcal{W}(\cdot) \psi \rangle_{\mathcal{S}, \mathcal{S}^*}) \leq q(\varphi)q'(\psi), \]
where \( p \) is a continuous seminorm on \( \mathcal{S}(\mathcal{G}) \) and \( q, q' \) are continuous seminorms on \( \mathcal{S} \).

The mappings
\[ \mathcal{W} : \mathcal{S}^*(\mathcal{G}) \rightarrow \mathcal{B}(\mathcal{S}, \mathcal{S}^*), \quad \mu \mapsto \mathcal{W}(\mu), \]
\[ \mathcal{O} : \mathcal{S}^*(\mathcal{G}) \rightarrow \mathcal{B}(\mathcal{S}, \mathcal{S}^*), \quad a \mapsto \mathcal{O}(a), \]
are well defined linear and continuous if we consider on \( \mathcal{S}^*(\mathcal{G}) \) the weak* topology
\( \sigma(\mathcal{S}^*(\mathcal{G}), \mathcal{S}(\mathcal{G})) \) and on \( \mathcal{B}(\mathcal{S}, \mathcal{S}^*) \) the topology defined by the seminorms \( \{ p_{\varphi, \psi} \}_{\varphi, \psi \in \mathcal{S}} \)

\[ p_{\varphi, \psi}(A) = |\langle \varphi, A\psi \rangle|, \quad A \in \mathcal{B}(\mathcal{S}, \mathcal{S}^*). \]

A subspace \( E \subset \mathcal{G} \) is called isotropic if \( E \subset E^\sigma \) and involutive if \( E^\sigma \subset E \). If both are valid, i.e. \( E = E^\sigma \), then \( E \) is lagrangian. An isotropic subspace \( X \subset \mathcal{G} \) is lagrangian if and only if \( 2 \dim X = \dim \mathcal{G} \).

Next we shall make the connection with Weyl calculus familiar to users of pseudodifferential calculus as presented, for example, in Chapter XVIII of [15] or in [11].

Let \( X \) be an \( n \) dimensional vector space over \( \mathbb{R} \) and \( X^* \) its dual. Denote \( x, y, \ldots \) the elements of \( X \) and \( k, p, \ldots \) those of \( X^* \). Let \( \langle \cdot, \cdot \rangle : X \times X^* \rightarrow \mathbb{R} \) be the duality form, which is a non-degenerate bilinear form. The symplectic space is
defined by $\mathcal{G} = T^*(X) = X \times X^*$ the symplectic form being $\sigma((x, p), (x', p')) = \langle x', p \rangle - \langle x, p' \rangle$. Observe that $X$ and $X^*$ are lagrangian subspaces of $\mathcal{G}$. Let us mention that there is a kind of converse to this construction. Let $(X, X^*)$ be a couple of lagrangian subspaces of $\mathcal{G}$ such that $X \cap X^* = 0$ or, equivalently, $X + X^* = \mathcal{G}$. If for $x \in X$ and $p \in X^*$ we define $\langle x, p \rangle = \sigma(p, x)$, then we get a non-degenerate bilinear form on $X \times X^*$ which allows us to identify $X^*$ with the dual of $X$. A couple $(X, X^*)$ of subspaces of $\mathcal{G}$ with the preceding properties is called a holonomic decomposition of $\mathcal{G}$.

We define the (Fourier) transforms

$$\mathcal{F}_X, \mathcal{F}_X^{-1} : S^*(X) \to S^*(X),$$

$$\mathcal{F}_{X^*}, \mathcal{F}_{X^*}^{-1} : S^*(X^*) \to S^*(X),$$

by

$$(\mathcal{F}_X u)(p) = \int_X e^{-i\langle x, p \rangle} u(x) dx, \quad (\mathcal{F}_X^{-1} u)(p) = \int_X e^{i\langle x, p \rangle} u(x) dx,$$

$$(\mathcal{F}_{X^*} v)(x) = \int_{X^*} e^{-i\langle x, p \rangle} v(p) dp, \quad (\mathcal{F}_{X^*}^{-1} v)(x) = \int_{X^*} e^{i\langle x, p \rangle} v(p) dp.$$

Here $dx$ is a Haar measure in $X$ and $dp$ is the dual one in $X^*$ such that Fourier’s inversion formulas $\mathcal{F}_{X^*} \circ \mathcal{F}_X = 1_{S^*(X)}$, $\mathcal{F}_X \circ \mathcal{F}_{X^*} = 1_{S^*(X^*)}$ hold. Replacing $dx$ by $cdx$ one must change $dp$ to $c^{-1} dp$ so $d\xi = dx \otimes dp$ is invariantly defined and it is exactly the Fourier measure on $\mathcal{G}$. Then the symplectic Fourier transform is given by

$$(\mathcal{F}_\mathcal{G} a)(x, p) = \int_{X \times X^*} e^{-i \langle y, p \rangle - \langle x, k \rangle} a(y, k) dy dk = (\mathcal{F}_X \otimes \mathcal{F}_{X^*}) a(p, x),$$

so $\mathcal{F}_\mathcal{G} = \mathcal{I} \circ (\mathcal{F}_X \otimes \mathcal{F}_{X^*})$, where $\mathcal{I} : S^*(X^* \times X) \to S^*(X \times X^*)$ is given by $\mathcal{I} b(x, p) = b(p, x)$.

To each finite dimensional vector space $X$ over $\mathbb{R}$ and to each Haar measure $dx$ on $X$, one associates a representation of the symplectic space $\mathcal{G} = T^*(X) = X \times X^*$, the Schrödinger representation, defined as follows: $\mathcal{H}(X) = L^2(X, dx)$ and for $\xi = (x, p)$ and $\psi \in \mathcal{H}(X)$

$$\mathcal{W}(\xi) \psi(\cdot) = e^{i\langle \cdot, -\xi \rangle} \psi(\cdot - x).$$

Equivalently,

$$\mathcal{W}(\xi) = e^{-\frac{i}{2} \langle x, p \rangle} e^{i\langle Q, p \rangle} e^{-i \langle x, p \rangle} = e^{\frac{i}{2} \langle x, p \rangle} e^{-i \langle x, p \rangle} e^{i\langle Q, p \rangle} = e^{i \langle Q, p \rangle - \langle x, p \rangle}.$$

**Remark 1.3.** (i) If $\{\epsilon_1, \ldots, \epsilon_n\}$ is a basis in $X$ and $\{\epsilon^1, \ldots, \epsilon^n\}$ is the dual basis in $X^*$, then $\mathcal{B} = \{\epsilon_1, \ldots, \epsilon_n, \epsilon^1, \ldots, \epsilon^n\}$ is a symplectic basis and

$$P_j = -i \frac{\partial}{\partial \epsilon^j}, \quad Q^j = M_{x^j}, \quad j = 1, \ldots, n,$$
in the Schrödinger representation. Here $M_f$ is the multiplication operator by the function $f$.

(ii) The space $S = S(\mathcal{H}(X), \mathcal{W})$ of the $C^\infty$ vectors of the Schrödinger representation $(\mathcal{H}(X), \mathcal{W})$ is the space $S(X)$ of tempered test functions.

(iii) The Schrödinger representation $(\mathcal{H}(X), \mathcal{W})$ is irreducible (see Lemma 7.1.4 in [15]).

We recall that a symplectic space has only one irreducible representation (modulo unitary equivalence) and that each of its representations is a multiple of this one (see Theorem 15 in [18], Theorem C.38 in [20]).

In the rest of the section we shall work in the representation described above.

For a function $f$ on $\mathcal{S} = T^*(X)$, we denote by $f(P_{\mathcal{S}})$ the operator $F_{\mathcal{S}}M_fF_{\mathcal{S}}$ on $\mathcal{H}(X)$. Let $\tau$ be an endomorphism of $X$. For $(x, p) \in T^*(X)$ we set $\theta_{X, \tau}(x, p) = \langle \tau x, p \rangle$. We get a quadratic form on $T^*(X)$ which allows us to introduce the following definition.

**Definition 1.4.** Let $a \in S^*(X \times X^*)$. We can define the operator $a^\tau_{X}(R) = a^\tau_{X}(Q, P) : S(X) \to S^*(X)$ by

$$a^\tau_{X}(R) = \int \mathcal{W}(\xi) \hat{a}^\tau_{X}(\xi)d\xi,$$

where $\hat{a}^\tau_{X} = e^{i\theta_{X, \frac{1}{2} - \tau}(p_{\mathcal{S}})}a$. The above integral make sense if it is taken in the weak sense, i.e. for $\varphi, \psi \in S(X)$

$$\langle \varphi, a^\tau_{X}(R)\psi \rangle_{S(X), S^*(X)} = \langle \varphi, \mathcal{W}(\cdot)\psi \rangle_{S(X), S^*(X)},$$

$a \in S^*(X \times X^*)$ is called $(X, \tau)$-symbol of $a^\tau_{X}(Q, P)$ and $a^\tau_{X}$ is the Weyl symbol of this operator.

Let $a, b \in \mathcal{M}(\mathcal{S})$. Then the Weyl symbol of the operator $a^\tau_{X}(Q, P)b^\tau_{X}(Q, P)$ is $a^\tau_{X} \circ b^\tau_{X}$ while the $(X, \tau)$-symbol is $e^{-i\theta_{X, \frac{1}{2} - \tau}}a^\tau_{X} \circ b^\tau_{X}$ denoted by $a^\tau_{X} \circ b^\tau_{X}$ and called $(X, \tau)$-composition product.

We shall now check the action of $a^\tau_{X}(R)$ when $a \in S(X \times X^*)$. For $\varphi \in S(X)$ we have

$$(a^\tau_{X}(R)\varphi)(x) = \int_{X \times X^*} e^{i\theta_{X, \frac{1}{2} - \tau}(y, p)} \mathcal{W}(y, p) \varphi(x) \hat{a}(y, p)dydp$$

$$= \int_{X \times X^*} e^{i((\frac{1}{2} - \tau)y, p)} e^{i(x - \frac{1}{2}y, p)} \varphi(x - y) (F_{X} \otimes F_{X}^{-1})a(p, y)dydp$$

$$= \int_{X \times X^*} e^{i(x - \tau)y, p)} \varphi(x - y) (F_{X} \otimes F_{X}^{-1})a(p, y)dydp.$$
\[
= \int_{\mathcal{X}} \varphi(x-y)(\mathcal{F}_{X}^{-1} \otimes \text{id})(\mathcal{F}_{X} \otimes \mathcal{F}_{X}^{-1})a(x-\tau y, y)dy
\]
\[
= \int_{\mathcal{X}} \varphi(x-y)(\text{id} \otimes \mathcal{F}_{X}^{-1})a(x-\tau y, y)dy
\]
\[
= \int_{\mathcal{X}} (\text{id} \otimes \mathcal{F}_{X}^{-1})a((1-\tau)x+\tau y, x-y)\varphi(y)dy.
\]
It follows that the kernel of \(a_{X}^{\tau}(R)\) is given by
\[
(1.8) \quad \mathcal{K}_{a_{X}^{\tau}(R)} = ((\text{id} \otimes \mathcal{F}_{X}^{-1})a) \circ C_{\tau}
\]
where \(C_{\tau}\) is the map
\[
C_{\tau}: \mathcal{X} \times \mathcal{X} \to \mathcal{X} \times \mathcal{X}, \quad C_{\tau}(x,y) = ((1-\tau)x + \tau y, x-y).
\]
Let us note that (1.8) is true for \(a \in S^{\ast}(X \times X^{\ast})\) because \(S(X \times X^{\ast})\) is dense in \(S^{\ast}(X \times X^{\ast})\) in the weak topology and the mappings
\[
S^{\ast}(X \times X^{\ast}) \to S^{\ast}(X \times X^{\ast}) \to B(S(X), S^{\ast}(X)),
\]
\[
a \to a_{X}^{\tau} = e^{i\theta_{X}^{1-\tau}(P_{E})}a \to \text{Op}(a_{X}^{\tau}) = a_{X}^{\tau}(R),
\]
\[
S^{\ast}(X \times X^{\ast}) \to S^{\ast}(X \times X^{\ast}), \quad a \to ((\text{id} \otimes \mathcal{F}_{X}^{-1})a) \circ C_{\tau},
\]
are continuous if on \(S^{\ast}(X \times X^{\ast})\) we consider the \(\sigma(S^{\ast}(X \times X^{\ast}), S(X \times X^{\ast}))\) topology and on \(B(S(X), S^{\ast}(X))\) the topology defined by the family of seminorms \(\{p_{\varphi,\psi}\}_{\varphi,\psi \in S(X)}\)
\[
p_{\varphi,\psi}(A) = |\langle \varphi, A\psi \rangle|, \quad A \in B(S(X), S^{\ast}(X)).
\]
Since the equation in \(a \in S^{\ast}(X \times X^{\ast}), ((\text{id} \otimes \mathcal{F}_{X}^{-1})a) \circ C_{\tau} = \mathcal{K}\), has a unique solution for each \(\mathcal{K} \in S^{\ast}(X \times X^{\ast})\), a consequence of the kernel theorem is the fact that the map
\[
S^{\ast}(X \times X^{\ast}) \to B(S(X), S^{\ast}(X)), \quad a \to a_{X}^{\tau}(R)
\]
is linear, continuous and bijective. Hence to each \(A \in B(S(X), S^{\ast}(X))\) we associate a distribution \(a \in S^{\ast}(X \times X^{\ast})\) such that \(A = a_{X}^{\tau}(R)\). This distribution is called \((X, \tau)\)-symbol of \(A\) and we shall use the notation \(a = \sigma_{X}^{\tau}(A)\). When \(\tau = \frac{1}{2}\) then \(\sigma_{X}^{\frac{1}{2}}(A)\) is just the Weyl symbol of \(A\).

2. KATO’S IDENTITY

In this section we state and prove an extension of a formula due to T. Kato which is a basic tool in this paper. For a finite dimensional vector space \(E\) over \(\mathbb{R}\), we shall use the notation \(C_{\text{pol}}^{\infty}(E)\) for the subalgebra of \(C^{\infty}(E)\) consisting of functions whose derivatives have at most polynomial growth at infinity. We shall need the following auxiliary result.
LEMMA 2.1. Let \((\mathcal{H}, \mathcal{W})\) be a Weyl system associated to the symplectic space \(\mathcal{S}\) and let \(\varphi, \psi \in \mathcal{S}\).

(i) If \(a \in S^*(\mathcal{S})\), then the following map belongs to \(C^\infty_{\text{pol}}(\mathcal{S})\):
\[
\mathcal{S} \ni \xi \rightarrow \langle \varphi, \mathcal{W}(\xi) a(R) \mathcal{W}(-\xi) \psi \rangle_{S, S^*} \in \mathbb{C}.
\]
(ii) If \(a \in \mathcal{S}(\mathcal{S})\), then the following map belongs to \(\mathcal{S}(\mathcal{S})\):
\[
\mathcal{S} \ni \xi \rightarrow \langle \varphi, \mathcal{W}(\xi) a(R) \mathcal{W}(-\xi) \psi \rangle_{S, S^*} \in \mathbb{C}.
\]
(iii) If \(a \in S^*(\mathcal{S})\) and \(\xi \in \mathcal{S}\), then
\[
\mathcal{W}(\xi) a(R) \mathcal{W}(-\xi) = (T_\xi a)(R),
\]
where \(T_\xi a\) denote the translate by \(\xi\) of the distribution \(a\), i.e. \((T_\xi a)(\eta) = a(\eta - \xi)\).

Proof. We know that \(w = w_{\varphi, \psi} = \langle \varphi, \mathcal{W}(\cdot) \psi \rangle_{S, S^*} \in \mathcal{S}(\mathcal{S})\). Assume that \(a \in S^*(\mathcal{S})\). Then from Corollary 1.2, (1.1) and (1.2) we get
\[
\langle \varphi, \mathcal{W}(\xi) a(R) \mathcal{W}(-\xi) \psi \rangle_{S, S^*} = (a * \hat{w})(-\xi), \quad \xi \in \mathcal{S}
\]
and (i) and (ii) follows at once from this equality.

(iii) We shall prove the equality for \(a \in \mathcal{S}(\mathcal{S})\), then the general case follows by continuity. For \(a \in \mathcal{S}(\mathcal{S})\) we have
\[
\mathcal{W}(\xi) a(R) \mathcal{W}(-\xi) = \mathcal{W}(\delta_\xi) \mathcal{W}(\hat{a}) \mathcal{W}(\delta_{-\xi}) = \mathcal{W}(\delta_\xi \times \hat{a} \times \delta_{-\xi})
\]
and the proof is complete. \(\blacksquare\)

We shall now prove two extensions of some important identities due to T. Kato.

LEMMA 2.2 (Kato). Let \((\mathcal{H}, \mathcal{W})\) be a Weyl system associated to the symplectic space \(\mathcal{S}\).

(i) If \(b \in \mathcal{S}(\mathcal{S}), c \in S^*(\mathcal{S})\), then \(b * c \in S^*(\mathcal{S})\) and
\[
(b * c)(R) = \int b(\xi) \mathcal{W}(\xi) c(R) \mathcal{W}(-\xi) d\xi = \int c(\xi) \mathcal{W}(\xi) b(R) \mathcal{W}(-\xi) d\xi,
\]
for \(\xi \in \mathcal{S}\).
where the first integral is weakly absolutely convergent while the second one must be interpreted in the sense of distributions and represents the operator defined by
\[ \langle \varphi, (\int c(\xi)W(\xi)b(R)W(-\xi)d\xi)\psi \rangle_{S,S^*} = \langle \langle \varphi, W(\cdot)b(R)W(-\cdot)\psi \rangle_{S,S^*}, c \rangle_{S(\mathbb{R})^*} \]
for all \( \varphi, \psi \in S \).

(ii) Let \( h \in C_{\text{pol}}^\infty(\mathbb{R}) \). If \( b \in L^p(\mathbb{R}) \) and \( c \in L^q(\mathbb{R}) \), where \( 1 \leq p, q \leq \infty \) and \( p^{-1} + q^{-1} \geq 1 \), then \( b \ast c \in L^r(\mathbb{R}) \), \( r^{-1} = p^{-1} + q^{-1} - 1 \) and
\[
(h(P_R)(b \ast c))(R) = \int_\mathbb{R} b(\xi)W(\xi)(h(P_R)c)(R)W(-\xi)d\xi,
\]
where the integral is weakly absolutely convergent.

**Proof.** (i) Let \( \varphi, \psi \in S \). Then \( w = w_{\varphi,\psi} = \langle \varphi, W(\cdot)\psi \rangle_{S,S^*} \in S(\mathbb{R}) \). First we consider the case when \( b, c \in S(\mathbb{R}) \). Then
\[
\langle \varphi, (b \ast c)(R)\psi \rangle_{S,S^*} = \int_\mathbb{R} \hat{\varphi}(\eta)\hat{w}(\eta)d\eta = \int_\mathbb{R} \hat{b}(\xi)\hat{c}(\xi)\hat{\psi}(\xi)d\xi
\]
where in the last equality we used the formula \( (T_\xi c)(R) = W(\xi)c(R)W(-\xi) \).

Let \( c \in S^*(\mathbb{R}) \) and let \( \{c_j\} \subset S(\mathbb{R}) \) be such that \( c_j \to c \) weakly in \( S^*(\mathbb{R}) \).

The uniform boundedness principle and Peetre’s inequality imply that:
- \( \langle \varphi, W(\xi)a_j(R)W(-\xi)\psi \rangle_{S,S^*} \to \langle \varphi, W(\xi)a(R)W(-\xi)\psi \rangle_{S,S^*} \)
- There are \( M \in \mathbb{N}, C = C(M, w) > 0 \) such that
\[
|\langle \varphi, W(\xi)c_j(R)W(-\xi)\psi \rangle_{S,S^*}| \leq C(\xi)^M, \quad \xi \in \mathbb{R},
\]
where \( \langle \xi \rangle = (1 + |\xi|^2)^{1/2} \) and \( |\cdot| \) is an euclidean norm on \( \mathbb{R} \).

The general case can be deduced from the above case if we observe that:
\[
\langle \varphi, (b \ast c_j)(R)\psi \rangle_{S,S^*} \to \langle \varphi, (b \ast c)(R)\psi \rangle_{S,S^*},
\]

\[
\langle \langle \varphi, W(\cdot)b(R)W(-\cdot)\psi \rangle_{S,S^*}, c_j \rangle_{S(\mathbb{R})^*} \to \langle \langle \varphi, W(\cdot)b(R)W(-\cdot)\psi \rangle_{S,S^*}, c \rangle_{S(\mathbb{R})^*} \,
\]

\[
\text{CHATTEN - CLASS PROPERTIES OF PSEUDODIFFERENTIAL OPERATORS,}
\]
and that the sequence \( \{b(\cdot)\langle \varphi, W(\cdot)c_j(R)W(-\cdot)\psi \}_{S,S^*} \} \) converge dominated to \( b(\cdot)\langle \varphi, W(\cdot)c(R)W(-\cdot)\psi \}_{S,S^*} \).

(ii) We recall the Young inequality. If \( 1 \leq p, q \leq \infty, p^{-1} + q^{-1} \geq 1, r^{-1} = p^{-1} + q^{-1} - 1, b \in L^p(\mathcal{S}) \) and \( c \in L^q(\mathcal{S}) \), then \( b \ast c \in L^r(\mathcal{S}) \) and

\[
\|b \ast c\|_{L^r(\mathcal{S})} \leq \|b\|_{L^p(\mathcal{S})}\|c\|_{L^q(\mathcal{S})}.
\]

Let \( p', r' \geq 1 \) such that \( p^{-1} + p'^{-1} = r^{-1} + r'^{-1} = 1 \). Then \( r'^{-1} + q^{-1} \geq 1 \) and \( p'^{-1} + q'^{-1} = 1 \).

If \( b \in S(\mathcal{S}) \) and \( c \in S(\mathcal{S}) \), then \( g = h(P_{\mathcal{S}})c \in S(\mathcal{S}) \) and \( h(P_{\mathcal{S}})(b \ast c) = b \ast h(P_{\mathcal{S}})c = b \ast g \). Using (i) it follows that

\[
\langle \varphi, (b \ast g)(R)\psi \rangle_{S,S^*} = \int_{\mathcal{S}} b(\xi) \langle \varphi, W(\xi)g(R)W(-\xi)\psi \rangle_{S,S^*} d\xi.
\]

Similarly, if \( b \in L^p(\mathcal{S}) \subset S^*(\mathcal{S}) \) and \( c \in S(\mathcal{S}) \), then \( g = h(P_{\mathcal{S}})c \in S(\mathcal{S}) \) and \( h(P_{\mathcal{S}})(b \ast c) = b \ast h(P_{\mathcal{S}})c = b \ast g \). Using (i) again, it follows that

\[
\langle \varphi, (b \ast g)(R)\psi \rangle_{S,S^*} = \langle \langle \varphi, W(\cdot)g(R)W(-\cdot)\psi \rangle_{S,S^*}, b \rangle_{S(\mathcal{S}), S^*(\mathcal{S})} = \int_{\mathcal{S}} b(\xi) \langle \varphi, W(\xi)g(R)W(-\xi)\psi \rangle_{S,S^*} d\xi.
\]

Hence we proved (ii) in the case when either \( b \in S(\mathcal{S}) \) or \( c \in S(\mathcal{S}) \).

The general case can be obtained from these particular cases by an approximation argument. Observe that condition \( p^{-1} + q^{-1} \geq 1 \) implies that \( p < \infty \) or \( q < \infty \). For the convergence of the left-hand side of (2.2) we use the Young inequality and the continuity of the map \( Op \) in Corollary 1.2. To estimate right-hand side of (2.2) we use (2.1) and the Hölder and Young inequalities. We have

\[
\langle \varphi, W(\xi)(h(P_{\mathcal{S}})c)(R)W(-\xi)\psi \rangle_{S,S^*} = \langle h(P_{\mathcal{S}})c \ast \hat{w}, (\xi) \rangle = \langle c \ast \hat{\text{w}}, (\xi) \rangle
\]

and

\[
\begin{align*}
\left| \left( \int_{\mathcal{S}} b(\xi)W(\xi)(h(P_{\mathcal{S}})c)(R)W(-\xi)d\xi \right) \right|_{S,S^*} & = \left| \left( \int_{\mathcal{S}} b(\xi)(c \ast \hat{\text{w}})(\xi)d\xi \right) \right|_{L^p(\mathcal{S})} \\
& \leq \|b\|_{L^p(\mathcal{S})}\|c \ast \hat{\text{w}}\|_{L^{p'}(\mathcal{S})} \\
& \leq \|b\|_{L^p(\mathcal{S})}\|c\|_{L^q(\mathcal{S})}\|\hat{\text{w}}\|_{L^{q'}(\mathcal{S})}.
\end{align*}
\]

where \( w = w_{\varphi, \psi} = \langle \varphi, W(\cdot)\psi \rangle_{S,S^*} \in S(\mathcal{S}) \).
where the integral is weakly absolutely convergent.

If \( \mathcal{S} = T^*(X) = X \times X^* \) and \( \tau \) an endomorphism of \( X \), then we have

\[
(b * c)_{X}^{\tau}(R) = \int_{\mathcal{S}} b(\xi)W(\xi)c_{X}^{\tau}(R)W(-\xi)d\xi,
\]

where the integral is weakly absolutely convergent.

**Proof.** We take \( h = e^{i\theta_{X} \frac{1}{2} - \tau^{(\cdot)}} \) in the previous lemma. \( \square \)

### 3. Kato's Operator Calculus

In [10], H.O. Cordes noticed that the \( L^2 \)-boundedness of an operator \( a(x, D) \) in \( OPS_{0,0}^0 \) could be deduced by a synthesis of \( a(x, D) = a(R) \) from trace-class operators. In [19], T. Kato extended this argument to the general case \( OPS_{\rho,\rho}^0 \), \( 0 < \rho < 1 \), and abstracted the functional analysis involved in Cordes’ argument. This operator calculus can be extended further to investigate the Schatten-class properties of pseudodifferential operators in \( OPS_{0,0}^0 \).

Let \( \mathcal{H} \) be a separable Hilbert space. For \( 1 \leq p < \infty \), we denote by \( B_p(\mathcal{H}) \) the Schatten ideal of compact operators on \( \mathcal{H} \) whose singular values lie in \( l^p \) with the associated norm \( \| \cdot \|_p \). For \( p = \infty \), \( B_\infty(\mathcal{H}) \) is the ideal of compact operators on \( \mathcal{H} \) with \( \| \cdot \|_\infty = \| \cdot \| \).

**Definition 3.1.** Let \( T, A, B \in B(\mathcal{H}) \), \( A \geq 0, B \geq 0 \). We write

\[
T \ll (A; B) \overset{\text{def}}{\iff} |(u, Tv)|^2 \leq (u, Au)(v, Bv), \quad \text{for } u, v \in \mathcal{H}.
\]

**Lemma 3.2.** Let \( S, T, A, B \in B(\mathcal{H}) \), \( A \geq 0, B \geq 0 \). Then

(i) \( T \ll (|T^*|; |T|) \).

(ii) \( T \ll (A; B) \implies T^* \ll (B; A) \).

(iii) \( T \ll (A; B) \implies S^*TS \ll (S^*AS; S^*BS) \).

(iv) Let \( \{T_j\}, \{A_j\}, \{B_j\} \subset B(\mathcal{H}) \), \( A_j \geq 0, B_j \geq 0 \), \( j = 1, 2, \ldots \). If \( T_j \ll (A_j; B_j) \), \( j = 1, 2, \ldots \), then

\[
\sum T_j \ll \left( \sum A_j; \sum B_j \right)
\]

in the sense that whenever the series \( \sum A_j \) and \( \sum B_j \) converge in the strong sense, the same is true for the series \( \sum T_j \) and the inequality holds.

**Proof.** (i) If \( T \geq 0 \), then \( T = |T| = |T^*| \) and

\[
|(u, Tv)|^2 = |(T^{\frac{1}{2}}u, T^{\frac{1}{2}}v)|^2 \leq \|T^{\frac{1}{2}}u\|^2 \|T^{\frac{1}{2}}v\|^2 = (u, Tu)(v, Tv), \quad u, v \in \mathcal{H}.
\]

In the general case, \( T \in B(\mathcal{H}) \), we shall use the polar decomposition of \( A \). Let \( T = V|T| \) with \( |T| = (T^*T)^{\frac{1}{2}} \) and \( V \in B(\mathcal{H}) \) a partial isometry such that \( \text{Ker}V = \text{Ker}T \). Then \( V^*V \) is the projection onto the initial space of \( V \), \( (\text{Ker}V)^\perp \)
\[(= \text{Ker} T) = \overline{\text{Ran} T^*}\). It follows that \(|T^*| = V|T|V^*\ since \(T^* = V^*TV^*\), \(V|T|V^* \geq 0\ and \)
\[(V|T|V^*)^2 = V|T|V^*V|T|V^* = TV^*VT^* = TT^* = |T^*|^2.\]
Then we have
\[|(u, Tv)|^2 = |(u, V|T|v)|^2 = |(V^*u, |T|v)|^2 \leq (V^*u, |T|V^*u)(v, |T|v) = (u, |T^*|u)(v, |T|v), \ u, v \in \mathcal{H}.\]

(ii), (iii) are obvious.

(iv) Assume that there are \(A, B \in \mathcal{B}(\mathcal{H})\) such that \(A = s - \lim_{n \to \infty} \sum_{j=1}^{n} A_j, B = s - \lim_{n \to \infty} \sum_{j=1}^{n} B_j\). For \(n \geq 1\), we set \(A(n) = \sum_{j=1}^{n} A_j, B(n) = \sum_{j=1}^{n} B_j, T(n) = \sum_{j=1}^{n} T_j\) and for \(n = 0\), we set \(A(0) = B(0) = T(0) = 0\). Since \(A = s - \lim_{n \to \infty} A(n)\) by the uniform boundedness principle there is a constant \(C > 0\) such that \(|A(n)| \leq C\) for all \(n \in \mathbb{N}\).

Let \(m > n\). Then
\[|(u, (T(m) - T(n))v)| \leq \sum_{j=n+1}^{m} |(u, T_jv)| \leq \sum_{j=n+1}^{m} (u, A_ju)^\frac{1}{2}(v, B_jv)^\frac{1}{2} \leq \left( \sum_{j=n+1}^{m} (u, A_ju) \right)^\frac{1}{2} \left( \sum_{j=n+1}^{m} (v, B_jv) \right)^\frac{1}{2}\]
\[= (u, (A(m) - A(n))u)^\frac{1}{2}(v, (B(m) - B(n))v)^\frac{1}{2} \leq (u, A(m)u)^\frac{1}{2}(v, (B(m) - B(n))v)^\frac{1}{2}\]
\[\leq C^{\frac{1}{2}} u\|v\|^\frac{1}{2} \|(B(m) - B(n))v\|^\frac{1}{2}, \ u, v \in \mathcal{H}.\]
Thus
\[\|T(m)v - T(n)v\| \leq C^{\frac{1}{2}} \|v\|^\frac{1}{2} \|(B(m) - B(n))v\|^\frac{1}{2}, \ v \in \mathcal{H},\]
which implies that \(\{T(n)v\}\) is a Cauchy sequence for any \(v \in \mathcal{H}\), so there is \(T \in \mathcal{B}(\mathcal{H})\) such that \(T = s - \lim_{n \to \infty} T(n)\). By passing to the limit in the estimate
\[|(u, T(m)v)| \leq (u, A(m)u)^\frac{1}{2}(v, (B(m)v)^\frac{1}{2}), \ u, v \in \mathcal{H},\]
we conclude that \(T \ll (A; B)\).  

**Lemma 3.3.** Let \(Y\) be a measure space and \(Y \ni y \to U(y) \in \mathcal{B}(\mathcal{H})\) a weakly measurable map.

(i) Assume that there is \(C > 0\) such that
\[\int_{Y} |(\varphi, U(y)\psi)|^2 dy \leq C\|\varphi\|^2\|\psi\|^2, \ \varphi, \psi \in \mathcal{H}.\]

If \(b \in L^\infty(Y)\) and \(G \in \mathcal{B}_1(\mathcal{H})\), then the integral
\[b\{G\} = \int_{Y} b(y)U(y)^*GU(y)dy\]
is weakly absolutely convergent and defines a bounded operator such that
\[ \|b\{G\}\| \leq C\|b\|_{L^\infty}\|G\|_1. \]

(ii) Assume that there is \( C > 0 \) such that
\[ \|U(y)\| \leq C^\frac{1}{2} \quad \text{a.e. } y \in Y. \]
If \( b \in L^1(Y) \) and \( G \in B_1(H) \), then the integral
\[ b\{G\} = \int_Y b(y)U(y)^*GU(y)dy \]
is absolutely convergent and defines a trace class operator such that
\[ \|b\{G\}\|_1 \leq C\|b\|_{L^1}\|G\|_1. \]

(iii) Assume that there is \( C > 0 \) such that
\[ \int_Y |(\varphi, U(y)\psi)|^2dy \leq C\|\varphi\|^2\|\psi\|^2, \quad \varphi, \psi \in H \quad \text{and} \quad \|U(y)\| \leq C^\frac{1}{2} \quad \text{a.e. } y \in Y. \]
If \( b \in L^p(Y) \) with \( 1 \leq p < \infty \) and \( G \in B_1(H) \), then the integral
\[ b\{G\} = \int_Y b(y)U(y)^*GU(y)dy \]
is weakly absolutely convergent and defines an operator \( b\{G\} \) in \( B_p(H) \) which satisfies
\[ \|b\{G\}\|_p \leq C\|b\|_{L^p}\|G\|_1. \]

**Proof.** (i) We do this in several steps.

**Step 1.** Suppose \( G \geq 0 \). Write \( G = \sum_{j=1}^\infty \lambda_j|\varphi_j\rangle\langle \varphi_j| = \sum_{j=1}^\infty \lambda_j(\varphi_j, \cdot)\varphi_j \) with \( \{\varphi_j\} \)
an orthonormal basis of \( H \), \( \lambda_j \geq 0 \), \( \text{Tr}(G) = \sum_{j=1}^\infty \lambda_j = \sum_{j=1}^\infty |\lambda_j| = \|G\|_1 \). Then
\[ (U(y)\varphi, GU(y)\psi) = \sum_{j=1}^\infty \lambda_j(U(y)\varphi, \varphi_j)(\varphi_j, U(y)\psi), \quad y \in Y, \varphi, \psi \in H \]
and
\[ |(\varphi, B\psi)| = \left| \int_Y b(y)(U(y)\varphi, GU(y)\psi)dy \right| \leq \|b\|_{L^\infty} \sum_{j=1}^\infty \lambda_j \left( \int_Y |(U(y)\varphi, \varphi_j)|^2dy \right)^\frac{1}{2} \left( \int_Y |(\varphi_j, U(y)\psi)|^2dy \right)^\frac{1}{2} \]
\[ \leq \|b\|_{L^\infty} \sum_{j=1}^\infty \lambda_j C^\frac{1}{2} \|\varphi\||\varphi_j||\psi||^2 \]
\[ \leq C\|b\|_{L^\infty} \sum_{j=1}^\infty \lambda_j^\frac{1}{2}\|\varphi\|^2|\varphi_j||\psi||^2 \]
\[ = C\|b\|_{L^\infty} \left( \sum_{j=1}^\infty \lambda_j \right)^\frac{1}{2}\|\varphi\||\psi||^2 = C\|b\|_{L^\infty} \|G\|_1\|\varphi\||\psi||, \quad \varphi, \psi \in H. \]
Step 2. The general case, $G \in B_1(\mathcal{H})$, can be reduced to the above case by using Lemma 3.2 and the Schwarz inequality to evaluate the integrand. We have $U(y)^*GU(y) \ll (U(y)^*|G^*|U(y); U(y)^*|G|U(y))$ and

$$|b(y)(U(y)\varphi, GU(y)\psi)| \leq \|b\|_{L^\infty}(U(y)\varphi, |G^*|U(y)\varphi)^{\frac{1}{2}}(U(y)\psi, |G|U(y)\psi)^{\frac{1}{2}}.$$ 

Thus

$$|(\varphi, b\{G\}\psi)| = \left| \int_Y b(y)(U(y)\varphi, GU(y)\psi)dy \right|$$

$$\leq \|b\|_{L^\infty} \left( \int_Y (U(y)\varphi, |G^*|U(y)\varphi)dy \right)^{\frac{1}{2}} \left( \int_Y (U(y)\psi, |G|U(y)\psi)dy \right)^{\frac{1}{2}}$$

$$\leq \|b\|_{L^\infty} C_2 \|G^*\|_1^{\frac{1}{2}} \|\varphi\|_1 \|G\|_1^{\frac{1}{2}} \|\psi\|_1 = C \|b\|_{L^\infty} \|G\|_1 \|\varphi\|_1 \|\psi\|_1, \quad \varphi, \psi \in \mathcal{H}.$$

(ii) Let $\{\psi_\alpha\}_{\alpha \in I}$ and $\{\varphi_\alpha\}_{\alpha \in I}$ be two orthonormal systems in $\mathcal{H}$. Then we have

$$\sum_{\alpha \in I} |(\psi_\alpha, b\{G\}\varphi_\alpha)| \leq \sum_{\alpha \in I} \int_Y |b(y)||U(y)^*GU(y)\varphi_\alpha||dy$$

$$= \int_Y |b(y)||\sum_{\alpha \in I} |(\psi_\alpha, U(y)^*GU(y)\varphi_\alpha)||dy$$

$$\leq \int_Y |b(y)||U(y)^*GU(y)||_1^2dy \leq \int_Y |b(y)||U(y)||^2||G||_1dy$$

$$\leq C\|G\|_1 \int_Y |b(y)||dy = C\|b\|_{L^1}^2\|G\|_1 < +\infty.$$ 

It follows that

$$\sum_{\alpha \in I} |(\psi_\alpha, b\{G\}\varphi_\alpha)| \leq C\|b\|_{L^1} \|G\|_1 < +\infty$$

for any orthonormal systems $\{\psi_\alpha\}_{\alpha \in I}, \{\varphi_\alpha\}_{\alpha \in I}$. We obtain that $b\{G\} \in B_1(\mathcal{H})$ and

$$\|b\{G\}\|_1 \leq C\|b\|_{L^1} \|G\|_1.$$

See p. 246–247, Theorems 3 and 4 in [3].

(iii) is a consequence of (i) and (ii) since the $p$-Schatten classes interpolate like $L^p$-spaces: $[B_1(\mathcal{H}), B_\infty(\mathcal{H})]_\theta = [B_1(\mathcal{H}), B(\mathcal{H})]_\theta = B_\frac{1}{1-\theta}(\mathcal{H}), 0 < \theta < 1,$ (see p. 147 in [24]).
We now give a direct elementary proof of this part. Let \( \{ \psi_{\alpha} \}_{\alpha \in I} \) and \( \{ \varphi_{\alpha} \}_{\alpha \in I} \) be two orthonormal systems in \( \mathcal{H} \). Then by Hölder inequality and part (i):

\[
\sum_{\alpha \in I} |(\psi_{\alpha}, b\{G\} \varphi_{\alpha})|^p \\
\leq \sum_{\alpha \in I} \left( \int_Y |b(y)| |(\psi_{\alpha}, U(y)^* GU(y) \varphi_{\alpha})| dy \right)^p \\
\leq \sum_{\alpha \in I} \left( \int_Y |(\psi_{\alpha}, U(y)^* GU(y) \varphi_{\alpha})| dy \right)^{\frac{p}{q}} \int_Y |b(y)|^p |(\psi_{\alpha}, U(y)^* GU(y) \varphi_{\alpha})| dy \\
\leq (C\|G\|_1)^{\frac{p}{q}} \int_Y |b(y)|^p \sum_{\alpha \in I} |(\psi_{\alpha}, U(y)^* GU(y) \varphi_{\alpha})| dy \\
\leq (C\|G\|_1)^{\frac{p}{q}} \int_Y |b(y)|^p \|U(y)^* GU(y)\|_1 dy \\
\leq (C\|G\|_1)^{\frac{p}{q} + 1} \int_Y |b(y)|^p dy = (C\|G\|_1)^{\frac{p}{q} + 1} \|b\|_{L^p}^p < +\infty.
\]

Hence

\[
\left( \sum_{\alpha \in I} |(\psi_{\alpha}, b\{G\} \varphi_{\alpha})|^p \right)^{\frac{1}{p}} \leq C\|G\|_1 \|b\|_{L^p} < +\infty
\]

for any orthonormal systems \( \{ \psi_{\alpha} \}_{\alpha \in I} \) and \( \{ \varphi_{\alpha} \}_{\alpha \in I} \). Thus, by using Proposition 2.6 of Simon [22], we conclude that \( b\{G\} \in \mathcal{B}_p(\mathcal{H}) \) and

\[
\|b\{G\}\|_p \leq C\|b\|_{L^p}\|G\|_1. \quad \blacksquare
\]

**Lemma 3.4.** Let \( (\mathcal{H}, \mathcal{W}) \) be an irreducible Weyl system associated to the symplectic space \( \mathcal{G} \). Then for any \( \varphi, \psi \) in \( \mathcal{H} \) the map

\[
\mathcal{G} \ni \xi \to (\varphi, \mathcal{W}(\xi) \psi) \in \mathbb{C},
\]

belongs to \( L^2(\mathcal{G}) \cap C_\infty(\mathcal{G}) \) and

\[
\int_{\mathcal{G}} |(\varphi, \mathcal{W}(\xi) \psi)|^2 d\xi = \|\varphi\|^2 \|\psi\|^2 \quad \text{and} \quad \|(\varphi, \mathcal{W}(\cdot) \psi)\|_{L^\infty} \leq \|\varphi\| \|\psi\|.
\]

For \( \varphi, \psi, \varphi', \psi' \in \mathcal{H} \) we have

\[
\int_{\mathcal{G}} (\varphi', \mathcal{W}(\xi) \psi')(\varphi, \mathcal{W}(\xi) \psi) d\xi = (\varphi, \varphi')(\psi', \psi).
\]

**Proof.** Since (modulo a unitary equivalence) a symplectic space has only one irreducible representation, we may assume that \( \mathcal{G} = T^*(X) \) and that \( (\mathcal{H}, \mathcal{W}) \) is the Schrödinger representation \( (\mathcal{H}(X), \mathcal{W}) \). If \( \varphi, \psi \in \mathcal{H}(X) \equiv L^2(X, dx) \), then

\[
(\varphi, \mathcal{W}(x, p) \psi) = \int_X \overline{\varphi(y)} e^{i(y - \frac{x}{2}, p)} \psi(y - x) dy = e^{-\frac{i}{2}(x, p)} \overline{\mathcal{F}_X(\overline{\varphi} T_x \psi)(p)},
\]
where $T_x \psi(\cdot) = \psi(\cdot - x)$, and
\[
\int |(\varphi, \mathcal{W}(\zeta) \psi)|^2 d\zeta
\]
\[
= \int \int |(\varphi, \mathcal{W}(x, p) \psi)|^2 dxdp = \int \mathcal{F}_X(\varphi T_x \psi)|^2_{L^2(X^*)} dX = \int |\varphi T_x \psi|^2_{L^2(X^*)} dX
\]
\[
= \int \left( \int |\varphi(y) \psi(y - x)|^2 dy \right) dx = \int \left( \int |\varphi(y) \psi(y - x)|^2 dx \right) dy = \| \varphi \|^2 \| \psi \|^2.
\]
Let $\{\varphi_n\}, \{\psi_n\} \subset S(X)$ such that $\varphi_n \to \varphi$ and $\psi_n \to \psi$ in $\mathcal{H}(X)$. Since
\[
\{(\varphi_n, \mathcal{W}(\cdot) \psi_n)\} \subset S(\mathfrak{G})
\]
and
\[
\|(\varphi_n, \mathcal{W}(\cdot) \psi_n) - (\varphi, \mathcal{W}(\cdot) \psi)\|_\infty = \|(\varphi_n - \varphi, \mathcal{W}(\cdot) \psi_n) + (\varphi, \mathcal{W}(\cdot) (\psi_n - \psi))\|
\leq \|\varphi_n - \varphi\| \|\psi_n\| + \|\varphi\| \|\psi_n - \psi\|,
\]
\(n \in \mathbb{N},\)

it follows that $(\varphi_n, \mathcal{W}(\cdot) \psi_n) \to (\varphi, \mathcal{W}(\cdot) \psi)$ in $C_\infty(\mathfrak{G})$ as $n \to \infty$. Hence
\[
(\varphi, \mathcal{W}(\cdot) \psi) \in L^2(\mathfrak{G}) \cap C_\infty(\mathfrak{G}), \quad \|(\varphi, \mathcal{W}(\cdot) \psi)\|^2_{L^2(\mathfrak{G})} = \|\varphi\|^2 \|\psi\|^2
\]
and $\|(\varphi, \mathcal{W}(\cdot) \psi)\|_\infty \leq \|\varphi\| \|\psi\|$. The last formula is a consequence of polarization identity. \[\Box\]

**Theorem 3.5.** Let $(\mathcal{H}, \mathcal{W})$ be an irreducible Weyl system associated to the symplectic space $\mathfrak{G}$.  

(a) If $b \in L^\infty(\mathfrak{G})$ and $G \in \mathcal{B}_1(\mathcal{H})$, then the integral
\[
b\{G\} = \int_{\mathfrak{G}} b(\zeta) \mathcal{W}(\zeta) GW(-\zeta) d\zeta
\]
is weakly absolutely convergent and defines a bounded operator such that
\[
\|b\{G\}\| \leq \|b\|_{L^\infty(\mathfrak{G})} \|G\|_1.
\]
Moreover, if $b$ vanishes at $\infty$ in the sense that for any $\varepsilon > 0$ there is a compact subset $K$ of $\mathfrak{G}$ such that
\[
\|b\|_{L^\infty(\mathfrak{G}\setminus K)} \leq \varepsilon,
\]
then $b\{G\}$ is a compact operator.

The mapping $(b, G) \to b\{G\}$ has the following properties: 

(i) $b \geq 0, G \geq 0 \rightarrow b\{G\} \geq 0$;

(ii) $1\{G\} = \text{Tr}(G) i d_\mathcal{H}$;

(iii) $(b_1 b_2)\{G\} \leq \left(\|b_1\|^2\{G^*\}; \|b_2\|^2\{G\}\right)$.

(b) If $b \in L^p(\mathfrak{G})$ with $1 \leq p < \infty$ and $G \in \mathcal{B}_1(\mathcal{H})$, then the integral
\[
b\{G\} = \int_{\mathfrak{G}} b(\zeta) \mathcal{W}(\zeta) GW(-\zeta) d\zeta
\]
is weakly absolutely convergent and defines an operator \( b\{ G \} \) in \( \mathcal{B}_p(\mathcal{H}) \) which satisfies
\[
\| b\{ G \} \|_p \leq \| b\|_{L^p} \| G \|_1.
\]

Proof. (ii) Let \( G = |\varphi)(\psi| = (\psi, \cdot)\varphi, \varphi, \psi \in \mathcal{H} \). Then
\[
\mathcal{W}(\xi)G\mathcal{W}(-\xi) = |\mathcal{W}(\xi)\varphi)(\mathcal{W}(\xi)\psi|
\]
and
\[
\int_{\mathfrak{S}} (u, \mathcal{W}(\xi)\varphi)(\mathcal{W}(\xi)\psi, v) d\xi = (\psi, \varphi)(u, v) = (u, Tr(|\varphi)(\psi|v)).
\]
So the equality holds for operators of rank 1. Next we extend this equality by linearity and continuity.

(iii) We have \( \mathcal{W}(\xi)G\mathcal{W}(-\xi) \ll (\mathcal{W}(\xi)|G^*|\mathcal{W}(-\xi); \mathcal{W}(\xi)|G|\mathcal{W}(-\xi)) \) which gives
\[
|b_1(\xi)b_2(\xi)(\varphi, \mathcal{W}(\xi)G\mathcal{W}(-\xi)\psi)| \leq (|b_1(\xi)|^2(\varphi, \mathcal{W}(\xi)|G^*|\mathcal{W}(-\xi)\varphi))^{\frac{1}{2}}(|b_2(\xi)|^2(\psi, \mathcal{W}(\xi)|G|\mathcal{W}(-\xi)\psi))^{\frac{1}{2}}.
\]
Now we just use the Schwarz inequality to conclude that (iii) is true. □

4. SOME SPECIAL SYMBOLS

To apply Theorem 3.5 in combination with Corollary 2.3, we need some special symbols \( g \) for which \( g_X^\tau(R) \) has an extension \( G \in \mathcal{B}_1(\mathcal{H}) \). Such symbols have been constructed by Cordes [10].

Let \((E, |\cdot|)\) be an euclidean space. If \( x \in E \), we set \( \langle x \rangle = (1 + |x|^2)^{\frac{1}{2}} \). Sometimes, in order to avoid confusions, we shall add a subscript specifying the space, e.g. \( \langle \cdot, \cdot \rangle_E, |\cdot|_E \) or \( \langle \cdot \rangle_E \).

Let \( \mathfrak{S} = T^*(X) \) with the standard symplectic structure and \( (X, |\cdot|_X) \) an euclidean space. We shall work in the Schrödinger representation \((\mathcal{H}(X), \mathcal{W})\).

Let \( s > 0 \). Then \( (1 - \triangle_X)^{\frac{s}{2}} = \mathcal{F}_X^{-1}M_{\langle \cdot \rangle_X^s} \mathcal{F}_X = a_X^s(Q, P) \), where \( a(x, p) = \langle p \rangle_X^s, (x, p) \in X \times X^* \). Let \( \psi_s = \psi_s^X \) be the unique solution within \( S^s(X) \) for
\[
(1 - \triangle_X)^{\frac{s}{2}} \psi_s = \delta.
\]
Similarly, \( (1 - \triangle_{X^*})^{\frac{s}{2}} = \mathcal{F}_X^{-1}M_{\langle \cdot \rangle_X^s} \mathcal{F}_{X^*} \). We shall denote by \( \chi_s = \chi_s^{X^*} \) the unique solution in \( S^s(X^*) \) for
\[
(1 - \triangle_{X^*})^{\frac{s}{2}} \chi_s = \delta.
\]
Let \( n = \dim X \). We recall that \( \psi_s \in L^1(X) \cap C^\infty(X \setminus \{0\}) \), \( \psi_s(x) \) and its derivatives decay exponentially as \( |x| \to \infty \), and that \( \partial^\alpha \psi_s(x) = O(1 + |x|^{s-n-|\alpha|}) \) as \( |x| \to 0 \), except when \( s - n - |\alpha| = 0 \) in which case we have \( \partial^\alpha \psi_s(x) = O(1 + \log \left(\frac{1}{|x|}\right)) \). \( \chi_s \) has similar properties.
LEMMA 4.1 (Cordes [10]). Let $t, s > \frac{n}{2}$ and
\[ g : X \times X^* \to \mathbb{R}, \quad g(x, p) = \psi_t(x)\chi_s(p), \quad (x, p) \in X \times X^*, \]
i.e. $g = \psi_t \otimes \chi_s$. Then $g^0_X(Q, P)$ has an extension $G \in \mathcal{B}_1(\mathcal{H}(X))$. Moreover, for any $\alpha \in \mathbb{N}^n, g^{\alpha}_X(Q, P)P_\alpha$ has an extension $G_\alpha \in \mathcal{B}_1(\mathcal{H}(X))$.

LEMMA 4.2 (Cordes [10]). Let $t, s > n$ and
\[ g : X \times X^* \to \mathbb{R}, \quad g(x, p) = \psi_t(x)\chi_s(p), \quad (x, p) \in X \times X^*, \]
i.e. $g = \psi_t \otimes \chi_s$. If $\tau \in \mathbb{R} \equiv \mathbb{R} \cdot 1_X$, then $g^\tau_X(Q, P)$ has an extension in $\mathcal{B}_1(\mathcal{H}(X))$ denoted also by $g^\tau_X(Q, P)$. The following mapping is continuous:
\[ \mathbb{R} \ni \tau \to g^\tau_X(Q, P) \in \mathcal{B}_1(\mathcal{H}(X)). \]

For the proof of these two lemmas see [10].

Let $X = X_1 \oplus \cdots \oplus X_k$ be an orthogonal decomposition with the canonical injections $j_1 : X_1 \hookrightarrow X, \ldots, j_k : X_k \hookrightarrow X$ and $\pi_1 : X \to X_1, \ldots, \pi_k : X \to X_k$ the orthogonal projections. Then $X^* = X_1^* \oplus \cdots \oplus X_k^*$ is an orthogonal decomposition with $\pi^*_1 : X_1^* \hookrightarrow X^*, \ldots, \pi^*_k : X_k^* \hookrightarrow X^*$ as the canonical injections and $j^*_1 : X^* \to X_1^*, \ldots, j^*_k : X^* \to X_k^*$ as the orthogonal projections.

The mapping
\[ J : X \times X^* \to (X_1 \times X_1^*) \times \cdots \times (X_k \times X_k^*), \]
\[ J(x, p) = ((\pi_1 x, j^*_1 p), \ldots, (\pi_k x, j^*_k p)), \]
is an isometry with the inverse given by
\[ J^{-1} : (X_1 \times X_1^*) \times \cdots \times (X_k \times X_k^*) \to X \times X^*, \]
\[ J^{-1}((x_1, p_1), \ldots, (x_k, p_k)) = (x_1 + \cdots + x_k, p_1 + \cdots + p_k). \]

If $a_1 \in S^*(X_1 \times X_1^*)$, $a_k \in S^*(X_k \times X_k^*)$, then
\[ a = (a_1 \otimes \cdots \otimes a_k) \circ J \in S^*(X \times X^*) \]
and
\[ a^\tau_X(Q, P) = (a_1)^{X_1}(Q_1, P_1) \otimes \cdots \otimes (a_k)^{X_k}(Q_k, P_k), \]
where $Q = (Q_1, \ldots, Q_k)$ and $P = (P_1, \ldots, P_k)$.

Let $s_1, \ldots, s_k > 0$ and $\mathbf{s} = (s_1, \ldots, s_k)$. Then $\psi^X_{\mathbf{s}} = \psi^X_{s_1} \otimes \cdots \otimes \psi^X_{s_k} \in S^*(X)$ is the unique solution within $S^*(X)$ for
\[ (1 - \Delta_{X_1})^{\frac{s_1}{2}} \otimes \cdots \otimes (1 - \Delta_{X_k})^{\frac{s_k}{2}} \psi = \delta. \]

Similarly, $\chi^X_{\mathbf{s}} = \chi^X_{s_1} \otimes \cdots \otimes \chi^X_{s_k} \in S^*(X^*)$ is the unique solution within $S^*(X^*)$ for
\[ (1 - \Delta_{X_1}^*)^{\frac{s_1}{2}} \otimes \cdots \otimes (1 - \Delta_{X_k}^*)^{\frac{s_k}{2}} \chi^X = \delta. \]
For \( t = (t_1, \ldots, t_k), t_1, \ldots, t_k > 0 \) and \( s = (s_1, \ldots, s_k), s_1, \ldots, s_k > 0 \) we introduce the distributions \( g = g_{t,s} = \psi_t \otimes \chi_s \in S^*(X \times X^*) \) and \( g_j = g_{t_j,s_j} = \psi_{t_j} \otimes \chi_{s_j} \in S^*(X_j \times X_j^*) \), for \( j = 1, \ldots, k \). Then \( g = (g_1 \otimes \cdots \otimes g_k) \circ f \)
and
\[
g^X_{t}(Q,P) = (g_1)^X_{t_1}(Q_1,P_1) \otimes \cdots \otimes (g_k)^X_{t_k}(Q_k,P_k).
\]

We recall that the mapping
\[
B_1(\mathcal{H}) \times B_1(\mathcal{H}) \to B_1(\mathcal{H} \otimes \mathcal{H}), \quad (A,B) \to A \otimes B,
\]
is well defined and
\[
\|A \otimes B\|_1 = \|A\|_1 \|B\|_1, \quad (A,B) \in B_1(\mathcal{H}) \times B_1(\mathcal{H}).
\]

**Corollary 4.3.** Let \( t = (t_1, \ldots, t_k), s = (s_1, \ldots, s_k) \) such that \( t_1, s_1 > \frac{\dim X_1}{4}, \ldots, t_k, s_k > \frac{\dim X_k}{4} \).

(i) Let \( g = g_{2t,2s} = \psi_{2t} \otimes \chi_{2s} \in S^*(X \times X^*) \). Then \( g^0_X(Q,P) \) has an extension \( G \in B_1(\mathcal{H}(X)) \). Moreover, for any \( \alpha \in \mathbb{N}^n \), \( g^0_X(Q,P)P_\alpha \) has an extension \( G_\alpha \in B_1(\mathcal{H}(X)) \).

(ii) Let \( g = g_{4t,4s} = \psi_{4t} \otimes \chi_{4s} \in S^*(X \times X^*) \). If \( \tau \in \mathbb{R} \ni \tau \cdot 1_X \), then \( g^\tau_X(Q,P) \) has an extension in \( B_1(\mathcal{H}(X)) \) denoted also by \( g^\tau_X(Q,P) \). The following mapping is continuous:
\[
\mathbb{R} \ni \tau \to g^\tau_X(Q,P) \in B_1(\mathcal{H}(X)).
\]

5. SCHATTEN-CLASS PROPERTIES OF PSEUDODIFFERENTIAL OPERATORS

We are now able to consider \( L^2 \)-boundedness and Schatten-class properties of certain pseudodifferential operators. In the notation of the previous section, \( \mathcal{S} \) will be \( T^*(X) \) with the standard symplectic structure and \( (X, | \cdot |_X) \) an euclidean space, \( (\mathcal{H}(X), \mathcal{W}) \) will be the Schrödinger representation associated to the symplectic space \( \mathcal{S} \). Let \( X = X_1 \oplus \cdots \oplus X_k \) be an orthogonal decomposition and \( X^* = X_1^* \oplus \cdots \oplus X_k^* \) be the dual orthogonal decomposition.

**Theorem 5.1.** Let \( a \in S^*(\mathcal{S}) \) and \( 1 \leq p < \infty \).

(i) Assume that there are \( t = (t_1, \ldots, t_k), s = (s_1, \ldots, s_k) \) such that \( t_1, s_1 > \frac{\dim X_1}{4}, \ldots, t_k, s_k > \frac{\dim X_k}{4} \) and that
\[
b = (1 - \Delta X_1)^{t_1} \otimes \cdots \otimes (1 - \Delta X_k)^{t_k} \otimes (1 - \Delta X_1^*)^{s_1} \otimes \cdots \otimes (1 - \Delta X_k^*)^{s_k} \]
is in \( L^p(\mathcal{S}) \). Then \( a^0_X(Q,P) \) has an extension in \( B_p(\mathcal{H}(X)) \) denoted also by \( a^0_X(Q,P) \) and there is \( C > 0 \) such that
\[
\|a^0_X(Q,P)\|_p \leq C \|b\|_{L^p(\mathcal{S})}.
\]

(ii) Assume that there are \( t = (t_1, \ldots, t_k), s = (s_1, \ldots, s_k) \) such that \( t_1, s_1 > \frac{\dim X_1}{4}, \ldots, t_k, s_k > \frac{\dim X_k}{4} \) and that
\[
c = (1 - \Delta X_1)^{2t_1} \otimes \cdots \otimes (1 - \Delta X_k)^{2t_k} \otimes (1 - \Delta X_1^*)^{2s_1} \otimes \cdots \otimes (1 - \Delta X_k^*)^{2s_k} \]
is in $L^p(\mathcal{G})$. If $\tau \in \mathbb{R} \equiv \mathbb{R} \cdot 1_X$, then $a^\tau_X(Q,P)$ has an extension in $B_p(H(X))$ denoted also by $a^\tau_X(Q,P)$. The mapping

$$\mathbb{R} \ni \tau \mapsto a^\tau_X(Q,P) \in B_p(H(X))$$

is continuous and for any $K$ a compact subset of $\mathbb{R}$, there is $C_K > 0$ such that, for any $\tau \in K$,

$$\|a^\tau_X(Q,P)\|_p \leq C_K\|c\|_{L^p(\mathcal{G})}.$$  

\textbf{Proof.} (i) For $t = (t_1, \ldots, t_k)$ and $s = (s_1, \ldots, s_k)$ we consider $g = g_{2t,2s} = \psi_{2t} \otimes \chi_{2s} \in S^*(X \times X^*)$. Recall that we use the notations of the previous section. Then $g \in L^1(X \times X^*)$ and $a = b \ast g$ because

$$(1 - \triangle_X)^{t_1} \otimes \cdots \otimes (1 - \triangle_X)^{t_k} \otimes (1 - \triangle_X)^{s_1} \otimes \cdots \otimes (1 - \triangle_X)^{s_k} g = \delta.$$  

It follows from Corollary 4.3 (i) that $g^0_X(Q,P) \subset G \in B_1(H(X))$. On the other hand, by using Corollary 2.3 and Theorem 3.5 (ii) we conclude that

$$a^0_X(Q,P) = (b \ast g)^0_X(Q,P) \subset b\{G\} \in B_p(H(X))$$

and we have

$$\|a^0_X(Q,P)\|_p \leq \|b\|_{L^p(\mathcal{G})}\|g^0_X(Q,P)\|_1.$$  

(ii) The proof of this point is essentially the same. For $t = (t_1, \ldots, t_k)$ and $s = (s_1, \ldots, s_k)$ we set $g = g_{4t,4s} = \psi_{4t} \otimes \chi_{4s} \in S^*(X \times X^*)$. Then $g \in L^1(X \times X^*)$ and $a = c \ast g$. Now we apply Corollary 4.3 (ii) to obtain that $g^\tau_X(Q,P)$ has an extension in $B_1(H(X))$ and that the mapping

$$\mathbb{R} \ni \tau \mapsto g^\tau_X(Q,P) \in B_1(H(X))$$

is continuous. From Corollary 2.3 and Theorem 3.5 (ii) it follows that

$$a^\tau_X(Q,P) = (c \ast g)^\tau_X(Q,P) \subset c\{g^\tau_X(Q,P)\} \in B_p(H(X))$$

and we have

$$\|a^\tau_X(Q,P)\|_p \leq \|c\|_{L^p(\mathcal{G})}\|g^\tau_X(Q,P)\|_1,$$

$$\|a^\tau_X(Q,P) - a^\tau'(Q,P)\|_p \leq \|c\|_{L^p(\mathcal{G})}\|g^\tau_X(Q,P) - g^\tau'(Q,P)\|_1,$$

for any $\tau, \tau' \in \mathbb{R}$. Hence the mapping

$$\mathbb{R} \ni \tau \mapsto a^\tau_X(Q,P) \in B_p(H(X))$$

is continuous and for any $K$ a compact subset of $\mathbb{R}$, there is $C_K > 0$ such that

$$\|a^\tau_X(Q,P)\|_p \leq \sup\{\|g^\tau_X(Q,P)\|_1 : \tau \in K\}\|c\|_{L^p(\mathcal{G})},$$

for any $\tau \in K$.  

If we replace the $L^p$-conditions by $L^\infty$-conditions, then we obtain the theorems due to Cordes.
THEOREM 5.2. Let \( a \in S^*(\mathcal{G}) \).

(i) Assume that there are \( t = (t_1, \ldots, t_k) \), \( s = (s_1, \ldots, s_k) \) such that \( t_1, s_1 > \frac{\dim X_1}{4} \), \( \ldots \), \( t_k, s_k > \frac{\dim X_k}{4} \) and that

\[
\begin{align*}
   b = (1 - \Delta X_1)^{t_1} \otimes \cdots \otimes (1 - \Delta X_k)^{t_k} \otimes \cdots \otimes (1 - \Delta X_1)^{s_1} \otimes \cdots \otimes (1 - \Delta X_k)^{s_k} a
\end{align*}
\]

is in \( L^\infty(\mathcal{G}) \). Then \( a_{X}^0(Q, P) \) has an extension in \( \mathcal{B}(\mathcal{H}(X)) \) denoted also by \( a_{X}^0(Q, P) \) and there is \( C > 0 \) such that

\[
\|a_{X}^0(Q, P)\|_{\mathcal{B}(\mathcal{H}(X))} \leq C \|b\|_{L^\infty(\mathcal{G})}.
\]

(ii) Assume that there are \( t = (t_1, \ldots, t_k) \), \( s = (s_1, \ldots, s_k) \) such that \( t_1, s_1 > \frac{\dim X_1}{4} \), \( \ldots \), \( t_k, s_k > \frac{\dim X_k}{4} \) and that

\[
\begin{align*}
   c = (1 - \Delta X_1)^{2t_1} \cdots \otimes (1 - \Delta X_1)^{2t_k} \otimes \cdots \otimes (1 - \Delta X_1)^{2s_1} \cdots \otimes (1 - \Delta X_1)^{2s_k} a
\end{align*}
\]

is in \( L^\infty(\mathcal{G}) \). If \( \tau \in \mathbb{R} = \mathbb{R} \cdot 1_X \), then \( a_{X}^\tau(Q, P) \) has an extension in \( \mathcal{B}(\mathcal{H}(X)) \) denoted also by \( a_{X}^\tau(Q, P) \). The mapping

\[
\mathbb{R} \ni \tau \rightarrow a_{X}^\tau(Q, P) \in \mathcal{B}(\mathcal{H}(X))
\]

is continuous and for any \( K \) a compact subset of \( \mathbb{R} \), there is \( C_K > 0 \) such that, for any \( \tau \in K \),

\[
\|a_{X}^\tau(Q, P)\|_{\mathcal{B}(\mathcal{H}(X))} \leq C_K \|c\|_{L^\infty(\mathcal{G})}.
\]

The proof of this theorem is essentially the same, the only change we must do is the reference to part (i) instead of part (ii) in Theorem 3.5.

Let \( (X, \cdot \cdot \cdot \cdot \cdot X) \) be an euclidean space in which we choose an orthonormal basis. Let \( m = \lfloor \frac{\dim X}{2} \rfloor + 1 \) and \( 1 \leq p \leq \infty \). Then there are \( \tau = \tau(\dim X) > \frac{\dim X}{4} \) and \( \gamma > 0 \) such that

\[
\|(1 - \Delta X)^\tau \|_{L^p} \leq \gamma \| \varphi \|_{p,m}, \quad \varphi \in \mathcal{S}(X),
\]

where \( \| \cdot \|_{p,m} \) is the norm on \( \mathcal{S}(X) \) defined by

\[
\| \varphi \|_{p,m} = \max \{ \| P^\beta \varphi \|_{L^p} : |\beta| \leq m \} = \max \{ \| \partial^\beta \varphi \|_{L^p} : |\beta| \leq m \}, \quad \varphi \in \mathcal{S}(X).
\]

For the proof see p. 118–119 in [10].

Let \( X = X_1 \oplus \cdots \oplus X_k \) be an orthogonal decomposition, \( m_1 = \lfloor \frac{\dim X_1}{2} \rfloor + 1 \), \( \ldots \), \( m_k = \lfloor \frac{\dim X_k}{2} \rfloor + 1 \) and \( 1 \leq p \leq \infty \). By an induction argument we conclude that there are \( \tau_1 > \frac{\dim X_1}{4}, \ldots, \tau_k > \frac{\dim X_k}{4} \) and \( \gamma > 0 \) such that

\[
(1 - \Delta X_1)^{\tau_1} \cdots \otimes (1 - \Delta X_k)^{\tau_k} \varphi \leq \gamma \| \varphi \|_{p,m_1,\ldots,m_k}, \quad \varphi \in \mathcal{S}(X),
\]

where \( \| \cdot \|_{p,m_1,\ldots,m_k} \) is the norm on \( \mathcal{S}(X) \) defined by

\[
\| \varphi \|_{p,m_1,\ldots,m_k} = \max \{ \| P_{X_1}^{\beta_1} \cdots P_{X_k}^{\beta_k} \varphi \|_{L^p} : |\beta_1| \leq m_1, \ldots, |\beta_k| \leq m_k \}
\]

\[
= \max \{ \| (\partial_{X_1})^{\beta_1} \cdots (\partial_{X_k})^{\beta_k} \varphi \|_{L^p} : |\beta_1| \leq m_1, \ldots, |\beta_k| \leq m_k \}, \quad \varphi \in \mathcal{S}(X).
\]
We introduce the space
\[ C_{m_1, \ldots, m_k}^p = \{ u \in S^*(X) : P_{X_{m_1}}^1 \cdots P_{X_{m_k}}^p u \in L^p(X), |\beta_1| \leq m_1, \ldots, |\beta_k| \leq m_k \} \]
\[ = \{ u \in S^*(X) : (\partial X_{m_1})^{\beta_1} \cdots (\partial X_{m_k})^{\beta_k} u \in L^p(X), |\beta_1| \leq m_1, \ldots, |\beta_k| \leq m_k \}, \]
on which the norm \( \| \cdot \|_{p,m_{1},\ldots,m_{k}} \) has a natural extension. With this norm \( C_{m_1,\ldots,m_k}^p \) becomes a Banach space.

**Lemma 5.3.** There are \( \tau_1 > \frac{\dim X_1}{4}, \ldots, \tau_k > \frac{\dim X_k}{4} \) and \( \gamma > 0 \) such that if \( u \in C_{m_1,\ldots,m_k}^p \), then \( (1 - \Delta X_1)^{\tau_1} \cdots (1 - \Delta X_k)^{\tau_k} u \in L^p(X) \) and
\[ \| (1 - \Delta X_1)^{\tau_1} \cdots (1 - \Delta X_k)^{\tau_k} u \|_{L^p(X)} \leq \gamma \| u \|_{p,m_{1},\ldots,m_{k}}. \]

**Proof.** The constants \( \tau_1 > \frac{\dim X_1}{4}, \ldots, \tau_k > \frac{\dim X_k}{4} \) and \( \gamma > 0 \) will be those in (5.1). We denote by \( L \) the operator \( (1 - \Delta X_1)^{\tau_1} \cdots (1 - \Delta X_k)^{\tau_k} \). Let \( \{ \varphi_e \} \) be a family of smooth functions such that \( 0 \leq \varphi_e \in C_0^\infty(X), \int \varphi_e(x) \, dx = 1 \) and \( \text{supp} \varphi_e \subset B(0;\epsilon) \). Let \( \chi \in C_0^\infty(X) \) be such that \( 0 \leq \chi \leq 1 \), \( \text{supp} \varphi \subset B(0;2) \) and \( \chi|B(0;1) = 1 \). For \( j \in \mathbb{N} \), \( j \geq 1 \), we set \( \chi_j = \chi(j \cdot) \).

Let \( u \in C_{m_1,\ldots,m_k}^p \). Then \( u_j = (\chi_j u) * \varphi_{\frac{1}{j}} \in C_0^\infty(X) \) and \( u_j \to u \) in \( S^*(X) \).

If \( 1 \leq p < \infty \), then \( u_j \to u \) in \( C_{m_1,\ldots,m_k}^p \), because
\[ \| (\chi_j u) * \varphi_{\frac{1}{j}} - u \|_{p,m_{1},\ldots,m_{k}} \leq \| (\chi_j u) - u \|_{p,m_{1},\ldots,m_{k}} + \| u * \varphi_{\frac{1}{j}} - u \|_{p,m_{1},\ldots,m_{k}}. \]
Since
\[ \| Lu_j \|_{L^p(X)} \leq \gamma \| u_j \|_{p,m_{1},\ldots,m_{k}}, \quad \| Lu_j - Lu \|_{L^p(X)} \leq \gamma \| u_j - u \|_{p,m_{1},\ldots,m_{k}}, \]
it follows that \( Lu_j \to Lu \) in \( L^p(X) \) and \( \| Lu_j \|_{L^p(X)} \leq \gamma \| u \|_{p,m_{1},\ldots,m_{k}}. \)

If \( p = \infty \), then \( u_j \to u \) only in \( S^*(X) \) and
\[ |\langle \psi, Lu_j \rangle| \leq \| \psi \|_{L^1} \| Lu_j \|_{L^\infty(X)} \leq \gamma \| \psi \|_{L^1} \| u_j \|_{\infty,m_{1},\ldots,m_{k}}, \]
\[ \leq \gamma \| \psi \|_{L^1} \| \chi_j u \|_{\infty,m_{1},\ldots,m_{k}} \leq \gamma (1 + C(\chi)^{-1}) \| \psi \|_{L^1} \| u \|_{\infty,m_{1},\ldots,m_{k}}, \]
for all \( \psi \in S(X) \) and \( j \in \mathbb{N}, j \geq 1 \). If we let \( j \to \infty \) we obtain that
\[ |\langle \psi, Lu \rangle| \leq \gamma \| \psi \|_{L^1} \| u \|_{\infty,m_{1},\ldots,m_{k}}, \quad \psi \in S(X). \]
It follows that \( Lu \in (L^1(X))^* = L^\infty(X) \) and \( \| Lu \|_{L^\infty(X)} \leq \gamma \| u \|_{\infty,m_{1},\ldots,m_{k}}. \)

We return to the symplectic space \( \mathcal{G} = T^*(X) \). An orthogonal decomposition of \( X, X_j = X_1 \oplus \cdots \oplus X_k \), gives an orthogonal decomposition of \( \mathcal{G}, \mathcal{G} = X_1 \oplus \cdots \oplus X_k \oplus X_1^* \oplus \cdots \oplus X_k^* \), if on \( \mathcal{G} \) we consider the euclidean norm \( \| (x,p) \|_{\mathcal{G}}^2 = \| x \|_{X_k}^2 + \| p \|_{X_k^*}^2 \). We shall choose an orthonormal basis in each space \( X_j, j = 1, \ldots, k \), while in \( X_j^*, j = 1, \ldots, k \) we shall consider the dual basis. Then \( \partial X = (\partial X_1, \ldots, \partial X_k), \partial X^* = (\partial X_1^*, \ldots, \partial X_k^*). \)

For \( 1 \leq p \leq \infty \) and \( t = (t_1, \ldots, t_k) \in \mathbb{N}^k \) we set \( \mathcal{M}_t^p = \mathcal{M}_{t_1,\ldots,t_k}^p \) for the space of all distributions \( a \in S^*(\mathcal{G}) \) whose derivatives
\[ (\partial X_1)^{t_1} \cdots (\partial X_k)^{t_k} (\partial X_1^*)^{\beta_1} \cdots (\partial X_k^*)^{\beta_k} a. \]
belong to $L^p(\mathcal{S})$ when $\alpha_j, \beta_j \in \mathbb{N}^{\dim X_j}$, $|\alpha_j|, |\beta_j| \leq t_j$, $j = 1, \ldots, k$. On this space we shall consider the natural norm $|a|_{p,t} = \max_{|\alpha|, |\beta| \leq t} \| (\partial X_1)^{\alpha_1} \cdots (\partial X_k)^{\alpha_k} (\partial X_1^*)^{\beta_1} \cdots (\partial X_k^*)^{\beta_k} a \|_{L^p}$.

Let $m_1 = \lfloor \frac{\dim X_1}{2} \rfloor + 1, \ldots, m_k = \lfloor \frac{\dim X_k}{2} \rfloor + 1$ and $m = (m_1, \ldots, m_k)$.

A consequence of Theorem 5.1 and of Lemma 5.3 is the following

**Theorem 5.4.** Assume that $1 \leq p < \infty$ and let $a \in S^*(\mathcal{S})$.

(i) If $a \in M^p_{m_1, \ldots, m_k}$, then $a^0_X(Q, P)$ has an extension in $B_p(\mathcal{H}(X))$ denoted also by $a^0_X(Q, P)$ and there is $C > 0$ such that

$$\| a^0_X(Q, P) \|_p \leq C |a|_{p,m_1, \ldots, m_k}.$$ 

(ii) If $a \in M^p_{2m_1, \ldots, 2m_k}$, then for any $\tau \in \mathbb{R} \equiv \mathbb{R} \cdot 1_X$, $a^\tau_X(Q, P)$ has an extension in $B_p(\mathcal{H}(X))$ denoted also by $a^\tau_X(Q, P)$. The mapping

$$\mathbb{R} \ni \tau \mapsto a^\tau_X(Q, P) \in B_p(\mathcal{H}(X))$$

is continuous and for any $K$ a compact subset of $\mathbb{R}$, there is $C_K > 0$ such that, for any $\tau \in K$,

$$\| a^\tau_X(Q, P) \|_p \leq C_K |a|_{p,2m_1, \ldots, 2m_k}.$$ 

Similarly, for $p = \infty$, a consequence of Theorem 5.2 and of Lemma 5.3 is the celebrated Calderón-Vaillancourt Theorem.

**Theorem 5.5 (Calderón, Vaillancourt).** Assume that $p = \infty$ and let $a \in S^*(\mathcal{S})$.

(i) If $a \in M^\infty_{m_1, \ldots, m_k}$, then $a^0_X(Q, P)$ is $L^2$-bounded and there is $C > 0$ such that

$$\| a^0_X(Q, P) \|_{B(\mathcal{H}(X))} \leq C |a|_{\infty,m_1, \ldots, m_k}.$$ 

(ii) If $a \in M^\infty_{2m_1, \ldots, 2m_k}$, then $a^\tau_X(Q, P)$ is $L^2$-bounded for any $\tau \in \mathbb{R} \equiv \mathbb{R} \cdot 1_X$. The mapping

$$\mathbb{R} \ni \tau \mapsto a^\tau_X(Q, P) \in B(\mathcal{H}(X))$$

is continuous and for any $K$ a compact subset of $\mathbb{R}$, there is $C_K > 0$ such that, for any $\tau \in K$,

$$\| a^\tau_X(Q, P) \|_{B(\mathcal{H}(X))} \leq C_K |a|_{\infty,2m_1, \ldots, 2m_k}.$$ 

To obtain other results we shall use the following

**Lemma 5.6.** Let $X = X_1 \oplus \cdots \oplus X_k$ be an orthogonal decomposition. If $s_1, \ldots, s_k \geq 0$, $\varepsilon > 0$, then for any $1 \leq p \leq \infty$,

$$(1 - \Delta_{X_1} \otimes 1)^{s_1} \cdots (1 - 1 \otimes \Delta_{X_k})^{s_k} (1 - \Delta_X)^{-(s_1 + \cdots + s_k) - \varepsilon} \in B(L^p(X)).$$

The proof of this lemma is given in the appendix.

The next two theorems are consequences of this lemma, Theorem 5.1 and Theorem 5.2. Recall that the Sobolev space $H^s_p(\mathcal{S})$, $s \in \mathbb{R}$, $1 \leq p \leq \infty$, consists of all $a \in S^*(\mathcal{S})$ such that $(1 - \Delta_{\mathcal{S}})^{\frac{s}{2}} a \in L^p(\mathcal{S})$, and we set $\|a\|_{H^s_p(\mathcal{S})} \equiv \| (1 - \Delta_{\mathcal{S}})^{\frac{s}{2}} a \|_{L^p(\mathcal{S})}$. 
Theorem 5.7. Assume that $1 \leq p < \infty$.
(i) If $s > \dim X$ and $a \in H_p^s(\mathcal{G})$, then $a_X^0(Q, P)$ has an extension in $\mathcal{B}_p(\mathcal{H}(X))$ denoted also by $a_X^0(Q, P)$ and there is $C > 0$ such that
$$\|a_X^0(Q, P)\|_p \leq C\|a\|_{H_p^s(\mathcal{G})}.$$ 
(ii) If $s > 2\dim X$ and $a \in H_p^s(\mathcal{G})$, then for any $\tau \in \mathbb{R} \equiv \mathbb{R} \cdot 1_X$, $a_X^{\tau}(Q, P)$ has an extension in $\mathcal{B}_p(\mathcal{H}(X))$ denoted also by $a_X^{\tau}(Q, P)$. The mapping
$$\mathbb{R} \ni \tau \rightarrow a_X^{\tau}(Q, P) \in \mathcal{B}_p(\mathcal{H}(X))$$
is continuous and for any $K$ a compact subset of $\mathbb{R}$, there is $C_K > 0$ such that, for any $\tau \in K$,
$$\|a_X^{\tau}(Q, P)\|_p \leq C_K\|a\|_{H_p^s(\mathcal{G})}.$$ 

Theorem 5.8. Assume that $p = \infty$.
(i) If $s > \dim X$ and $a \in H_\infty^s(\mathcal{G})$, then $a_X^0(Q, P)$ is $L^2$-bounded and there is $C > 0$ such that
$$\|a_X^0(Q, P)\|_{\mathcal{B}(\mathcal{H}(X))} \leq C\|a\|_{H_\infty^s(\mathcal{G})}.$$ 
(ii) If $s > 2\dim X$ and $a \in H_\infty^s(\mathcal{G})$, then $a_X^{\tau}(Q, P)$ is $L^2$-bounded for any $\tau \in \mathbb{R} \equiv \mathbb{R} \cdot 1_X$. The mapping
$$\mathbb{R} \ni \tau \rightarrow a_X^{\tau}(Q, P) \in \mathcal{B}(\mathcal{H}(X))$$
is continuous and for any $K$ a compact subset of $\mathbb{R}$, there is $C_K > 0$ such that, for any $\tau \in K$,
$$\|a_X^{\tau}(Q, P)\|_{\mathcal{B}(\mathcal{H}(X))} \leq C_K\|a\|_{H_\infty^s(\mathcal{G})}.$$ 

If we note that $a_X^{\tau}(Q, P) \in \mathcal{B}_2(\mathcal{H}(X))$ whenever $a \in L^2(\mathcal{G}) = H_2^0(\mathcal{G})$, then the last two theorems and standard interpolation results in Sobolev spaces (see Theorem 6.4.5 of [2]) give us the following theorem.

Theorem 5.9. Let $\mu > 1$, $1 \leq p < \infty$ and $n = \dim X$.
(i) If $a \in H_p^{\mu n[1-\frac{2}{p}]}(\mathcal{G})$, then $a_X^0(Q, P)$ has an extension in $\mathcal{B}_p(\mathcal{H}(X))$ denoted also by $a_X^0(Q, P)$ and there is $C > 0$ such that
$$\|a_X^0(Q, P)\|_p \leq C\|a\|_{H_p^{\mu n[1-\frac{2}{p}]}(\mathcal{G})}.$$ 
(ii) If $a \in H_p^{2\mu n[1-\frac{2}{p}]}(\mathcal{G})$, then for any $\tau \in \mathbb{R} \equiv \mathbb{R} \cdot 1_X$, $a_X^{\tau}(Q, P)$ has an extension in $\mathcal{B}_p(\mathcal{H}(X))$ denoted also by $a_X^{\tau}(Q, P)$. The mapping
$$\mathbb{R} \ni \tau \rightarrow a_X^{\tau}(Q, P) \in \mathcal{B}_p(\mathcal{H}(X))$$
is continuous and for any $K$ a compact subset of $\mathbb{R}$, there is $C_K > 0$ such that, for any $\tau \in K$,
$$\|a_X^{\tau}(Q, P)\|_p \leq C_K\|a\|_{H_p^{2\mu n[1-\frac{2}{p}]}(\mathcal{G})}.$$
APPENDIX A. A CLASS OF FOURIER MULTIPLIERS

In this appendix we shall prove Lemma 5.6.

Let us assume that the lemma has been proved for \( k = 2 \). Let \( k \geq 3 \). For \( 1 \leq l \leq k \), we set \( T_l = (1 - 1 \otimes \Delta X_1 \otimes 1)^{s_l} (1 - \Delta X)^{-s_l - \frac{m}{n}} \). Then \( T_l \in B(L^p(X)) \) and

\[
(1 - \Delta X_1 \otimes 1)^{s_1} \cdots (1 - 1 \otimes \Delta X_k)^{s_k} (1 - \Delta X)^{-s_1 - \cdots - s_k - \varepsilon} = T_1 \cdots T_k \in B(L^p(X)).
\]

This shows that the lemma is true provided that it holds for \( k = 2 \).

By choosing an orthonormal basis in each space \( X_1 \) and \( X_2 \) we identify \( X_1 \) with \( \mathbb{R}^{n_1} \times \mathbb{R}^{n_2} \) and \( X_2 \) with \( \mathbb{R}^{n_1} \times \mathbb{R}^{n_2} \). Then the operator

\[
(1 - \Delta_1 \otimes 1)^{s_1} (1 - 1 \otimes \Delta_2)^{s_2} (1 - \Delta_1 \otimes 1 - 1 \otimes \Delta_2)^{-s_1 - s_2 - \varepsilon}
\]

has the symbol \( a : \mathbb{R}^{n_1} \times \mathbb{R}^{n_2} \to \mathbb{R} \) defined by

\[
a(\xi_1, \xi_2) = \langle \xi_1 \rangle^{s_1} \langle \xi_2 \rangle^{s_2} \langle (\xi_1, \xi_2) \rangle^{-s_1 - s_2 - \varepsilon}, \quad (\xi_1, \xi_2) \in \mathbb{R}^{n_1} \times \mathbb{R}^{n_2}
\]

where \( \langle \xi \rangle = (1 + |\xi|^2)^{1/2}, \xi \in \mathbb{R}^n \). If we write \( \varepsilon = \varepsilon_1 + \varepsilon_2 \) with \( \varepsilon_1, \varepsilon_2 > 0 \), then it can be easy check that for any \( (\alpha_1, \alpha_2) \in \mathbb{N}^{n_1} \times \mathbb{N}^{n_2} \), there is \( C_{\alpha_1, \alpha_2} = C(n_1, n_2, s_1, s_2, \alpha_1, \alpha_2, \varepsilon) > 0 \) such that

\[
|\partial_{\xi_1}^{\alpha_1} \partial_{\xi_2}^{\alpha_2} a(\xi_1, \xi_2)| \leq C_{\alpha_1, \alpha_2} \langle \xi_1 \rangle^{-|\alpha_1|} \langle \xi_2 \rangle^{-|\alpha_2|}, \quad (\xi_1, \xi_2) \in \mathbb{R}^{n_1} \times \mathbb{R}^{n_2}.
\]

DEFINITION A.1. Let \( m = (m_1, m_2) \in \mathbb{R}^2 \). We shall say that \( a : \mathbb{R}^{n_1} \times \mathbb{R}^{n_2} \to \mathbb{C} \) is a symbol of degree \( m \) if \( a \in C^\infty(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2}) \) and for any \( (\alpha_1, \alpha_2) \in \mathbb{N}^{n_1} \times \mathbb{N}^{n_2} \), there is \( C_{\alpha_1, \alpha_2} > 0 \) such that

\[
|\partial_{\xi_1}^{\alpha_1} \partial_{\xi_2}^{\alpha_2} a(\xi_1, \xi_2)| \leq C_{\alpha_1, \alpha_2} \langle \xi_1 \rangle^{m_1 - |\alpha_1|} \langle \xi_2 \rangle^{m_2 - |\alpha_2|}, \quad (\xi_1, \xi_2) \in \mathbb{R}^{n_1} \times \mathbb{R}^{n_2}.
\]

We denote by \( S^m(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2}) = S^{m_1, m_2}(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2}) \) the vector space of all symbols of degree \( m \) and observe that

\[
S(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2}) \subset S^m(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2}) \subset S^*(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2}),
\]

\[
S^m(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2}) \cdot S^m(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2}) \subset S^{m+\bar{m}}(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2}),
\]

\[
m \leq \bar{m} \rightarrow S^m(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2}) \subset S^{\bar{m}}(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2}),
\]

where \( m \leq \bar{m} \) means \( m_1 \leq m_1, m_2 \leq m_2 \).

Let \( \varphi \in C_0^\infty(\mathbb{R}^n) \) such that \( 0 \leq \varphi \leq 1, \varphi(\xi) = 1 \) for \( |\xi| \leq 1, \varphi(\xi) = 0 \) for \( |\xi| \geq 2 \). We define \( \psi \in C_0^\infty(\mathbb{R}^n) \) by

\[
\psi(\xi) = -\xi \cdot \nabla \varphi(\xi) = -\sum_{j=1}^n \xi_j \partial_j \varphi(\xi), \quad \xi \in \mathbb{R}^n.
\]

Then \( \text{supp} \psi \subset \{ \xi \in \mathbb{R}^n : 1 \leq |\xi| \leq 2 \} \).
If \( r > 1 \) and \( \xi \in \mathbb{R}^n \setminus 0 \), then
\[
\varphi(\frac{\xi}{r}) - \varphi(\xi) = \int_{1}^{r} \frac{d}{dt} \varphi(\frac{\xi}{t}) \, dt = -\int_{1}^{r} \frac{\xi}{t} \cdot \nabla \varphi(\frac{\xi}{t}) \, \frac{dt}{t} = \int_{1}^{r} \frac{\varphi(\frac{\xi}{t})}{t} \, dt.
\]
We take \( r \to \infty \) to obtain that
\[
1 = \varphi(\xi) + \int_{1}^{\infty} \frac{\varphi(\frac{\xi}{t})}{t} \, dt, \quad \xi \in \mathbb{R}^n.
\]
If we use such identities in each space \( \mathbb{R}^n_1 \) and \( \mathbb{R}^n_2 \), we get
\[
1 = \varphi_1(\xi_1) \varphi_2(\xi_2) + \int_{1}^{\infty} \psi_1(\frac{\xi_1}{t_1}) \varphi_2(\frac{\xi_2}{t_2}) \, \frac{dt_1}{t_1} + \int_{1}^{\infty} \varphi_1(\frac{\xi_1}{t_1}) \psi_2(\frac{\xi_2}{t_2}) \, \frac{dt_2}{t_2}
\]
\[
+ \int_{1}^{\infty} \int_{1}^{\infty} \psi_1(\frac{\xi_1}{t_1}) \psi_2(\frac{\xi_2}{t_2}) \, \frac{dt_1}{t_1} \frac{dt_2}{t_2}, \quad (\xi_1, \xi_2) \in \mathbb{R}^n_1 \times \mathbb{R}^n_2,
\]
where for \( j = 1, 2 \) the functions \( \varphi_j, \psi_j \in \mathcal{C}_0^\infty(\mathbb{R}^n_j) \) satisfy \( 0 \leq \varphi_j \leq 1, \varphi_j(\xi_j) = 1 \) for \( |\xi_j| \leq 1 \), \( \varphi_j(\xi_j) = 0 \) for \( |\xi_j| \geq 2 \), \( \text{supp} \psi_j \subset \{ \xi_j \in \mathbb{R}^n_j : 1 \leq |\xi_j| \leq 2 \} \), and \( \psi_j \) is given by \( \psi_j(\xi_j) = -\frac{\xi_j}{|\xi_j|} \cdot \nabla \psi_j(\xi_j), \xi_j \in \mathbb{R}^n_j \).

We make use of the following simple but important remark. If
\[
a \in \mathcal{S}^m(\mathbb{R}^n_1 \times \mathbb{R}^n_2) = \mathcal{S}^{m_1,m_2}(\mathbb{R}^n_1 \times \mathbb{R}^n_2),
\]
then the families of functions
\[
\{ (\psi_1 \otimes \psi_2) a_{t_1,t_2} \}_{1 \leq t_1 < \infty} \{ (\varphi_1 \otimes \varphi_2) a_{t_1,t_2} \}_{1 \leq t_2 < \infty} \{ (\psi_1 \otimes \varphi_2) a_{t_1,t_2} \}_{1 \leq t_1 < \infty, 1 \leq t_2 < \infty}
\]
are bounded in \( \mathcal{S}(\mathbb{R}^n_1 \times \mathbb{R}^n_2) \), where for \( t_1, t_2 > 0 \)
\[
a_{t_1,t_2}(\xi_1, \xi_2) = t_1^{-m_1} t_2^{-m_2} a(t_1 \xi_1, t_2 \xi_2), \quad (\xi_1, \xi_2) \in \mathbb{R}^n_1 \times \mathbb{R}^n_2.
\]
Let \( a_0 = (\varphi_1 \otimes \varphi_2) a \in \mathcal{C}_0^\infty(\mathbb{R}^n_1 \times \mathbb{R}^n_2) \). Since
\[
a(\xi_1, \xi_2) = t_1^{m_1} t_2^{m_2} a_{t_1,t_2}(\frac{\xi_1}{t_1}, \frac{\xi_2}{t_2}), \quad (\xi_1, \xi_2) \in \mathbb{R}^n_1 \times \mathbb{R}^n_2, 1 \leq t_1, t_2 < \infty,
\]
we can write
\[
a(\xi_1, \xi_2) = a_0(\xi_1, \xi_2) + \int_{1}^{\infty} t_1^{m_1} ((\psi_1 \otimes \varphi_2) a_{t_1,1})(\frac{\xi_1}{t_1}, \xi_2) \, \frac{dt_1}{t_1}
\]
\[
+ \int_{1}^{\infty} t_2^{m_2} ((\varphi_1 \otimes \psi_2) a_{1,t_2})(\xi_1, \xi_2) \, \frac{dt_2}{t_2}
\]
\[
+ \int_{1}^{\infty} \int_{1}^{\infty} t_1^{m_1} t_2^{m_2} ((\psi_1 \otimes \psi_2) a_{t_1,t_2})(\frac{\xi_1}{t_1}, \frac{\xi_2}{t_2}) \, \frac{dt_1}{t_1} \frac{dt_2}{t_2}, \quad (\xi_1, \xi_2) \in \mathbb{R}^n_1 \times \mathbb{R}^n_2,
\]
We have
\[ |t_1^{m_1}t_2^{m_2}((\psi_1 \otimes \psi_2)a_{t_1,t_2})(\frac{\xi_1}{t_1}, \frac{\xi_2}{t_2})| \]
\[ \leq t_1^{m_1-2N_1}t_2^{m_2-2N_2}(\sup |(\psi_1 \otimes \psi_2)a_{t_1,t_2}|)|\xi_1|^{2N_1}|\xi_2|^{2N_2}, (\xi_1, \xi_2) \in \mathbb{R}^{n_1} \times \mathbb{R}^{n_2}, 1 \leq t_1, t_2 < \infty; \]
\[ |t_1^{m_1}((\psi_1 \otimes \varphi_2)a_{t_1,t_1})(\frac{\xi_1}{t_1}, \frac{\xi_2}{t_2})| \]
\[ \leq t_1^{m_1-2N_1}(\sup |(\psi_1 \otimes \varphi_2)a_{t_1,t_1}|)|\xi_1|^{2N_1}, (\xi_1, \xi_2) \in \mathbb{R}^{n_1} \times \mathbb{R}^{n_2}, 1 \leq t_1 < \infty; \]
\[ |t_2^{m_2}((\varphi_1 \otimes \psi_2)a_{t_1,t_2})(\frac{\xi_1}{t_1}, \frac{\xi_2}{t_2})| \]
\[ \leq t_2^{m_2-2N_2}(\sup(\varphi_1 \otimes \psi_2)a_{t_1,t_2})|\xi_2|^{2N_2}, (\xi_1, \xi_2) \in \mathbb{R}^{n_1} \times \mathbb{R}^{n_2}, 1 \leq t_2 < \infty; \]
so if we choose \(N_1, N_2 \in \mathbb{N}\) such that \(m_1 < 2N_1\) and \(m_2 < 2N_2\), it follows that the representation
\[
a = a_0 + \int \frac{df_1}{t_1} + \int \frac{df_2}{t_2}
\]
\[ + \int \int \frac{df_1 df_2}{t_1 t_2}, \]
is valid also as equality in \(S^*(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2})\) with the integrals weakly absolutely convergent. If we apply \(\mathcal{F}^{-1}\) to this formula we obtain a decomposition of \(\mathcal{F}^{-1}a\). Thus
\[
\mathcal{F}^{-1}a = \mathcal{F}^{-1}a_0 + \int \mathcal{F}^{-1}((\psi_1 \otimes \varphi_2)a_{t_1,t_1}) \frac{df_1}{t_1} + \int \mathcal{F}^{-1}((\varphi_1 \otimes \psi_2)a_{t_1,t_2}) \frac{df_2}{t_2}
\]
\[ + \int \int \mathcal{F}^{-1}((\psi_1 \otimes \varphi_2)a_{t_1,t_2}) \frac{df_1 df_2}{t_1 t_2}, \]
with the integrals weakly absolutely convergent.

We have
\[
\mathcal{F}^{-1}(((\psi_1 \otimes \varphi_2)a_{t_1,t_2})^{-1})^{-1}(x_1, x_2) = t_1^{m_1+n_1}t_2^{m_2+n_2} \mathcal{F}^{-1}((\psi_1 \otimes \varphi_2)a_{t_1,t_2})(t_1x_1, t_2x_2),
\]
\[
\mathcal{F}^{-1}(((\psi_1 \otimes \varphi_2)a_{t_1,t_1})^{-1})(x_1, x_2) = t_1^{m_1+n_1} \mathcal{F}^{-1}((\psi_1 \otimes \varphi_2)a_{t_1,t_1})(t_1x_1, x_2),
\]
\[
\mathcal{F}^{-1}(((\varphi_1 \otimes \psi_2)a_{t_1,t_2})^{-1})(x_1, x_2) = t_2^{m_2+n_2} \mathcal{F}^{-1}((\varphi_1 \otimes \psi_2)a_{t_1,t_2})(x_1, t_2x_2),
\]

Since the families of functions
\[ \{(\psi_1 \otimes \varphi_2)a_{t_1,t_1}\}_{1 \leq t_1 < \infty}, \{(\varphi_1 \otimes \psi_2)a_{t_1,t_2}\}_{1 \leq t_2 < \infty}, \{(\psi_1 \otimes \psi_2)a_{t_1,t_2}\}_{1 \leq t_1, t_2 < \infty}, \]
are bounded in $S(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2})$, it follows that the families

$$\{\mathcal{F}^{-1}((\psi_1 \otimes \varphi_2)a_{t_1,t_2})\}_{1 \leq t_1 < \infty}, \quad \{\mathcal{F}^{-1}((\varphi_1 \otimes \psi_2)a_{t_1,t_2})\}_{1 \leq t_2 < \infty}$$

are also bounded in $S(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2})$. In particular, for any $N \in \mathbb{N}$, there is $C_N > 0$ such that if $(x_1, x_2) \in \mathbb{R}^{n_1} \times \mathbb{R}^{n_2}$, then

$$|\mathcal{F}^{-1}((\psi_1 \otimes \varphi_2)a_{t_1,t_2})(x_1, x_2)| \leq C_N \langle (x_1, x_2) \rangle^{-2N-M},$$

$$|\mathcal{F}^{-1}((\varphi_1 \otimes \psi_2)a_{t_1,t_2})(x_1, x_2)| \leq C_N \langle (x_1, x_2) \rangle^{-2N-M},$$

$$|\mathcal{F}^{-1}((\psi_1 \otimes \psi_2)a_{t_1,t_2})(x_1, x_2)| \leq C_N \langle (x_1, x_2) \rangle^{-2N-2M},$$

where $M \in \mathbb{N}$, $M \geq 1 + \max\{0, m_1 + n_1, m_2 + n_2\}$ is fixed.

It follows that

$$|\mathcal{F}^{-1}(((\psi_1 \otimes \varphi_2)a_{t_1,t_2})_{t_1^{-1},t_2^{-1}})(x_1, x_2)|$$

$$\leq C_{N_1}^{-m_1+n_1} t_1^{-m_2+n_2} \langle x_1 \rangle^{-N} \langle t_1 x_1 \rangle^{-M} \langle t_2 x_2 \rangle^{-M}, \quad (x_1, x_2) \in \mathbb{R}^{n_1} \times \mathbb{R}^{n_2}, t_1, t_2 \geq 1;$$

$$|\mathcal{F}^{-1}(((\psi_1 \otimes \psi_2)a_{t_1,t_2})_{t_1^{-1},t_2^{-1}})(x_1, x_2)|$$

$$\leq C_{N_1}^{-m_1+n_1} \langle x_2 \rangle^{-N} \langle t_1 x_1 \rangle^{-M}, \quad (x_1, x_2) \in \mathbb{R}^{n_1} \times \mathbb{R}^{n_2}, t_1 \geq 1;$$

$$|\mathcal{F}^{-1}(((\varphi_1 \otimes \psi_2)a_{t_1,t_2})_{t_1^{-1},t_2^{-1}})(x_1, x_2)|$$

$$\leq C_{N_1}^{-m_2+n_2} \langle x_2 \rangle^{-N} \langle t_2 x_2 \rangle^{-M}, \quad (x_1, x_2) \in \mathbb{R}^{n_1} \times \mathbb{R}^{n_2}, t_2 \geq 1.$$

To calculate the restriction of the temperate distribution $\mathcal{F}^{-1}a$ to the complement of the singular subset $\mathcal{M} = \{(x_1, x_2) \in \mathbb{R}^{n_1} \times \mathbb{R}^{n_2} : |x_1||x_2| = 0\}$, we need the following easy consequence of Fubini theorem.

**Lemma A.2.** Let $(T, \mu)$ be a measure space, $\Omega \subset \mathbb{R}^n$ an open set and $f : \Omega \times T \to \mathbb{C}$ a measurable function.

(i) If for any $\varphi \in \mathcal{C}_0^\infty(\Omega)$ the function $\Omega \times T \ni (x,t) \to \varphi(x)f(x,t) \in \mathbb{C}$ belongs to $L^1(\Omega \times T)$, then the mapping

$$\mathcal{C}_0^\infty(\Omega) \ni \varphi \to \iint \varphi(x)f(x,t)dx \mu(t) \in \mathbb{C}$$

define a distribution, the function $\Omega \ni x \to \iint f(x,t)dx \mu(t) \in \mathbb{C}$, defined a.e., belongs to $L^1_{loc}(\Omega)$ and we have

$$\langle \varphi, \iint f(\cdot,t)dx \mu(t) \rangle = \iint \varphi(x)f(x,t)dx \mu(t) = \int \left( \int \varphi(x)f(x,t)dx \right) dx \mu(t), \quad \varphi \in \mathcal{C}_0^\infty(\Omega).$$

(ii) Assume that $\Omega = \mathbb{R}^n$. If there is $\tau \in \mathbb{R}$ such that the function $\mathbb{R}^n \times T \ni (x,t) \to \langle x \rangle^{-\tau}f(x,t) \in \mathbb{C}$ belongs to $L^1(\mathbb{R}^n \times T)$, then the mapping

$$S(\mathbb{R}^n) \ni \varphi \to \iint \varphi(x)f(x,t)dx \mu(t) \in \mathbb{C}$$
define a temperate distribution, the function \( \mathbb{R}^n \ni x \rightarrow \int f(x,t)\,d\mu(t) \in \mathbb{C}, \) defined a.e., belongs to \( L^1_{\text{loc}}(\mathbb{R}^n), \langle \cdot \rangle^{-\tau} \int f(\cdot,t)\,d\mu(t) \in L^1(\mathbb{R}^n) \) and we have

\[
\langle \varphi, \int f(\cdot,t)\,d\mu(t) \rangle = \int \varphi(x) f(x,t)\,dx \,d\mu(t) = \int \left( \int \varphi(x) f(x,t)\,dx \right) \,d\mu(t), \quad \varphi \in \mathcal{S}(\mathbb{R}^n).
\]

From the representation formula of \( F^{-1}a \), the above estimates and part (i) of the previous lemma we conclude that \( F^{-1}a \) is a \( C^\infty \)-function on \( \mathbb{R}^{n_1} \times \mathbb{R}^{n_2} \setminus \mathcal{M} \) which decays at infinity, together with all its derivatives, more rapidly than any power of \( \langle (x_1, x_2) \rangle^{-1} \). We shall denote by \( \mathcal{S}(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2} \setminus \mathcal{M}) \) the space of elements of \( \mathcal{S}^*(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2}) \) that are smooth outside the set \( \mathcal{M} \) and rapidly decreasing at infinity. Since on \( \mathbb{R}^{n_1} \times \mathbb{R}^{n_2} \setminus \mathcal{M} \) we have for \( (x_1, x_2) \in \mathbb{R}^{n_1} \times \mathbb{R}^{n_2} \setminus \mathcal{M} \)

\[
F^{-1}a(x_1, x_2) = F^{-1}a_0(x_1, x_2) + \int_1^\infty F^{-1}(\langle \psi_1 \otimes \psi_2 \rangle a_{1\cdot}, 1)(x_1, x_2) \frac{dt_1}{t_1} + \int_1^\infty F^{-1}(\langle \psi_1 \otimes \psi_2 \rangle a_{1\cdot}, 2)(x_1, x_2) \frac{dt_2}{t_2} + \int_1^\infty \int_1^\infty F^{-1}(\langle \psi_1 \otimes \psi_2 \rangle a_{1\cdot}, 2)(x_1, x_2) \frac{dt_1 dt_2}{t_1 t_2},
\]

it follows that for \( (x_1, x_2) \in \mathbb{R}^{n_1} \times \mathbb{R}^{n_2} \setminus \mathcal{M} \)

\[
|F^{-1}a(x_1, x_2)| \leq |F^{-1}a_0(x_1, x_2)| + C_N \langle x_1 \rangle^{-N} \langle x_2 \rangle^{-N} \int_1^\infty t_1^{m_1+n_1} \langle t_1 x_1 \rangle^{-M} \frac{dt_1}{t_1} + C_N \langle x_1 \rangle^{-N} \langle x_2 \rangle^{-N} \int_1^\infty t_2^{m_2+n_2} \langle t_2 x_2 \rangle^{-M} \frac{dt_2}{t_2} + C_N \langle x_1 \rangle^{-N} \langle x_2 \rangle^{-N} \int_1^\infty \int_1^\infty t_1^{m_1+n_1} t_2^{m_2+n_2} \langle t_1 x_1 \rangle^{-M} \langle t_2 x_2 \rangle^{-M} \frac{dt_1 dt_2}{t_1 t_2},
\]

or equivalently, for \( (x_1, x_2) \in \mathbb{R}^{n_1} \times \mathbb{R}^{n_2} \setminus \mathcal{M} \)

\[
\langle x_1 \rangle^N \langle x_2 \rangle^N |F^{-1}a(x_1, x_2)| \leq \langle x_1 \rangle^N \langle x_2 \rangle^N |F^{-1}a_0(x_1, x_2)| + C_N |x_1|^{-m_1-n_1} \int_1^\infty \langle t_1 x_1 \rangle^{-M} \frac{dt_1}{t_1} + C_N |x_2|^{-m_2-n_2} \int_1^\infty \langle t_2 x_2 \rangle^{-M} \frac{dt_2}{t_2} + C_N \langle x_1 \rangle^{-m_1-n_1} \langle x_2 \rangle^{-m_2-n_2} \int_1^\infty \int_1^\infty \langle t_1 x_1 \rangle^{-M} \langle t_2 x_2 \rangle^{-M} \frac{dt_1 dt_2}{t_1 t_2}.
\]
If \( m_1 + n_1 > 0 \) and \( m_2 + n_2 > 0 \), then
\[
C = \max \left\{ \int_0^\infty t_1^{m_1+n_1} \langle t_1 \rangle^{-M} \frac{dt_1}{t_1} \int_0^\infty t_2^{m_2+n_2} \langle t_2 \rangle^{-M} \frac{dt_2}{t_2} \right\} < \infty
\]
and for \((x_1, x_2) \in \mathbb{R}^{n_1} \times \mathbb{R}^{n_2} \setminus \mathcal{M}\)
\[
\langle x_1 \rangle^N \langle x_2 \rangle^N |\mathcal{F}^{-1}a(x_1, x_2)|
\leq \langle x_1 \rangle^N \langle x_2 \rangle^N |\mathcal{F}^{-1}a_0(x_1, x_2)| + C\langle x_1 \rangle^{-m_1-n_1} + |x_2|^{-m_2-n_2} + C|x_1|^{-m_1-n_1} |x_2|^{-m_2-n_2}.
\]
Assume that \(-n_1 < m_1 < 0, -n_2 < m_2 < 0\) and \( N > \max\{n_1 + 1, n_2 + 1\}\).
By using part (ii) of the previous lemma we conclude that \( \mathcal{F}^{-1}a \in L^1(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2})\).
Since \( m_1 \leq \overline{m} = S^m(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2}) \subset S^\overline{m}(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2})\), it follows that \( \mathcal{F}^{-1}a \) belongs to \( L^1(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2})\) for any \( a \in S^m(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2})\) with \( m_1 < 0 \) and \( m_2 < 0\).
Thus we have proved the following result.

**PROPOSITION A.3.** Let \( a \in S^m(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2}) = S^{m_1,m_2}(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2})\). We denote by \( \mathcal{M} \) the set \( \{(x_1, x_2) \in \mathbb{R}^{n_1} \times \mathbb{R}^{n_2} : |x_1||x_2| = 0\} \). Then:

(i) \( \mathcal{F}^{-1}a \in S(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2} \setminus \mathcal{M})\).

(ii) If \( m_1 + n_1 > 0 \) and \( m_2 + n_2 > 0\), then for any \( N \in \mathbb{N}\), there is \( C_N > 0 \) such that for \((x_1, x_2) \in \mathbb{R}^{n_1} \times \mathbb{R}^{n_2} \setminus \mathcal{M}\)
\[
|\mathcal{F}^{-1}a(x_1, x_2)| \leq C_N \langle x_1 \rangle^{-N} \langle x_2 \rangle^{-N} (1 + |x_1|^{-m_1-n_1})(1 + |x_2|^{-m_2-n_2}).
\]

(iii) If \( m_1 < 0 \) and \( m_2 < 0\), then \( \mathcal{F}^{-1}a \in L^1(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2})\).

**COROLLARY A.4.** Let \( a \in S^m(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2}) = S^{m_1,m_2}(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2})\) and \( 1 \leq p \leq \infty\). If \( m_1 < 0 \) and \( m_2 < 0\), then \( a(P_{\mathbb{R}^{n_1}}, P_{\mathbb{R}^{n_2}}) \in B(L^p(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2}))\).

**COROLLARY A.5.** If \( s_1, s_2 > 0, \varepsilon > 0 \) and \( 1 \leq p \leq \infty\), then
\[
(I - \triangle_1 \otimes I)^{s_1}(I - I \otimes \triangle_2)^{s_2}(I - \triangle_1 \otimes I - I \otimes \triangle_2)^{-s_1-s_2-\varepsilon} \in B(L^p(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2})).
\]

**REFERENCES**


GRUIA ARSU, INSTITUTE OF MATHEMATICS OF THE ROMANIAN ACADEMY, BUCHAREST, P.O. BOX 1-174, 014700, ROMANIA

E-mail address: Gruia.Arsu@imar.ro, agmilro@yahoo.com

Received October 27, 2005.