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Translations of

MATHEMATICAL MONOGRAPHS

Volume 105

Mathematical Scattering Theory

General Theory

D. R. Yafaev
<table>
<thead>
<tr>
<th>Contents</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>vi. CONTENTS</td>
<td></td>
</tr>
<tr>
<td>§4. Equations of second order in time</td>
<td>107</td>
</tr>
<tr>
<td>§5. IP for Abelian WO</td>
<td>109</td>
</tr>
<tr>
<td>CHAPTER 4. Scattering for Relatively Smooth Perturbations</td>
<td></td>
</tr>
<tr>
<td>§1. The Friedrichs-Faddeev model</td>
<td>113</td>
</tr>
<tr>
<td>§2. Scattering in the Friedrichs-Faddeev model</td>
<td>114</td>
</tr>
<tr>
<td>§3. Kato smoothness</td>
<td>128</td>
</tr>
<tr>
<td>§4. Sufficient conditions for smoothness</td>
<td>135</td>
</tr>
<tr>
<td>§5. The WO for smooth perturbations</td>
<td>138</td>
</tr>
<tr>
<td>§6. Smoothness with respect to the full Hamiltonian</td>
<td>143</td>
</tr>
<tr>
<td>§7. The absolutely continuous and point spectra of the operator $H$</td>
<td>146</td>
</tr>
<tr>
<td>CHAPTER 5. The General Scheme in Stationary Scattering Theory</td>
<td></td>
</tr>
<tr>
<td>§1. Weak smoothness</td>
<td>153</td>
</tr>
<tr>
<td>§2. Justification of the stationary method</td>
<td>157</td>
</tr>
<tr>
<td>§3. Connection with the time-dependent approach. IP</td>
<td>164</td>
</tr>
<tr>
<td>§4. Integral operators in direct decompositions</td>
<td>169</td>
</tr>
<tr>
<td>§5. The scattering matrix</td>
<td>174</td>
</tr>
<tr>
<td>§6. The decomposition theorem</td>
<td>178</td>
</tr>
<tr>
<td>§7. Scattering for relatively compact perturbations</td>
<td>181</td>
</tr>
<tr>
<td>§8. A local version of the stationary scheme</td>
<td>183</td>
</tr>
<tr>
<td>CHAPTER 6. Scattering for Perturbations of Trace Class Type</td>
<td></td>
</tr>
<tr>
<td>§1. Weak smoothness of Hilbert-Schmidt operators</td>
<td>187</td>
</tr>
<tr>
<td>§2. The Kato-Rosenblum theorem. “Negative” results</td>
<td>193</td>
</tr>
<tr>
<td>§3. Time-dependent proofs</td>
<td>196</td>
</tr>
<tr>
<td>§4. Local criteria for the existence of the WO</td>
<td>203</td>
</tr>
<tr>
<td>§5. Further generalizations</td>
<td>210</td>
</tr>
<tr>
<td>§6. An example. Perturbation by an integral operator of Fourier type</td>
<td>215</td>
</tr>
<tr>
<td>§7. One-dimensional perturbation</td>
<td>219</td>
</tr>
<tr>
<td>§8. Double Stieltjes operator integrals</td>
<td>225</td>
</tr>
<tr>
<td>CHAPTER 7. Properties of the Scattering Matrix (SM)</td>
<td></td>
</tr>
<tr>
<td>§1. The multiplication theorem for scattering operators and scattering matrices</td>
<td>229</td>
</tr>
<tr>
<td>§2. The invariance principle for SM. The SM in the unitary case</td>
<td>232</td>
</tr>
<tr>
<td>§3. Stationary representations for the WO and the scattering operator</td>
<td>236</td>
</tr>
<tr>
<td>§4. The SM for smooth perturbations</td>
<td>239</td>
</tr>
<tr>
<td>§5. Trace class integral operators</td>
<td>241</td>
</tr>
<tr>
<td>§6. The SM for trace class perturbations</td>
<td>246</td>
</tr>
<tr>
<td>§7. The structure of the stationary representation of the SM</td>
<td>250</td>
</tr>
<tr>
<td>§8. The spectrum of the SM for perturbations of definite sign</td>
<td>256</td>
</tr>
<tr>
<td>§9. The scattering cross section. Upper bounds</td>
<td>260</td>
</tr>
</tbody>
</table>
Preface

At present in mathematical scattering theory a large number of directions and methods have been combined (trace class and smooth theories, stationary and time-dependent approaches, etc.) which at first glance have little to do with one another. Various versions of the theory are related in a rather complicated manner already at the abstract level. Their interrelations are still more complicated in applications to the theory of differential operators. In this connection there arose the idea of giving a systematic exposition, while oriented toward concrete applications, of the methods of abstract scattering theory. In the second volume we intend to apply these methods to the theory of differential operators, primarily to the Schrödinger operator. The book is devoted mainly to basic areas of scattering theory. In the choice of material we have oriented ourselves toward results which are now considered classical.

In writing the book the author set for himself the task of organizing the exposition of the various ideas and methods of scattering theory according to a definite system. This is achieved by a systematic application of the stationary approach. Within this framework it is possible to some extent to combine the two basic methods of scattering theory—trace class and smooth perturbations. Simultaneously with the proofs of various facts, the stationary approach makes it possible to give formula representations for the basic objects of the theory. Along with the wave operators, we also investigate in detail the properties of the scattering matrix, the spectral shift function, etc.

Of course, in the choice of material we have considered the texts available. Therefore, some well-known areas well described in the literature are not found in the book. In particular, this applies to methods specially adapted to the study of differential operators. As examples, we mention the Lax-Phillips theory and Enss' method which are expounded in detail in the monographs [12] and [34] respectively.

We believe that the present book differs basically from the texts available on scattering theory. In connection with this we mention first of all the third volume of the well-known course of M. Reed and B. Simon [18]. This volume of the course has become a desktop copy for many, in particular, the author of the present book. However, in view of the broad compass of material,
the course [18] was necessarily written in encyclopedic style and apparently
cannot replace a systematic exposition of the theory. Our book differs from
another familiar monograph [30] on scattering theory in its direction toward
applications and the choice of concrete material. The last circumstance dis-
tinguishes it also from the books [28] and [29]. Moreover, as compared with
[28], [29], our exposition is of a more “advanced” character. We further men-
tion the monographs [3], [27], [32] in which various areas of scattering theory
for differential operators are presented. We have not undertaken a physical
exposition of scattering theory which can be found, for example, in the books
[13], [16]. We make special mention of the textbook [23], which contains an
exposition of quantum scattering theory intended for mathematicians.

In working on the book the author has tried to resolve two opposite prob-
lems. The first of them is a systematic exposition of the material beginning
“from zero”. The second problem is the exposition of a number of topics to
a degree of completeness which might possibly be of interest to specialists in
the area of spectral theory. We have also tried to fill certain gaps present in
the monograph literature. This pertains especially to the exposition of works
of Soviet and, in particular, Leningrad mathematicians. As an example, we
note the rather detailed description of the spectral properties of the scatter-
ing matrix. Another example is the theory of the spectral shift function of
M. G. Krein.

As a whole the book is oriented toward a reader (for example, a student of
higher courses or a graduate student in mathematical physics) interested in a
deeper study of scattering theory. The book may be considered an expanded
version of the course on scattering theory given by the author in the Leningrad
University. The exposition is presented under the assumption that the reader
is familiar with the theory of selfadjoint operators within the framework of
the textbook of M. Sh. Birman and M. Z. Solomyak [4]. Also of possible
use is the textbook of N. I. Akhiezer and I. M. Glazman [1] where there are
elementary facts concerning scattering theory.

In references in the book we use the “three-stage” enumeration of formulas
and theorems, and a “two-stage” enumeration of sections. The first number
is omitted within a chapter, while within a section the second number is also
omitted.

The conception and structure of the entire book were formed under the
influence of the author’s teacher M. Sh. Birman. The exposition of many
specific questions was also discussed with him. L. D. Faddeev also had an
influence on the mathematical development and tastes of the author. In
the book this is especially notable in the exposition of the theory of
smooth perturbations. The author is deeply grateful to M. Sh. Birman and
L. D. Faddeev.

Basic Notation

In this book we consider a pair of selfadjoint operators $H_0$ and $H$ acting
in Hilbert spaces $H_0$ and $H$, respectively. We denote by $\gamma$ a fixed (iden-
tification) operator acting from $H_0$ to $H$. The perturbation $V = H\gamma - 3H_0$
can often be factored as the product $V = G^*G$, where $G_0 : H_0 \to \Theta$, $G : H \to \Theta$, $\Theta$ is an auxiliary Hilbert space and $G^*$ is the operator ad-
joint to $G$.

As a rule, the notation for various objects associated with $H_0$ are provided
with the subindex “0”, while objects pertaining to the absolutely continuous
part of the operator are provided with the superindex “a”. Much of the
notation and definitions of the selfadjoint theory are retained in the unitary
case. In the notation of various function spaces the brackets usually denote
the set on which the functions considered are defined. In the case of vector-
valued functions the space in which the functions have their range is also
indicated. The letters $C$ and $c$ denote various estimation constants whose
precise value is immaterial. We use the following

Abbreviations

<table>
<thead>
<tr>
<th>Abbreviation</th>
<th>Meaning</th>
</tr>
</thead>
<tbody>
<tr>
<td>BS</td>
<td>Borel support</td>
</tr>
<tr>
<td>WO</td>
<td>wave operator</td>
</tr>
<tr>
<td>SM</td>
<td>scattering matrix</td>
</tr>
<tr>
<td>PD</td>
<td>perturbation determinant</td>
</tr>
<tr>
<td>a.e.</td>
<td>almost everywhere</td>
</tr>
<tr>
<td>IP</td>
<td>invariance principle</td>
</tr>
<tr>
<td>SSF</td>
<td>spectral shift function</td>
</tr>
</tbody>
</table>

We present a list of the most frequently encountered notations. If the
notation is not standard, the page is given on which it is introduced.

<table>
<thead>
<tr>
<th>Notation</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\overline{A}$</td>
<td>the closure of a linear operator $A$</td>
</tr>
<tr>
<td>$A : H_1 \to H_2$</td>
<td>a linear operator acting from the space $H_1$ to $H_2$</td>
</tr>
<tr>
<td>$\mathfrak{A}, \mathfrak{B}$</td>
<td>the strong and weak Abelian WO (pp. 73)</td>
</tr>
<tr>
<td>arg</td>
<td>the argument of a complex number</td>
</tr>
<tr>
<td>$\mathfrak{B}$</td>
<td>the set of all bounded, linear operators</td>
</tr>
</tbody>
</table>
\[ B(z) = GR(z)G^* \]
\[ \hat{B}(z) = G_0R(z)G^* \]
\[ B_0(z) = -G_0R_0(z)G^* \]
\[ B^{(0)}(z) = GR_0(z)G^* \]

- \( C \): the space of continuous functions
- \( C^\infty(\Omega) \): the set of infinitely differentiable functions on an open set \( \Omega \)
- \( C^\infty_0(\Omega) \): the subset of \( C^\infty(\Omega) \) consisting of functions with compact support
- \( \mathbb{C} \): the field of complex numbers
- \( D(z) \): PD (p. 265)
- \( \mathcal{D}(z), \mathcal{D}(z), D_\rho(z) \): the generalized (p. 269), modified (p. 270) and regularized (p. 271) PD
- \( \mathcal{D}^* \): the space dual to the Hilbert space \( \mathcal{D} \subset \mathcal{H} \) relative to the scalar product in \( \mathcal{H} \)
- \( \mathcal{D}(A) \): the domain of the operator \( A \)
- \( \det \): determinant
- \( \det_\rho \): regularized determinant (p. 44)
- \( \dim \): the dimension of a linear set
- \( \text{dist} \): the distance between sets
- \( \delta_{ij} \): Kronecker’s symbol
- \( \delta(\lambda, \varepsilon) \): the operator Poisson integral (p. 29)
- \( E(\cdot) \): the spectral measure of a selfadjoint operator
- \( \mathcal{E} \): the set of compactly supported elements (p. 60)
- \( \exp \): the exponential of a number or of an operator
- \( \mathcal{F} \): the unitary mapping of a Hilbert space \( \mathcal{H} \) (or \( \mathcal{H}^{(a)} \)) onto the direct integral \( \mathcal{F} \) (or \( \mathcal{F}^{(a)} \)) diagonalizing a selfadjoint operator (p. 31)
- \( \Phi \): the Fourier transform
- \( \mathcal{H}^{(a)}, \mathcal{H}^{(s)} \): the absolutely continuous and singular subspaces of a selfadjoint operator
- \( \mathcal{H}^{(p)} \): the subspace spanned by the eigenvectors of a selfadjoint operator
- \( H^p \): the Hardy class (p. 18)
- \( \mathcal{H} \): the direct integral of Hilbert spaces (p. 31)
- \( \mathcal{H}^{(a)} \): the direct integral corresponding to the absolutely continuous part of a selfadjoint operator (p. 32)
- \( \mathcal{H}^{(p)} \): the "infinitesimal subspace" of a direct integral (p. 31)
- \( I \): the identity operator

- \( \text{Im} \): the imaginary part of a complex number or operator
- \( \inf \): the infimum
- \( L_p \): the space of functions whose \( p \)th power is integrable
- \( L_p^{(loc)} \): the set of functions integrable in \( p \)th power only locally (on compact subsets)
- \( \mathcal{L} \): the set of finite-dimensional operators
- \( \Lambda \): a Borel set on the line
- \( N(A) \): the set of zeros (kernel) of the operator \( A \)
- \( \mathbb{R}^d \): the Euclidean space dual to \( \mathbb{R}^d \)
- \( P \) (or \( P^{(a)} \)): the projection onto the absolutely continuous subspace
- \( \mathbb{R}_+^d \): Euclidean space of dimension \( d \)
- \( \mathbb{R}_+ = [0, \infty) \)
- \( R(A) \): the range (image) of an operator \( A \)
- \( R_A(z) = (A - z)^{-1} \): the resolvent of an operator \( A \)
- \( \text{Re} \): the real part of a complex number or operator
- \( \mathcal{R} \): see (p. 85)
- \( r(f) \): see (p. 85)
- \( \rho(A) \): the set of regular points of the operator \( A \)
- \( S \): the scattering operator
- \( \sigma(A) \): the spectrum of a linear operator \( A \)
- \( \sigma^{(a)}, \sigma^{(s)}, \sigma^{(p)}, \sigma^{(ess)} \): absolutely continuous, singular, point, and essential spectra of a selfadjoint operator
- \( \mathcal{T} \): the unit circle in \( \mathbb{C} \)
- \( \text{Tr} \): the trace of an operator
- \( T(z) = V - VR(z)V \)
- \( s-\lim \): the strong limit of vectors and operators
- \( \mathbb{Z} \): the set of integers
- \( T_{\pm}(z) \): see (p. 95)
- \( \mathcal{W}_\pm \): stationary WO (p. 92)
- \( \mathcal{W}_\pm(A) \): local stationary WO (p. 184)
Introduction

1. Mathematical scattering theory is an area of perturbation theory. The ideology of the latter is that detailed information regarding the “unperturbed” operator $H_0$ makes it possible to draw conclusions regarding another operator $H$ if $H_0$ and $H$ are close in some sense. In physical terms the Hamiltonian $H_0$ describes a “free” system (for example, quantum particles not interacting with one another), while the “full” Hamiltonian $H$ describes the real system including interactions.

Scattering theory is concerned only with the structure of the continuous spectrum (more precisely, the absolutely continuous spectrum) and resolves two related problems. The first of them is the investigation of the behavior for large times of solutions of the time-dependent Schrödinger equation

$$i\frac{\partial u}{\partial t} = Hu, \quad u(0) = f.$$  \hspace{1cm} (1)

The asymptotics as $t \to \pm \infty$ of solutions of equation (1) with the full Hamiltonian $H$ is studied in terms of solutions of the equation with the “free” Hamiltonian $H_0$. The second problem consists in finding conditions for the unitary equivalence of the operators $H_0$ and $H$—more precisely, of their absolutely continuous parts $H^{(a)}_0$ and $H^{(a)}$.

To begin we consider the first of these problems. Suppose $H_0$ and $H$ are selfadjoint operators in a Hilbert space $H$. Equation (1) then has a unique solution $u(t) = \exp(-iHt)f$, while the solution of the same equation with the operator $H_0$ is given by the formula $u_0(t) = \exp(-iH_0t)f_0$. From the viewpoint of scattering theory the function $u(t)$ has “free” asymptotics as $t \to \pm \infty$ if for appropriate initial data $f^{(\pm)}_0$, found on the basis of $f$, we have

$$\lim_{t \to \pm \infty} \|u(t) - u^{(\pm)}_0(t)\| = 0, \quad u^{(\pm)}_0(t) = \exp(-iH_0t)f^{(\pm)}_0. \hspace{1cm} (2)$$

Here and everywhere a relation containing the signs “$\pm$” is understood as two independent equalities. Relation (2)$_\pm$ leads to a connection between the corresponding initial data $f^{(\pm)}_0$ and $f$:

$$f = \lim_{t \to \pm \infty} \exp(iHt)\exp(-iH_0t)f^{(\pm)}_0. \hspace{1cm} (3)$$
Thus, a solution $u(t)$ of equation (1) has free asymptotics as $t \to \pm \infty$ if for some element $f_0^{(\pm)}$ equality (3) holds. The operator $W_\pm = W_\pm(H_0, H_0')$; $f_0^{(\pm)} \mapsto f$, if it exists, is called the wave operator (for a precise definition see §2.1).

Let us discuss necessary conditions for the validity of (2)$.\pm$. If $f$ is an eigenvector of the operator $H_0$, $H_0 f = \lambda f$, then $u(t) = \exp(-i\lambda t)f$, and the dependence of the solution of equation (1) on time is trivial. However, since eigenvalues are displaced under arbitrarily weak perturbations, generally speaking, the unperturbed problem has no solutions with the same behavior as $t \to \pm \infty$. In a similar way, it is not possible to expect that relations (2)$\pm$ are satisfied for $f$ of the singular continuous subspace of the operator $H_0$. By the way, for cases discussed in scattering theory the subspaces of singular continuous spectrum of both operators are typically absent.

A basic proposition of scattering theory is that under rather general assumptions regarding the pair $H_0$, $H$ for an initial datum $f$ of the absolutely continuous subspace $\mathcal{H}_0^{(a)}$ of the operator $H$ the function $u(t)$ has free asymptotics. In definition (3) of the wave operators (WO) $W_\pm$, $W_\pm f_0^{(\pm)} = f$, the elements $f_0^{(\pm)}$ are also taken from the absolutely continuous subspace $\mathcal{H}_0^{(a)}$ of the operator $H_0$. Under the condition of existence of the WO $W_\pm$, relation (2) is satisfied if and only if $f$ belongs to the range of $W_\pm$. In particular, if (2)$\pm$ is valid for any absolutely continuous element $f$, then the WO $W_\pm$ are said to be complete. In concrete problems the verification of completeness is the main content of the investigation. It is easy to see that completeness of the WO $W_\pm(H, H_0)$ is equivalent to the existence of the inverse WO $W_\mp(H_0, H)$.

The second problem, simultaneously solved in scattering theory, consists of the proof of unitary equivalence of the operators $H_0^{(a)}$ and $H^{(a)}$. The fact of the matter is that under the condition of existence the WO are isometric on $\mathcal{H}_0^{(a)}$ and possess the intertwining property $HW_\pm = W_\pm H_0$. From this it follows that the operator $H^{(a)}$ has a unitarily equivalent to $H_0^{(a)}$. For complete WO $W_\pm$ the operators $H_0^{(a)}$ and $H^{(a)}$ are unitarily equivalent to one another. We emphasize that within the framework of scattering theory only the “canonical” unitary equivalence realizable by the WO is studied.

The necessity often arises of generalizing the theory to the case of operators $H_0$ and $H$ acting in different spaces $\mathcal{H}_0$ and $\mathcal{H}_0'$. This generalization requires an identification $\mathcal{J}$ taking $\mathcal{H}_0$ into $\mathcal{H}_0'$. The basic definitions of scattering theory for a pair of spaces are basically the same as for the case of a single space, but the various objects of the theory (for example, the WO) are now constructed with respect to the triple $H_0, H, \mathcal{J}$. Sometimes the operator $\mathcal{J}$ is not fixed by the formulation of the problem, and it can be chosen for convenience depending on the properties of the pair $H_0, H$. Introduction of an operator $\mathcal{J} \neq I$ is often useful also in considering operators acting in a single space. In this case $\mathcal{J}$ should be a satisfactory approximation to the WO. The scheme $\mathcal{J} = I$ described above pertains to the class of problems where the identity operator is such an approximation.

Together with the WO an important role in scattering theory is played by the mapping $\mathcal{S}: f_0^{(\pm)} \mapsto f_0^{(\pm)}$ which relates in terms of the free problem the asymptotics of solutions of the Schrödinger equation (1) as $t \to -\infty$ and $t \to +\infty$. The operator $\mathcal{S} = W_+^{\dagger}W_-$ is called the scattering operator. If the WO $W_\pm$ exist the operator $\mathcal{S}: \mathcal{H}_0^{(a)} \to \mathcal{H}_0^{(a)}$ commutes with $H_0$ and is a contraction. Under the additional condition of completeness of the WO the scattering operator is unitary in $\mathcal{H}_0^{(a)}$. In view of the equality $H_0\mathcal{S} = \mathcal{S}H_0$, the scattering operator can naturally be considered in a representation of the space $\mathcal{F}_0$ which is diagonal for $H_0$. In this representation $\mathcal{H}_0$ acts as multiplication by the independent variable $\lambda$, while $\mathcal{S}$ acts as multiplication by the operator-valued function $S(\lambda)$, called the scattering matrix. It is the scattering operator and matrix that are the objects of greatest interest in problems of mathematical physics, since they relate the “initial” characteristics of a process with the “final” characteristics, while avoiding consideration of the process for finite times. This approach is useful in the study of the interaction (scattering) of waves (acoustic, electromagnetic) by an obstacle or the flux of quantum particles from a target. This explains the term “scattering theory” itself which is borrowed from physics.

2. The temporal asymptotics of the time-dependent Schrödinger equation are closely connected with the behavior of the corresponding stationary problem at large distances from the “scatterer” (coordinate asymptotics). We shall clarify this with the example of the Schrödinger operator $H = -\Delta + q(x)$ in the space $\mathcal{F} = L^2(\mathbb{R}^d)$, where $H_0 = -\Delta$. Under the condition $q(x) = q(x) = O(|x|^{-d-\varepsilon})$, $\varepsilon > 0$, $|x| \to \infty$, the differential equation

$$-\Delta \psi + q(x)\psi = \lambda \psi$$

(4)

for any $\lambda > 0$ and unit vector $\omega \in S^{d-1}$ has a (unique) solution with the asymptotics

$$\psi(x; \omega, \lambda) = \exp(i\lambda^{1/2}(\omega, x)) + a(x) |x|^{-d-1/2} \int \exp(i\lambda^{1/2}|x|) + o(|x|^{-d-1/2})$$

(5)

as $|x| \to \infty$. The coefficient $a = a(\hat{x}, \omega; \lambda)$ depends on the direction $\omega$ of the plane wave $\exp(i\lambda^{1/2}(\omega, x))$ incident on the scatterer, its energy $\lambda$, and the direction $\hat{x} = x|x|^{-1}$ of observation of the outgoing spherical wave $|x|^{-d-1/2} \exp(i\lambda^{1/2}|x|)$. The function $a$ is called the scattering amplitude.

From the viewpoint of quantum mechanics $H_0$ is the operator of the kinetic energy of the relative motion of a pair of particles; their reduced mass is considered equal to $1/2$, and the Planck constant $\hbar = 1$; $q(x)$ is the potential energy of the interaction of the particles; $H$ is the operator of the total energy. In this picture the plane wave $\exp(i\rho, x)$ describes
the flux of particles with momentum \( p = \lambda^{1/2} \omega \) incident on the scattering center; here the flux density is equal to the velocity \( v = 2|p| \). Far from the center the scattered particles are described by the sum of the second and third terms on the right-hand side of (5). Therefore, the flux density of scattered particles through an element of area \( dS_r \) is \( \varphi^{d-1} \varphi \) of the sphere of radius \( r \) is equal to \( v_0^{d-1} (|a|^2 + o(1)) dS_r \). As \( r \to \infty \) the ratio of this quantity to the density of the incident flux tends to \( |a(\theta, \omega, \lambda)|^2 \). The last quantity is called the (effective) scattering cross section in the solid angle \( d\theta \). It is the basic quantity observed in scattering experiments. Further details on the quantum-mechanical picture of scattering can be found in the textbooks of L. D. Landau, E. M. Lifshits [13], and L. D. Faddeev, O. A. Yakubovskii [23].

The WO can be constructed in terms of solutions of the scattering problem, (the wave functions) \( \psi \). We set \( \psi_+(x, p) = \psi(x, -\omega, \lambda) \), \( \psi_-(x, p) = \psi(x, \omega, \lambda) \) and consider the transformations

\[
(\Phi_\pm f)(p) = (2\pi)^{-d/2} \int \psi_\pm(x, p)f(x) \, dx
\]

of the space \( L_2(\mathbb{R}^d) \) into \( L_2(\mathbb{R}^d) \), where \( \mathbb{R}^d \) is the Euclidean space (momentum space) dual to \( \mathbb{R}^d \). It is clear that if \( q = 0 \) the operators \( \Phi_\pm \) coincide with the usual Fourier transform \( \Phi_0 \). It can be shown that the action of \( \Phi_\pm \) on the operator \( H \) goes over into multiplication by \( p^2 \), i.e.,

\[
\Phi_\pm H = p^2 \Phi_0,
\]

(6)

whereby the operator \( \Phi_\pm \) is isometric on the subspace \( \mathcal{H}^{(a)} \), takes it onto all of \( L_2(\mathbb{R}^d) \), and is zero on \( \mathcal{H} \). Moreover, for \( f \in L_2(\mathbb{R}^d) \) as \( t \to \pm\infty \)

\[
\Phi_\pm \exp(-ip^2 t) f \sim \Phi_0 \exp(-ip^2 t) f.
\]

(7)

By comparing (6), (7), we establish the existence of the WO in this problem. For them we simultaneously obtain the representations

\[
W_\pm = \Phi_\pm \Phi_0.
\]

(8)

Completeness of the WO constructed follows from this.

The scattering matrix for the pair \( H_0, H \) can be computed in terms of the scattering amplitude. Namely, \( S(\lambda) \) acts in the space \( L_2(\mathbb{R}^d) \), and \( S(\lambda) - I \) is an integral operator with kernel

\[
i\exp(\pi i(d-3)/4) \lambda^{(d-1)/4} (2\pi)^{-d/2} a(\varphi, \omega; \lambda).
\]

(9)

The scattering problem for the pair \( H_0 = -\Delta, H = -\Delta + q \) can also be considered in the momentum representation. For this it is necessary to study the perturbation of the operator of multiplication by \( p^2 \) in \( L_2(\mathbb{R}^d) \) by the integral operator with kernel \( (2\pi)^{-d/2} \hat{q}(p - p') \), where \( \hat{q} \) is the Fourier transform of the potential \( q \). In this formulation the problem reduces to the Friedrichs-Faddeev model (see §§4.1, 4.2). In the momentum representation the WO act as singular integral operators (with a singularity at \( p^2 = p'^2 \)).

In this scheme of arguments the proof of the existence and completeness of the WO was preceded by an investigation of problem (4), (5) not depending on time. In scattering theory this approach is called the stationary method. One of its advantages consists in obtaining along the way explicit expressions for the WO and the scattering matrix. We further note that the construction of the wave functions \( \psi(x; \omega, \lambda) \) and the study of their properties can be conveniently done by means of the resolvent of the operator \( H \).

Scattering theory for the pair \( H_0 = -\Delta, H = -\Delta + q \) can also be constructed by means of a time-dependent approach. It consists in the direct verification of relations of the type (9). If the spectral analysis of the operator \( H_0 \) can be carried out explicitly (for the operator \( H_0 = -\Delta \) this can be done by Fourier transform), then by the time-dependent method (see the Cook criterion in §2.5) the existence of the WO \( W_\pm(H, H_0) \) can easily be established. Completeness of the WO can be established by an ensuing development of the time-dependent approach. An exposition of the corresponding theory (Essen's method) can be found in the original paper [94] or in the book [34].

On the other hand, the stationary approach described briefly in application to the pair \( H_0 = -\Delta, H = -\Delta + q \) is basic for us. We shall now discuss it in a more general formulation. The idea of the stationary approach (see §§2.7, 2.8) consists in a preliminary change of the definition of the WO in which the unitary groups are replaced by their expressions in terms of the corresponding resolvents \( R_0(z) = (H_0 - z)^{-1} \) and \( R(z) = (H - z)^{-1} \). The representations (8) are obtained in just this way. In the stationary definition in place of the limits (3) as \( t \to \pm\infty \) it is necessary to study the limiting values (in a suitable topology) of the resolvents as the spectral parameter tends to the real axis. As soon as the existence of limit values of the resolvents has been established and the properties of the stationary WO defined in terms of them have been investigated, it is no problem to prove the existence of the time-dependent WO and the fact that they coincide with the stationary WO. More precisely, the existence of a weak limit in (3) as \( t \to \pm\infty \) is established directly, while the isometricity of the corresponding weak WO is verified by stationary means. From this it follows that in (3) the strong limit also exists. As already noted, an important merit of the stationary version is the advanced formula part.

3. The methods used in scattering theory are naturally subdivided into two classes—smooth (Chapter 4) and trace class (Chapter 6). The smooth methods make essential use of the possibility of an explicit spectral analysis of the unperturbed operator \( H_0 \). It is hereby required that the spectral decomposition of \( H_0 \) be sufficiently regular, while the perturbation \( V = H - H_0 \)
be "smooth" relative to $H_0$. Results regarding the full Hamiltonian $H$ are usually derived from this by means of perturbation theory.

The construction of a stationary version of scattering theory relies on the existence (in a suitable topology) and the limits of $R_{\pm}(z)$ and $R(z)$ for $z = \lambda \pm i\epsilon$ and $\epsilon \to 0$. This assertion is often called the principle of limiting absorption. For a differential operator with constant coefficients $H_0$, this principle can be verified directly. However, an analogous investigation for the full Hamiltonian $H$ (for example, for the Schrödinger operator) is a problem of some substance. One of the ways of resolving it (see §4.6) consists in the study of the equation for $R(z)$ and the use of the limiting absorption principle for $H_0$. We remark that the stationary scheme of constructing scattering theory for the Schrödinger operator described in the preceding part is based on the smooth approach.

There are several versions of the concept of smoothness of a perturbation. A unitarily invariant condition on the behavior of the resolvent in a neighborhood of the spectrum can be formulated in terms of so-called Kato smoothness (see §4.3). This concept can be equivalently formulated in terms of the corresponding unitary group. Due to this, in the theory of such perturbations the stationary and time-dependent versions essentially coalesce.

The trace class theory is different in principle from the smooth theory; there the roles of $H_0$ and $H$ even out. The stationary version of the trace class method is based on the fact that for an arbitrary selfadjoint operator $H$ and any operators $G_1, G_2$ of the Hilbert-Schmidt class $\Theta_2$, the product $G_1 R(\lambda \pm i\epsilon) G_2$ has a limit in $\Theta_2$ as $\epsilon \to 0$ for almost all $\lambda \in \mathbb{R}$. In the time-dependent version the role of this assertion is played by the estimate

$$\int_{-\infty}^{\infty} \left\| G \exp(-itH) f \right\|^2 dt \leq 2\pi \|G\|^2 \sup_{\lambda} d(E(\lambda), f, f)$$

(here $E(\cdot)$ is the spectral family of $H$, and $\| \cdot \|_2$ is the norm in $\Theta_2$) whereby the set of $f$ for which the right-hand side is finite is dense in $\mathcal{R}^{(a)}$. The Kato-Rosenblum theorem on the existence of the WO $W_\pm(H, H_0)$ for a difference $H - H_0$ of trace class is basic for the development of the trace class theory. In this theorem it is not possible to relax the condition that the difference $H - H_0$ be of trace class in terms of symmetrically normed ideals. At the same time the passage to various classes of "relatively trace class" operators is decisive for applications to the theory of differential operators.

The smooth and trace class approaches do not overlap one another, and both have useful applications. Their formula parts are identical, but the concrete realization of the formulas may have different meaning. When the unperturbed operator is sufficiently simple and well studied, it is convenient to use the smooth method. In the theory of differential operators this method requires fewer conditions on the behavior of the coefficients of the perturbation at infinity. The trace class approach "eaves out" the roles of the perturbed and unperturbed operators which makes it possible to consider cases where an explicit spectral analysis of the operator $H_0$ is not possible.

The difference in the results of the smooth and trace class theory are clearly manifest in the case of the Schrödinger operator. Suppose now that $H_0 = -\Delta + q_0$, $H = H_0 + q$ in the space $\mathcal{H} = L^2(\mathbb{R}^d)$. It is assumed that $q_0$ is bounded, and

$$|q(x)| \leq C(1 + |x|)^{-\alpha}.$$  \hspace{1cm} (10)

By means of assertions of trace class type the existence and completeness of the WO $W_\pm(H, H_0)$ can be established for $\alpha > d$. An explicit spectral analysis of the operator $H_0$ is required for application of the smooth theory. Such an analysis is possible only for special potentials $q_0$. In this case the WO $W_\pm(H, H_0)$ exist and are complete if the estimate (10) is satisfied for some $\alpha > 1$. Moreover, the smooth theory makes it possible to verify the absence in the spectrum of the operator $H$ of a singular continuous component. Basically, the required spectral analysis of the operator $H_0$ reduces to a description of the behavior at infinity of solutions of the equation $-\Delta \psi + q_0 \psi = \lambda \psi$. This problem can easily be solved, for example, for a periodic function $q_0$ and, in particular, for $q_0 = 0$. On the other hand, for an arbitrary bounded function $q_0$ an effective expansion in terms of the eigenfunctions of the operator $H_0$ cannot be constructed even in the one-dimensional case. The smooth theory is therefore not applicable to such a pair.

At present the question of a substantial union of the smooth and trace class approaches is open. However, such a union cannot apparently be "pushed too far". In any case the existence of the WO $W_\pm(H, H_0)$ for an arbitrary bounded function $q_0$ and any $\alpha > 1$ is doubtful for $d > 1$. For $d = 1$ the existence and completeness of the WO $W_\pm(H, H_0)$ can be verified by the trace class method. For this the structure of the spectrum of the unperturbed operator $H_0$ is inconsequential. In connection with this we note that, as shown in [56], for "almost all" bounded $q_0$ the operators $H_0$ have purely point spectra. The existence and completeness of the WO imply that this result has a certain stability. Namely, the operator $H$ does not have any absolutely continuous component.

To a certain extent the smooth and trace class theories can be united within the framework of the stationary scheme (see Chapter 5) of the so-called "axiomatic" scattering theory. In this theory the existence of the required limits of $R_{\pm}(z)$ and $R(z)$ is assumed, although the limits are understood in a very weak sense. Under this assumption formulas for the stationary WO are obtained, their properties are studied, and their connection with time-dependent definitions is established. The "general scheme" thus makes it possible to avoid duplicating arguments in various concrete situations. The existence of the limits of the resolvents itself in different analytic circumstances can be verified (and interpreted) in various ways.
4. From the completeness of the wave operators it follows that the scattering matrix \( S(\lambda) \) is a unitary function of the spectral parameter \( \lambda \). Moreover, in the theory of relatively compact perturbations \( S(\lambda) \) differs from the identity operator by a compact operator. For example, for the Schrödinger operator \( S(\lambda) \) is a unitary operator in \( L_2(S^{d-1}) \) for all \( \lambda > 0 \), while \( S(\lambda) - I \) is an integral operator with the kernel \( (9) \). Thus, the spectrum of the operator \( S(\lambda) \) consists of eigenvalues lying on the unit circle and accumulating only at the point 1.

Various spectral properties of the scattering matrix are discussed in detail in Chapter 7. The starting point for this is the stationary representation for \( S(\lambda) \). With its help we obtain, for example, estimates of the norm of \( S(\lambda) - I \) in symmetrically normed ideals of compact operators. We note that for the Schrödinger operator the quantity

\[
\int_{S^{d-1}} |a(\theta, \omega; \lambda)|^2 \, d\theta \, d\omega = (2\pi)^{d-1} \lambda^{-(d-1)/2} \|S(\lambda) - I\|_2^2,
\]

further divided by the area of the sphere \( S^{d-1} \) is called the total scattering cross section averaged with respect to the directions \( \omega \) of incidence of particles. Roughly speaking, the scattering cross section shows how much the potential perturbs the flux of particles of energy \( \lambda \) falling on the scattering center.

We discuss in some detail the dependence of the properties of the scattering matrix on the assumptions regarding the potential. The problem of recovering the perturbation on the basis of the scattering matrix is important in applications to differential operators. This is called the inverse problem. It is not considered in the present work.

Chapter 8 is associated with the trace class theory but goes beyond the framework of scattering theory itself. There we study the integral representation

\[
\text{Tr} [f(H) - f(H_0)] = \int_{-\infty}^{\infty} \xi(\lambda) f'(\lambda) \, d\lambda
\]

for the trace of the difference of rather arbitrary functions of the operators \( H_0 \) and \( H \). The function \( \xi(\lambda) \) on the right-hand side of the trace formula (11) is called the spectral shift function for the pair \( H_0, H \). On the continuous spectrum, \( \xi(\lambda) \) is connected with the scattering matrix by the relation \( \text{Det} S(\lambda) = \exp(-2\pi i \xi(\lambda)) \). On the discrete spectrum, \( \xi(\lambda) \) depends on the shift of the eigenvalues of the operator \( H \) relative to the eigenvalues of \( H_0 \). This explains the name of the function \( \xi(\lambda) \).

### CHAPTER 1

#### Preliminary Facts

Here we collect various facts needed which properly do not belong to scattering theory.

§1. Measure theory

Scattering theory requires a classification of the spectrum of a selfadjoint operator based on measure theory. We briefly recall the basic facts regarding Borel measures on the line \( \mathbb{R} \). Details can be found in any textbook on measure theory (see, for example, [26]). It suffices, however, to turn to the introductory chapter of [4] or to the textbook [15] on the theory of functions of a real variable. We shall also need some elementary facts from that theory, which we shall recall when needed.

1. Borel sets are formed from open and closed sets by taking countable unions and intersections of them. Relative to these operations the Borel sets form a \( \sigma \)-algebra. We sometimes use the notation \( X' = \mathbb{R} \setminus X \).

Any nonnegative, countably additive function \( m \) defined on this \( \sigma \)-algebra is called a (Borel) measure. It is also assumed that the measure of any bounded interval is finite. Sometimes to the supply of Borel sets on which a measure \( m \) is defined there are also added all possible subsets of all sets of \( m \)-measure zero. In this case the measure \( m \) is called complete. In the questions we address, completeness of the measure is of little consequence, and we do not dwell on it.

Any set \( Z \) of full \( m \)-measure (i.e., such that \( m(Z') = 0 \)) we call a Borel support (BS) of the measure \( m \). Among all BS there exists a (unique, of course) smallest closed set, called simply the support of \( m \) and denoted by \( \text{supp} \, m \).

2. A measure \( m_1 \) is called absolutely continuous relative to a measure \( m_2 \) if \( m_1(X) = 0 \) whenever \( m_2(X) = 0 \). The Radon-Nikodým theorem asserts that this is the case if and only if for some \( m_2 \)-measurable function \( f \) for any Borel set \( X \)

\[
m_1(X) = \int_X f(\lambda) \, d m_2(\lambda), \quad f(\lambda) \geq 0.
\]

The function \( f = \frac{d m_1}{d m_2} \) is called the Radon-Nikodým derivative of the
measure $m_1$ with respect to the measure $m_2$. If the measure $m_1$ is finite, i.e., $m_1(\mathbb{R}) < \infty$, then $f$ belongs to the space $L_1(\mathbb{R}; dm_2)$ of functions absolutely integrable on $\mathbb{R}$ with respect to the measure $m_2$. Measures that are absolutely continuous with respect to one another are considered equivalent. It is clear that the supports of equivalent measures coincide. All mutually equivalent measures are combined into a single class called the type.

Measures $m_1$ and $m_2$ are called mutually singular if they have nonintersecting BS $Z_1$ and $Z_2$, i.e., $m_1(Z_2) = 0$, $j = 1, 2$, $Z_1 \cap Z_2 = \emptyset$. A nonzero measure $m_1$ cannot simultaneously be absolutely continuous with respect to $m_2$ and also singular with respect to it. The Lebesgue decomposition theorem asserts that for any measures $m_1$ and $m_2$ the first of them can be uniquely represented in the form

$$m_1 = m_{1,a} + m_{1,s},$$

where $m_{1,a}$ is absolutely continuous with respect to $m_2$, and $m_{1,s}$ is singular with respect to $m_2$.

Lebesgue measure, denoted by $|\cdot|$, plays a special role among all measures on $\mathbb{R}$; integration with respect to it is denoted by $d\lambda$ (in place of $dm(\lambda)$). In particular, the term “almost everywhere” (a.e.) presumes a set of full Lebesgue measure. The relation $X = Y$ (mod 0) means that the sets $X$ and $Y$ coincide up to a set of Lebesgue measure zero.

Unless explicitly stipulated otherwise, absolute continuity and singularity of various measures are understood to be with respect to Lebesgue measure. The definition of absolute continuity can be formulated in an equivalent way in terms of sets $X$ consisting of a finite number of nonintersecting (open) intervals. Namely, a measure $m$ is absolutely continuous if for any $\varepsilon > 0$ there exists $\delta > 0$ such that $m(X) < \varepsilon$ for $|X| < \delta$. A measure $m$ is called (simply) continuous if $m(X) = 0$ for any singleton and therefore for any countable $X$.

For a Borel set $Y$ we denote by $m_Y(X) = m(X \cap Y)$ the restriction of $m$ to $Y$. By definition, absolute continuity of $m$ on $Y$ means that the measure $m_Y$ is absolutely continuous. An analogous agreement pertains to other properties of the measure $m$. We note that a measure is absolutely continuous if its restriction to any finite interval is absolutely continuous.

A decomposition of the form (2) of the measure $m$ with respect to Lebesgue measure can be constructed by knowing the set on which the singular component of $m$ is concentrated. Namely, suppose that $|Z_1| = 0$ and $m_2(Z_1) = 0$; then

$$m_a = m_{Z_1}, \quad m_s = m_2 - m_{Z_1}.\quad (3)$$

3. A Borel measure can be constructed on the basis of any nondecreasing, left-continuous function $F(\lambda)$, $\lambda \in \mathbb{R}$. For the interval $[\lambda_0, \lambda_1]$ we set

$$m([\lambda_0, \lambda_1]) = F(\lambda_1) - F(\lambda_0).\quad (4)$$

By countable additivity such a function $m(\cdot)$ can be extended to a measure $m$ defined on all Borel subsets of $\mathbb{R}$. A function $F$ for which (4) is satisfied is sometimes called a generating function of the measure $m$. A generating function $F = F_m$ exists for any measure $m$ and is determined uniquely up to an additive constant.

By Lebesgue's theorem (see, for example, Chapter 8.2 in [15]) the nondecreasing function $F(\lambda)$ has a derivative $f(\lambda) = F'(\lambda) > 0$ for a.e. $\lambda \in \mathbb{R}$, and $f \in L_1(\mathbb{R})$, i.e., it is integrable on any finite interval. The definitions of absolute continuity and singularity of the measure $m$ can be reformulated in an equivalent way in terms of $f = f_m$. Namely, $m$ is absolutely continuous if

$$m(X) = \int_X f(\lambda) d\lambda$$

for any interval (and hence for any Borel set) $X$. A measure $m$ is singular if $f(\lambda) = 0$ for a.e. $\lambda$. From this it follows that $f$ coincides with the Radon-Nikodým derivative of the absolutely continuous part of the measure $m$ (relative to Lebesgue measure). In other words,

$$m_a(X) = \int_X f(\lambda) d\lambda\quad (5)$$

and for a.e. $\lambda$

$$F_m'(\lambda) = F_m'(\lambda).\quad (6)$$

The BS $Z_1$ and $Z_2$ of the singular $m_2$ and absolutely continuous $m_a$ components of the measure $m$ can also be described in terms of the function $f$. More precisely, we now consider $f$ to be a symmetric derivative, i.e.,

$$f(\lambda) = \lim_{\varepsilon \to 0} (2\varepsilon)^{-1} [F(\lambda + e) - F(\lambda - e)].\quad (7)$$

Then $Z_1$ consists of points $\lambda$ where the limit (7) exists and is equal to $+\infty$. We denote by $Z_0$ the set of points $\lambda$ at which either the limit (7) does not exist or is equal to zero. It turns out that $m(Z_0) = 0$. Therefore, the absolutely continuous component is concentrated (i.e., $m_a = m_{Z_1}$) on the set

$$Z_0 = R\setminus Z_1 \cup Z_2,$$

where the symmetric derivative exists, is finite, and is not equal to zero. For continuous measures $m$ these assertions are preserved if $f$ is considered to be an ordinary (not symmetric) derivative.

4. Among all BS of a given measure $m$ in spectral theory it is necessary to distinguish, in addition to the support supp $m$, other sets which possess minimal properties (relative to Lebesgue measure). Let us accept the following definition.

**Definition.** A BS $Z^{(0)}$ of the measure $m$ is called minimal if for any other BS $Z$ of this measure $|Z^{(0)} \setminus Z| = 0$.

A minimal BS always exists but, of course, it is not unique. Indeed, suppose to begin that $m$ has a BS of finite Lebesgue measure, i.e., $z = \inf |Z| < \infty$ where the infimum is taken over all BS of $m$. Consider any
sequence of BS $Z_n$ for which $|Z_n| \to z$. Then $Z^{(0)} = \bigcap_n Z_n$ is a minimal BS of $m$, since the Lebesgue measure $|Z^{(0)}| = z$ is minimal among all BS of $m$. In the general case the construction of a minimal BS can be carried out on each interval $(n, n+1)$, $n = 0, \pm 1, \pm 2, \ldots$, separately and then the union is taken over all $n$.

Two minimal BS can differ by no more than a set of Lebesgue measure zero. Given minimal BS $Z^{(0)}$ a set of Lebesgue measure zero can always be added. In the case of an absolutely continuous measure $m$ it is also possible to subtract from $Z^{(0)}$ an arbitrary set of Lebesgue measure zero. For a singular measure the minimal BS has Lebesgue measure zero. Thus, a minimal BS of a Borel measure is a minimal BS also for its absolutely continuous component.

A minimal BS can always be chosen to belong to $\text{supp} m$. However, $\text{supp} m$ may not be a minimal BS. We present a simple example of such a measure. We consider the restriction $| \cdot |_{G}$ of Lebesgue measure to the Borel set $G$ (by definition $|X|_{G} = |G \cap X|$). For any open set $G$ the support of such a measure is the closure $\overline{G}$ of this set, while one of the minimal BS coincides with $G$ itself. Therefore, for $|G| > 0$ (and such open sets exist) $G$ is not a minimal BS.

In terms of the sets introduced in Part 3, a minimal BS can be given by the equality $Z^{(0)} = \mathbb{R} \setminus Z_0$. In particular, $Z^{(0)} = Z_0$ for an absolutely continuous measure $m$. In this case $m$ is equivalent to the restriction $| \cdot |_{Z_0}$ of Lebesgue measure to the minimal BS. An absolutely continuous measure $m$ is absolutely continuous with respect to $| \cdot |_{\text{supp} m}$ but it may not be equivalent to it.

5. Together with the usual nonnegative measures we consider arbitrary real $\sigma$-additive functions (charges) on Borel sets. Complex measures are also admitted. In this case by $|m|$ we mean the total variation of the measure $m$. We often identity Borel measures and their generating functions—complex functions of locally bounded variation. In addition to measures on the line we shall also need analogous objects on the unit circle.

6. We make systematic use of various theorems (for the proof of them see, for example, [15]) on passage to the limit under the integral sign. The most useful is Lebesgue’s theorem.

**Theorem 2.** Suppose $|f_\varepsilon(\lambda)| \leq \varphi(\lambda)$, where $\varphi \in L_1(\mathbb{R})$, for almost all $\lambda \in \mathbb{R}$ and

$$\lim_{\varepsilon \to 0} f_\varepsilon(\lambda) = f(\lambda), \quad a.a. \lambda \in \mathbb{R}. \quad (8)$$

Then

$$\lim_{\varepsilon \to 0} \int_{-\infty}^{\infty} f_\varepsilon(\lambda) d\lambda = \int_{-\infty}^{\infty} f(\lambda) d\lambda. \quad (9)$$

The next more general but less effective assertion is due to Vitali.

§2. Analytic Functions

**Theorem 3.** Suppose for functions $f_\varepsilon \in L_1(\mathbb{R})$ the integrals

$$\int_{-\infty}^{\infty} f_\varepsilon(\lambda) d\lambda \leq \sup_{\varepsilon \to 0} \int_{-\infty}^{\infty} |f_\varepsilon(\lambda)| d\lambda. \quad (10)$$

tend to zero uniformly with respect to $\varepsilon$ as $|X| \to 0$. Suppose also the same for $X = (-\infty, N) \cup (N, \infty)$ as $N \to \infty$. Then under condition (8) $f \in L_1(\mathbb{R})$ and the relation (9) holds.

We note that equicontinuity of the integrals (10) is necessary in some sense for the validity of (9).

The following theorem (Fatou’s lemma) is convenient for passage to the limit in inequalities.

**Theorem 4.** Under condition (8)

$$\int_{-\infty}^{\infty} \|f(\lambda)\| d\lambda \leq \sup_{\varepsilon \to 0} \int_{-\infty}^{\infty} \|f_\varepsilon(\lambda)\| d\lambda. \quad (11)$$

**Remark 5.** We shall also need a version of this theorem for the case of vector-valued functions $f_\varepsilon(\lambda)$ with values in a Hilbert space $\mathcal{H}$. Namely, if $f_\varepsilon(\lambda)$ converge weakly as $\varepsilon \to 0$ to $f(\lambda)$ for a.e. $\lambda$ then

$$\int_{-\infty}^{\infty} \|f(\lambda)\|^2 d\lambda \leq \sup_{\varepsilon \to 0} \int_{-\infty}^{\infty} \|f_\varepsilon(\lambda)\|^2 d\lambda. \quad (12)$$

We note that in Theorems 2–4 condition (8) can be replaced by the assumption of the convergence of $f_\varepsilon$ to $f$ in Lebesgue measure on any finite interval. The results stated hold for arbitrary Borel measures.

We often use Fubini’s theorem to interchange the order of integration in double integrals. It asserts that the change of order is legitimate if at least one of the integrated integrals converges absolutely.

§2. Analytic functions

In this section we collect results used below regarding analytic (holomorphic) functions.

1. The stationary formulation of scattering theory is based on the investigation of the boundary values of the resolvent of a selfadjoint operator as the spectral parameter tends to the real axis. The study of the behavior of analytic functions on the boundary of the domain of analyticity is therefore of special importance. Proofs of the results presented here can be found, for example, in the books [6] and [17].

To be specific we consider the case of functions $\varphi(z)$ analytic in the upper (lower) half-plane. We denote by $\Lambda$ an interval of the real axis. We say that at a point $\lambda \in \mathbb{R}$ the function $\varphi(z)$ has a (finite) angular limit value $\varphi(\lambda \pm i0)$ if $\varphi(z)$ converges to $\varphi(\lambda \pm i0)$ as $z \to \lambda$ in any sector $0 < \theta < \arg(z - \lambda) < \pi - \theta$. The notation $\varphi(\lambda \pm i0)$ has a similar meaning. As a rule, in scattering theory it suffices to consider radial limit values where $\varphi(\lambda \pm i0)$ is understood.
as the limit of \( \phi(\lambda \pm i\varepsilon) \) as \( \varepsilon \to 0 \). Therefore, unless otherwise mentioned the limit values \( \phi(\lambda \pm i0) \) are considered to be radial.

We first present a very general uniqueness theorem. It is called the Luzin-Privalov theorem.

**Theorem 1.** Suppose that the function \( \phi(z) \) is holomorphic in the rectangle \( \text{Re} \, z \in \Lambda, \text{Im} \, z \in (0, \varepsilon_0) \) and has angular limit values \( \phi(\lambda + i0) \) for almost all \( \lambda \in \Lambda \). If \( \phi(\lambda + i0) = 0 \) on a set of positive Lebesgue measure, then the function \( \phi \) is identically zero.

In particular, the uniqueness theorem is true for functions continuous up to the boundary. A more general theorem can be obtained in terms of Hardy classes. We recall that the Hardy class \( \mathcal{H}^p \), \( 1 \leq p \leq \infty \), consists of functions \( \phi(z) \) for which the following quantity is finite:

\[
||\phi||_p^p = \sup_{\varepsilon > 0} \int_{-\infty}^{\infty} |\phi(\lambda + i\varepsilon)|^p d\lambda
\]

(1)

for \( p = \infty \) the integral is, of course, to be understood as the norm of \( \phi(\lambda + i\varepsilon) \) in \( L_{\infty} \) with respect to \( \lambda \). The following result holds.

**Theorem 2.** A function \( \phi \in \mathcal{H}^p \), \( 1 \leq p \leq \infty \), has angular limit values \( \phi(\lambda + i0) \) for a.e. \( \lambda \in \mathbb{R} \).

Combining Theorems 1 and 2, we obtain an effective uniqueness theorem.

**Theorem 3.** If \( \phi \in \mathcal{H}^p \), \( 1 \leq p \leq \infty \), and \( \phi(\lambda + i0) = 0 \) on a set of positive measure, then \( \phi \) is identically equal to zero.

**Remark 4.** Theorems 2 and 3 remain in force for functions with non-negative imaginary part in the upper half-plane. This can easily be seen by applying Theorems 2 and 3 to the bounded function

\[
\phi(z) = |\phi(z) - i||\phi(z) + i|^{-1}.
\]

Just as Theorem 1, Theorems 2 and 3 have local character. This means that they are true for functions defined only in some rectangle \( \text{Re} \, z \in \Lambda, \text{Im} \, z \in (0, \varepsilon_0) \), when the integration in (1) is restricted to the interval \( \Lambda \). Of course, the definition of the Hardy spaces and all the results for them carry over in a natural way to the case of functions analytic in the unit disk.

2. The resolvent of a selfadjoint operator is the Cauchy-Stieltjes integral with respect to its spectral measure. The investigation of such integrals therefore plays an important role in scattering theory. The next two assertions are called Privalov theorems. Proofs of them can be found in the books [14], [17].

**Theorem 5.** Suppose \( F \) is a complex function of bounded variation, and

\[
\mathcal{C}(z) = \int_{-\infty}^{\infty} (\mu - z)^{-1} dF(\mu)
\]

(2)
is its Cauchy-Stieltjes transform. Then the angular limit values

\[
\mathcal{C}(\lambda + i0) = \pm \pi i \frac{dF(\lambda)}{d\lambda} + \text{p.v.} \int_{-\infty}^{\infty} (\mu - \lambda)^{-1} dF(\mu),
\]

(3)

exist for a.e. \( \lambda \in \mathbb{R} \), where the integral in the sense of principal value on the right-hand side also exists for a.e. \( \lambda \in \mathbb{R} \). Moreover, for any interval \( \Lambda = [\lambda_0, \lambda_1] \) and \( \overline{\Lambda} = [\lambda_0, \lambda_1] \) the measure \( m \) corresponding to \( F \) (see (1.4)) can be recovered by the Cauchy-Stieltjes inversion formula

\[
m(\Lambda) + m(\overline{\Lambda}) = (\pi i)^{-1} \lim_{\varepsilon \to 0} \int_{\Lambda} |\mathcal{C}(\lambda + i\varepsilon) - \mathcal{C}(\lambda - i\varepsilon)| \, d\lambda.
\]

If the function \( F \) is smooth the integral (2) has the same smoothness as the derivative of \( F \).

**Theorem 6.** Suppose the function \( F \) is constant outside a finite interval \( \Lambda = [a, b] \), \( F \) is continuously differentiable, and the derivative \( f(\lambda) = F'(\lambda) \) satisfies a Hölder condition with exponent \( \alpha \in (0, 1) \). Then the integral (2) is Hölder continuous in \( z \) in the complex plane with a cut along \( \Lambda \) with the same exponent \( \alpha \), and for \( |z' - z| \leq 1 \)

\[
|\mathcal{C}(z)| + |z' - z|^{-\alpha} |\mathcal{C}(z') - \mathcal{C}(z)| \leq C \sup_{\lambda, \lambda' \in \Lambda} (|f(\lambda)| + |\lambda' - \lambda|^{-\alpha} |f(\lambda') - f(\lambda)|).
\]

(4)

It is clear that under the conditions of this theorem

\[
\mathcal{C}(z) = \int_a^b (\mu - z)^{-1} f(\mu) \, d\mu.
\]

(5)

If \( f \) is Hölder continuous but the condition \( f(a) = f(b) = 0 \) does not hold, then the assertion of Theorem 6 holds outside arbitrary neighborhoods of the points \( a \) and \( b \).

The integral (2) is smoother the smoother the function \( F \) is. Namely, if under the conditions of Theorem 6 the function \( F \) has \( n + 1 \) derivatives and \( f^{(n)}(\lambda) \) satisfies a Hölder condition, then \( \mathcal{C}^{(n)}(z) \) is continuous up to \( \Lambda \). An estimate of the form (4) holds for the derivatives. This can be seen by differentiating (2), integrating by parts, and applying Theorem 6 to \( \mathcal{C}^{(n)}(z) \).

3. For the Poisson integral

\[
\mathcal{P}(\lambda, \varepsilon) = (2\pi i)^{-1} [\mathcal{C}(\lambda + i\varepsilon) - \mathcal{C}(\lambda - i\varepsilon)]
\]

(6)

the results of the preceding part admit further refinement. Thus, according to relation (3),

\[
\lim_{\varepsilon \to 0, \varepsilon > 0} \mathcal{P}(\lambda, \varepsilon) = f(\lambda), \quad f = F',
\]

(7)

for a.e. \( \lambda \). Fatou's theorem (see, for example, [6]) makes it possible to describe this set of full measure.
Theorem 7. Suppose $F$ is a function of bounded variation. Then the limit (7) exists and relation (7) is satisfied at those points $\lambda$ where $F$ has a symmetric derivative (1.7).

We emphasize that for a real, nondecreasing $F$ we here admit the case $F'(\lambda) = +\infty$. For such $\lambda$ the limit of $P(\lambda, e)$ as $e \to 0$ is also equal to $+\infty$. We shall further need an assertion [6] regarding convergence of Poisson integrals in the class $L_1$.

Theorem 8. Suppose that the function $F$ is absolutely continuous and $f = F' \in L_1(\mathbb{R})$. Then the relation (7) is satisfied in the sense of the space $L_1(\mathbb{R})$.

In terms of the Poisson integral it is easy to give conditions that the function $F$ (or the corresponding measure $m$) be absolutely continuous on any interval $\Lambda_0$. It suffices, for example, to suppose that the function $P(\lambda, e)$ is bounded for $\lambda \in \Lambda_0$ and $e > 0$. On the basis of Lebesgue's theorem, for $\Lambda \subset \Lambda_0$ and $e \to 0$, it is possible to pass to the limit under the integral sign in the relation

$$m(\Lambda) + m(\overline{\Lambda}) = 2 \lim_{\epsilon \to 0} \int_{\Lambda} P(\lambda, e) \, d\lambda.$$ 

By equality (7) this gives the representation

$$m(\Lambda) = \int_{\Lambda} f(\lambda) \, d\lambda, \quad f \in L_1,$$

and therefore the measure $m$ is absolutely continuous on $\Lambda_0$.

Relations (3) or (7) now show that the smoothness of the function $F(\lambda)$ can be described in the form of conditions on the Cauchy-Stieltjes or Poisson integrals. For example, if the function $\mathcal{G}(z)$ is continuous in the rectangles $\text{Re} \, z \in \Lambda_0$, $\pm \text{Im} \, z \geq 0$, then $F(\lambda)$ is continuously differentiable on $\Lambda_0$.

4. We shall indicate conditions that an analytic function be representable in the form of an integral (2) with a real nondecreasing function $F$ (with a finite measure $m$—see (1.4)). In this case $\mathcal{G}(z) = \mathcal{G}(\zeta)$, and the inversion formula of Theorem 5 can be written in the form

$$m(\Lambda) + m(\overline{\Lambda}) = 2 \pi \lim_{\epsilon \to 0} \int_{\Lambda} \text{Im} \mathcal{G}(\lambda + ie) \, d\lambda.$$ 

For the proof of the next assertion see, for example, the book [1].

Theorem 9. In order that a function $\mathcal{G}(z)$ holomorphic in the upper half-plane admit the representation (2) with a real, nondecreasing function $F$ of bounded variation it is necessary and sufficient that $\text{Im} \mathcal{G}(z) \geq 0$ for $\text{Im} z > 0$ and that

$$\sup_{\gamma \geq 1} |\mathcal{G}(i\gamma)| < \infty.$$ 

Considering further that for a bounded function $\text{Im} \mathcal{G}(z)$ the measure $m$ is absolutely continuous, we obtain

Corollary 10. Suppose the function $\mathcal{G}(z)$ is holomorphic in the upper half-plane, has nonnegative (or nonpositive) bounded imaginary part, and (9) is satisfied. Then

$$\mathcal{G}(z) = \int_{-\infty}^{\infty} (\lambda - z)^{-1} f(\lambda) \, d\lambda, \quad f \in L_1, \quad f \geq 0 \text{ (resp. } f \leq 0 \text{)}.$$ 

where $f$ is constructed from the inversion formula (7).

5. The resolvent of a unitary operator is holomorphic inside and outside the unit disk. The corresponding spectral representation is given by the Cauchy-Stieltjes integral over the unit circle.

In integrating over $T$ we always assume that $T$ is traversed in a counterclockwise manner. This direction is taken to be positive. For a function $\varphi(\zeta)$ analytic of $|\zeta| \neq 1$ we denote by $\varphi(\mu_+) \ (or \ \varphi(\mu_-)), \ |\mu| = 1$, its angular limit values as $\zeta$ tends to the point $\mu \in T$ from within (respectively, from without). The next assertion is analogous to Theorem 5.

Theorem 11. Suppose $F$ is a function of bounded variation (or the corresponding finite complex measure) and

$$\mathcal{G}(\zeta) = \int_{T} (\nu - \zeta)^{-1} \, dF(\nu).$$ 

Then for $a.e. \ \lambda \in T$ there exist the angular limit values

$$\mathcal{G}(\mu_+) \pm \pi i \frac{dF(\mu)}{d\mu} + \text{p.v.} \int_{T} (\nu - \mu)^{-1} \, dF(\nu),$$

where the integral in the sense of principal value on the right-hand side also exists for $a.e. \ \mu \in T$. In particular,

$$\mathcal{G}(\mu_+) - \mathcal{G}(\mu_-) = 2\pi i \frac{dF(\mu)}{d\mu}.$$ 

If in (11) $dF(\mu) = i\mu \, d\mu(\mu)$, where the measure $m$ is real, then there is the symmetry equality

$$\mathcal{G}(\zeta) = \overline{\mathcal{G}(\overline{\zeta})} + \mathcal{G}(0), \quad \text{Re} \mathcal{G}(0) = 0.$$ 

In this case, along with (13), we have the relations

$$2\pi i \frac{dF(\mu)}{d\mu} = 2i \text{Im} \mathcal{G}(\mu_+) - \mathcal{G}(0)$$

$$= -2i \text{Im} \mathcal{G}(\mu_-) + \mathcal{G}(0).$$

The next assertion can be obtained (see [1]) from Theorem 9 by means of a fractional linear transformation of the upper half-plane onto the unit disk.

Theorem 12. In order that a function $\mathcal{G}(\zeta)$ holomorphic for $|\zeta| < 1$ admit the representation

$$\mathcal{G}(\zeta) = \mathcal{G}(0) + i \int_{T} \mu(\mu - \zeta)^{-1} \, dF(\mu).$$
with a finite, nonnegative measure \( m \) on the circle \( T \) it is necessary and sufficient that 
\[ \Im \mathcal{S}(\zeta) \geq 0. \]
Then \( m(T) = 2 \Im \mathcal{S}(0) \).

As in the case of functions holomorphic in a half-plane, under the additional condition of boundedness of \( 1 + \mathcal{S}(\zeta) \) the measure \( m \) is absolutely continuous, so that
\[ i \mu d m(\mu) = \eta(\mu) d \mu, \quad \text{where } \eta \in L_1(T), \, \eta \geq 0. \]  
Moreover, if the nonnegativity of \( \eta \) is waved away, then the representation
\[ \mathcal{S}(\zeta) = \mathcal{S}(0) + \int_T (\mu - \zeta)^{-1} \eta(\mu) d \mu \]  
holds for any function \( \mathcal{S} \) holomorphic in \( |\zeta| < 1 \) with bounded imaginary part. It is easy to see this by going over to the new function \( \mathcal{S}(\zeta) = \mathcal{S}(\zeta) + i a \) where \( a \) is a suitable positive number. Finally, Theorem 12 implies

**Corollary 13.** Suppose the function \( \mathcal{S}(\zeta) \) is holomorphic for \( |\zeta| < 1 \), has bounded imaginary part there, and \( \Re \mathcal{S}(0) = 0 \). Then there is the representation
\[ \mathcal{S}(\zeta) = \int_T (\mu - \zeta)^{-1} \eta(\mu) d \mu, \quad \eta = \bar{\eta} \in L_1(T), \]  
where \( \eta \) can be recovered by the formula
\[ \eta(\mu) = \pi^{-1} \Im \mathcal{S}(\mu) - (2 \pi i)^{-1} \mathcal{S}(0). \]

**Proof.** We need only apply (18) to the function \( \mathcal{S}(\zeta) = \mathcal{S}(\zeta) - 2^{-1} \mathcal{S}(0) \). The inversion formula is a consequence of the first equality in (15). \( \square \)

6. The next assertion is usually called the Hadamard three lines theorem. It is the basis for complex interpolation of operators (see Part 6 of §6).

**Theorem 14.** Suppose the function \( \varphi(z) \) is holomorphic in the strip \( a_1 < \Re z < a_2 \), is continuous up to the boundary, and is uniformly bounded in this strip. Suppose for all \( y \in \mathbb{R} \)
\[ \varphi(a_n + iy) = C_n, \quad n = 1, 2. \]
Then for any \( x \in (a_1, a_2) \)
\[ |\varphi(x + iy)| \leq C_1^{-a} C_2^a, \quad a = a(x) = (x - a_1)(a_2 - a_1)^{-1}. \]
For the proof see, for example, Volume 2 of the course [18].

§3. Absolutely continuous and singular spectra

1. Let \( H \) be any selfadjoint operator with domain \( \mathcal{D}(H) \) in a separable Hilbert space \( \mathcal{H} \). We denote by \( E_\mu(X) \) the spectral measure of the operator \( H \) which is also called its partition of unity or its spectral family. The notation for the dependence of various objects on \( H \) is often omitted. For the spectral measure it is possible to use the usual terminology of measure

theory. The operator-valued measure \( E(X) \) is defined on all Borel sets \( X \subset \mathbb{R} \). We sometimes also use the notation \( E(\lambda) := E((\infty, \lambda)) \), so that \( E(\lambda) \) is the generating function of the measure \( E(X) \). The support \( \text{supp} E_H := \sigma(H) \) of the spectral measure is called the spectrum of the operator \( H \). The complementary set \( \rho(H) = \mathbb{C} \setminus \sigma(H) \) consists of regular points \( z \) at which the inverse operator \( R(z) = (H - z)^{-1} \) exists, is defined on all of \( \mathcal{H} \), and is bounded.

For what follows we note the "polarization" identity
\[ 4(f, g) = \| f + g \|^2 - \| f - g \|^2 + i\| f + ig \|^2 - i\| f - ig \|^2 \]  
and the Schwarz inequality
\[ |(E(X)f, g)|^2 \leq (E(X)f, f)(E(X)g, g). \]  

2. The classification of the spectrum needed for scattering theory can be obtained by considering the collection of measures \( m(\cdot; f) = (E_H(\cdot)f, f) \) for all possible \( f \in \mathcal{H} \). We note immediately that
\[ m(X; E(Y)f) = m(X \cap Y; f) = m_Y(X; f). \]  

**Definition 1.** An element \( f \in \mathcal{H} \) is called absolutely continuous (singular) relative to \( H \) if the measure \( m(\cdot; f) \) is absolutely continuous (singular). We denote the set of absolutely continuous (singular) elements by \( \mathcal{H}^{(a)} = \mathcal{H}^{(s)} \) (\( \mathcal{H}^{(s)} = \mathcal{H}^{(s)} \)).

**Lemma 2.** Suppose one of the elements \( f \) or \( g \) is absolutely continuous (singular), while the second is arbitrary. Then the complex measure \( m(\cdot; f, g) = (E(\cdot)f, g) \) is absolutely continuous (singular).

**Proof.** Suppose, for example, that the element \( f \) is absolutely continuous (singular). According to (2),
\[ |(E(X)f, g)|^2 \leq (E(X)f, f)\|g\|^2. \]  
If \( f \in \mathcal{H}^{(a)} \) then for \( |X| = 0 \) we have \( (E(X)f, f) = 0 \) and hence \( (E(X)f, g) = 0 \). If \( f \in \mathcal{H}^{(s)} \), then \( (E(Z')f, f) = 0 \) for some Borel set \( Z \) with \(|Z| = 0 \) and \( Z' = \mathbb{R}\setminus Z \). We now apply (4) for \( X = Z' \). Then \( (E(Z')f, f) = 0 \), so that the measure \( (E(\cdot)f, g) \) is singular. \( \square \)

**Lemma 3.** The sets \( \mathcal{H}^{(a)} \) and \( \mathcal{H}^{(s)} \) form subspaces. These subspaces are orthogonal and
\[ \mathcal{H}^{(a)} \oplus \mathcal{H}^{(s)} = \mathcal{H}. \]  

**Proof.** The linearity of the sets \( \mathcal{H}^{(a)} \) and \( \mathcal{H}^{(s)} \) follows directly from Lemma 2. Moreover, by the same lemma \( (E(X)f, g) = 0 \) for \( f \in \mathcal{H}^{(a)} \), \( g \in \mathcal{H}^{(s)} \) and any Borel set \( X \subset \mathbb{R} \). In particular, for \( X = \mathbb{R} \) this implies that the linear manifolds \( \mathcal{H}^{(a)} \) and \( \mathcal{H}^{(s)} \) are orthogonal to one another. It remains to show that any element \( \varphi \in \mathcal{H} \) can be represented in the form
3. We recall that, by definition, an element \( h \) has maximal spectral type relative to a selfadjoint operator \( H \) if all the measures \( m(\cdot; h) \), \( f \in \mathcal{H} \), are absolutely continuous with respect to the measure \( m(\cdot; h) \). In a separable Hilbert space \( \mathcal{H} \) there is necessarily such an element \( h \) (see the book [4]) although it may not be unique. For the measure \( m(\cdot; h) \) of maximal spectral type the equalities \( E(X) = 0 \) and \( m(X; h) = 0 \) are equivalent. From this it follows that \( \sigma(H) = \text{supp} m(\cdot; h) \). The type of the measure \( m(\cdot; h) \) is called the spectral type of the operator-valued measure \( E(\cdot) \) or of the operator \( H \) itself.

It is now not hard to establish the following elementary assertion.

**Lemma 6.** The projection onto the absolutely continuous subspace admits the representation

\[
P^{(a)} = E(Z_a)
\]

for some Borel set \( Z_a \).

**Proof.** Suppose \( h \) has maximal spectral type. We consider the singular part \( m_s(\cdot; h) \) of the measure \( m(\cdot; h) \) and a set \( Z_s \) with \( |Z_s| = 0 \) such that \( m_s(\mathbb{R} \setminus Z_s; h) = 0 \). We shall see that \((7)\) is satisfied for \( Z_a = \mathbb{R} \setminus Z_s \). It suffices to show that for any \( f \in \mathcal{H} \) the element \( E(Z_a)f \) is singular, while \( E(Z_a)f \) is absolutely continuous. The first of these assertions follows from the fact that according to \((3)\) the measure \( m(\cdot; E(Z_a)f) \) is concentrated on the set \( Z_a \) of Lebesgue measure zero. By \((1.3)\) for \( |X| = 0 \) we have \( m(X \cap Z_a; h) = m_s(X; h) = 0 \) and hence \( m(X \cap Z_a; f) = 0 \). In accordance with \((3)\), this implies that \( E(Z_a)f \in \mathcal{H}(n) \).

The set \( Z_a \) for which \((7)\) holds is not uniquely determined. We note that the set \( Z_a \) constructed in the proof of the lemma has full Lebesgue measure in \( \mathbb{R} \).

**Corollary 7.** Suppose \( h \) has maximal spectral type relative to \( H \). Then the elements \( P^{(a)}h \) and \( P^{(s)}h \) have maximal spectral types relative to \( H^{(a)} \) and \( H^{(s)} \) respectively. Moreover,

\[
\sigma^{(a)}(H) = \text{supp} m_s(h), \quad \sigma^{(s)}(H) = \text{supp} m_s(h).
\]

**Proof.** A part from Lemma 6 equalities \((3)\) and \((6)\) should be taken into account.

As in the scalar case, a minimal BS of a spectral measure \( E \) is a BS \( Z^{(0)} \) such that for any other set \( Z \) of full \( E \)-measure we have \( |Z^{(0)} \setminus Z| = 0 \). Of course, the minimal BS of the spectral measure \( E(\cdot) \) and of the ordinary measure \( E(\cdot)h \) coincide if \( h \) is an element of maximal spectral type. We now introduce

**Definition 8.** Each of the minimal BS of the spectral measure \( E \) belonging to \( \sigma(H) \) is called a core of the spectrum of the operator \( H \). The core of the spectrum is denoted by \( \delta(H) \).
A minimal BS can be introduced also for the absolutely continuous part $E^{(a)} = p^{(a)} E$ of the spectral measure of $H$. In this case it is naturally called a core of the absolutely continuous spectrum. By the way, $\tilde{\sigma} = \tilde{\sigma}(H)$ is a minimal BS also for the spectral measure $E^{(a)}$. The concept of a core of the spectrum is therefore sufficient for scattering theory.

We emphasize that the set obtained from any core of the spectrum by addition of a set of measure zero is also a core of the spectrum. The set obtained from $\tilde{\sigma}$ by subtraction of a set of Lebesgue measure zero remains a core of the absolutely continuous spectrum. In scattering theory it is thus natural to assume that $\tilde{\sigma}$ is determined up to a set of Lebesgue measure zero.

4. We illustrate the concepts introduced with the example of an operator with simple spectrum. We recall that an operator $H$ has simple spectrum if there exists an element $h$, called a generating (or cyclic) element, such that the linear hull of the elements $E(X)h$ over all possible Borel $X$ is dense in $\mathcal{H}$. For operators with simple spectrum an element $h$ is a generating element if and only if it has maximal spectral type.

Example 9. Suppose $m$ is a nonnegative Borel measure on $\mathbb{R}$, and $H_j$ is the operator of multiplication by the independent variable (say, $\lambda$) in the space $L_2(\mathbb{R}; dm)$. The spectral projection $E(X)$ is the operator of multiplication by the characteristic (indicator) function $\chi_X(\lambda)$ of the set $X$. The operator $H$ has simple spectrum, and the element $f$ is cyclic if $f(\lambda) \neq 0$ for a.e. $\lambda$ with respect to the measure $m$. The spectral type of the operator $H$ is thus equal to the type of the measure $m$. From this it follows that $\mathcal{H}^{(a)} = L_2(\mathbb{R}; dm), \mathcal{H}^{(s)} = L_2(\mathbb{R}; dm)$, and the spectra $\sigma, \sigma^{(a)}$ and $\sigma^{(s)}$ coincide with the supports of the measures $m, m_a$, and $m_s$. The core of the spectrum $\tilde{\sigma}$ coincides with the minimal BS of the measure $m$.

We now consider two multiplication operators.

Lemma 10. Suppose $H_j$ is the operator of multiplication by the independent variable in $\mathcal{H}_j = L_2(\mathbb{R}; dm_j), j = 1, 2$. Then $H_1$ contains a part unitarily equivalent to $H_2$ if the measure $m_j$ is absolutely continuous with respect to $m_1$. In particular, $H_1$ and $H_2$ are unitarily equivalent if $m_1$ and $m_2$ have the same type.

Proof. By hypothesis

$$m_j(X) = \int_X p(\lambda) \, dm_1(\lambda), \quad p \geq 0.$$ 

We set $(Uf)(\lambda) = p^{1/2}(\lambda) f(\lambda)$. The mapping $U: \mathcal{H}_1 \to \mathcal{H}_2$ is isometric, and $H_1 U = U H_2$. Therefore, the restriction of $H_1$ to the range $R(U)$ of this mapping is unitarily equivalent to $H_2$.

In Example 9 the measure $m_a$ is equivalent to the restriction of Lebesgue measure to the core $\tilde{\sigma}$ of the spectrum of the operator $H$. Therefore, by Lemma 10 the operator $H^{(a)}$ is unitarily equivalent to the operator of multiplication by $\lambda$ in the space $L_2(\tilde{\sigma}; d\lambda)$ of functions square-integrable on $\tilde{\sigma}$ with respect to Lebesgue measure. The operator $H^{(a)}$ is unitarily equivalent also to a part of the operator of multiplication in $L_1(\sigma^{(a)}; d\lambda)$ but for $|\sigma(\lambda)| > 0$ this part is not equal to the entire operator.

5. By Lebesgue's theorem the monotone function $(E(\lambda)f, f)$ has a derivative $d(E(\lambda)f, f)/d\lambda \in L_1(\mathbb{R})$ on a set of full Lebesgue measure (depending on $f$). Applying the identity (1) to the elements $E(\lambda)f, E(\lambda)g$, we find that the function $(E(\lambda)f, g)$ is also differentiable for a.e. $\lambda$. The corresponding set of full measure depends on $f$ and $g$, and $d(E(\lambda)f, g)/d\lambda$ as before belongs to $L_1(\mathbb{R})$. Moreover, from (2) we obtain the estimate

$$\left| \frac{d(E(\lambda)f, g)}{d\lambda} \right|^2 \leq \frac{d(E(\lambda)f, f)}{d\lambda} \frac{d(E(\lambda)g, g)}{d\lambda}, \quad \text{a.e. } \lambda \in \mathbb{R}. \quad (9)$$

The expression "for a.e. $\lambda \in X"$ everywhere means that the set $\tilde{X}$ of full measure in $X$ on which the relation in question is satisfied may depend on all the parameters contained in the formulation. Otherwise we use the term "for a.e. $x \in X$ independent of \ldots" with an indication of the parameters on which $\tilde{X}$ does not depend.

From equalities (6) and (1.6) it follows that for any $f, g \in \mathcal{H}$

$$\frac{d(E(\lambda)f, g)}{d\lambda} = \frac{d(E(\lambda)p^{(a)} f, p^{(a)} g)}{d\lambda}, \quad \text{a.e. } \lambda \in \mathbb{R}. \quad (10)$$

Moreover, by (1.5)

$$(p^{(a)} f, g) = \int_{-\infty}^{\infty} \frac{d(E(\lambda)f, g)}{d\lambda} \, d\lambda. \quad (11)$$

Further, for an arbitrary function $\varphi$ (see part 1 of the next section for the definition of a function of an operator $H$) there is the equality

$$\frac{d(E(\lambda)\varphi(H)f, g)}{d\lambda} = \varphi(\lambda) \frac{d(E(\lambda)f, g)}{d\lambda}, \quad \text{a.e. } \lambda \in \mathbb{R}. \quad (12)$$

Of course, the left-hand side here is understood as the derivative of the function $(E(\lambda_0, \lambda)\varphi(H)f, g)$ for some finite $\lambda_0$ (or, in other words, as the derivative of the measure $(E(\lambda)\varphi(H)f, g)$). According to (10) it suffices to verify equality (12) for $f \in \mathcal{H}^{(a)}$, when

$$(E(\lambda)\varphi(H)f, g) = \int_X \varphi(\lambda) \frac{d(E(\lambda)f, g)}{d\lambda} \, d\lambda.$$

Now one needs only note that the derivative of this measure must be equal to the integrand.

From (12) it follows, in particular, that

$$\frac{d(E(\lambda)E(X)f, g)}{d\lambda} = \chi_X(\lambda) \frac{d(E(\lambda)f, g)}{d\lambda}, \quad \text{a.e. } \lambda \in \mathbb{R}, \quad (13)$$

where $\chi_X(\cdot)$ is the indicator of the Borel set $X$.\]
§4. Functions of a selfadjoint operator: the unitary group and resolvent

1. For a selfadjoint operator $H$ the spectral theorem makes it possible to construct a function $\varphi(H)$ of it if $\varphi$ is measurable and finite a.e. with respect to the spectral measure $E = E_H$. The domain $D(\varphi(H))$ consists of elements $f \in \mathcal{H}$ for which

$$\int_{-\infty}^{\infty} |\varphi(\lambda)|^2 d(E(\lambda)f, f) < \infty.$$ 

The set of such $f$ is dense in $\mathcal{H}$. For $f \in D(\varphi(H))$ and any $g \in \mathcal{H}$ the sesquilinear form of the operator $\varphi(H)$ is defined by the equality

$$\langle \varphi(H)f, g \rangle = \int_{-\infty}^{\infty} \varphi(\lambda) E(\lambda f, g).$$

(1)

For $\varphi = \overline{\varphi}$ the operator $\varphi(H)$ is selfadjoint. For a bounded function $\varphi$ the operator $\varphi(H)$ is also bounded and $D(\varphi(H)) = \mathcal{H}$. In the last case, with the integrals understood in the weak sense, we often write representations of the form (1) for the operators themselves (instead of their sesquilinear forms).

The unitary group $U(t) = U_H(t) = \exp(-iHt)$ is of special importance in scattering theory. According to (1), its sesquilinear form

$$\langle U(t)f, g \rangle = \int_{-\infty}^{\infty} \exp(-i\lambda t) d(E(\lambda)f, g)$$

(2)

is the Fourier transform of the spectral measure. In terms of $U(t)$ it is possible to construct a solution of the Cauchy problem

$$i\frac{\partial u}{\partial t} = Hu, \quad u(0) = f,$$

for the time-dependent Schrödinger equation. It is assumed that the solution $u(t)$ is continuously differentiable in $\mathcal{H}$ with respect to $t$ and belongs to $D(\mathcal{H})$ for all $t \in \mathbb{R}$. Since $\partial\|u(t)\|^2/\partial t = 0$, the quantity $\|u(t)\|$ does not depend on $t$, and therefore the solution of the Cauchy problem is unique. For $f \in D(H)$ a solution exists and is given by the formula $u(t) = U(t)f$. For any $f \in \mathcal{H}$ this function may be taken as a generalized solution of the problem in question.

2. Another important function of the operator $H$ is its resolvent $R(z) = R_H(z) = (H - z)^{-1}$. Its sesquilinear form

$$\langle R(z)f, g \rangle = \int_{-\infty}^{\infty} (\lambda - z)^{-1} d(E(\lambda)f, g)$$

(3)

is a Cauchy-Stieltjes integral with respect to the spectral measure. The connection between the functions (2) and (3) is given by the relation

$$R(\lambda \pm ie) = \pm i \int_{0}^{\infty} \exp(-\varepsilon t \pm i\varepsilon t) U(\pm t) dt.$$ 

(4)

To prove (4) it suffices to substitute the representation (2) into the right-hand side and interchange the order of integration. Since the integrals here converge absolutely, the interchange is justified by Fubini’s theorem.

We set

$$\delta(\lambda, \varepsilon) = (2\pi i)^{-1} [R(\lambda + i\varepsilon) - R(\lambda - i\varepsilon)] = \pi^{-1} \varepsilon R(\lambda + i\varepsilon) R(\lambda - i\varepsilon) \geq 0, \quad \varepsilon > 0.$$ 

(5)

Again with the help of Fubini’s theorem it follows from (3) that for any $\varepsilon$

$$\int_{-\infty}^{\infty} \delta(\lambda, \varepsilon) d\lambda = 1.$$ 

(6)

From the inversion formula (2.8) it follows that for any interval $\Lambda = (\lambda_0, \lambda_1)$

$$\lim_{\varepsilon \to 0} \int_{\Lambda} \delta(\lambda, \varepsilon) d\lambda = E(\Lambda) + E(\Lambda^c),$$

(7)

where the weak limit on the left-hand side exists. The relation (7) is called Stone’s formula. We note that actually (see Volume I of the course [18]) the strong limit also exists in (7). However, it suffices for us to understand (7) in the weak sense.

From (3) there also follows a representation of the spectral measure in terms of the resolvent. Suppose the endpoints of the bounded interval $\Lambda = (\lambda_0, \lambda_1)$ do not belong to the spectrum of the operator $H$, while the simple (without selfintersections) closed contour $\Gamma$ encloses $\Lambda$, intersects the real axis at the points $\lambda_0$ and $\lambda_1$ (so that $\Gamma \cap \sigma(H) = \emptyset$), and is traversed counterclockwise. Then

$$E(\Lambda) = -(2\pi i)^{-1} \int_{\Gamma} R(z) dz.$$ 

(8)

3. By equality (3), Theorem 2.5 implies that for a.e. $\lambda \in \mathbb{R}$

$$\lim_{\varepsilon \to \pm 0} (R(z)f, g) = \pm \pi i \frac{d(E(\lambda)f, g)}{d\lambda} + \text{p.v.} \int_{-\infty}^{\infty} (\mu - \lambda)^{-1} d(E(\mu)f, g).$$

(9)

Thus

$$\lim_{\varepsilon \to \pm 0} (R(z)f, g) - \lim_{\varepsilon \to \pm 0} (R(z)f, g) = 2\pi i \frac{d(E(\lambda)f, g)}{d\lambda} \quad \text{a.e. } \lambda \in \mathbb{R},$$

(10)

and, in particular,

$$\lim_{\varepsilon \to 0} (\delta(\lambda, \varepsilon)f, g) = \frac{d(E(\lambda)f, g)}{d\lambda}, \quad \text{a.e. } \lambda \in \mathbb{R}.\quad (11)$$

The last relation shows that there is the estimate uniform with respect to $\varepsilon$

$$\varepsilon \pi^{-1} \|R(\lambda \pm ie)f\|^2 = (\delta(\lambda, \varepsilon)f, g) \leq C(\lambda; f), \quad \text{a.e. } \lambda \in \mathbb{R}.\quad (12)$$

We emphasize that the sets of full measure on which relations (9)-(12) are satisfied depend on $f$ and $g$. For this reason, for example, the relation
I. PRELIMINARY FACTS

(11) does not imply the existence of a weak limit of \( \delta(\lambda, \varepsilon) \) as \( \varepsilon \to 0 \). This operator actually has no weak limit for any \( \lambda \). This follows from the equality

\[ ||\delta(\lambda, \varepsilon)|| = c\sqrt{\sup_{\mu \in \mathbb{R}} (\mu - \lambda)^2 + \varepsilon^2}^{-1} = \frac{1}{c} \varepsilon. \]

4. We shall present a property of absolutely continuous elements often used in scattering theory. We note beforehand that for \( f \in \mathcal{H}^{(0)} \) (or \( g \in \mathcal{H}^{(a)} \)) the measure \( d(E(\cdot)f, g) \) is absolutely continuous, and hence in integrals of the form (1)

\[ d(E(\lambda)f, g) = \frac{d(E(\lambda)f, g)}{d\lambda} d\lambda, \quad \frac{d(E(\lambda)f, g)}{d\lambda} \in L_1(\mathbb{R}). \] (13)

LEMMA 1. Let \( P^{(a)} \) be the projection onto the absolutely continuous subspace of \( H \). Then

\[ \text{w-lim}_{|t| \to \infty} U(t)^n P^{(a)} = 0. \] (14)

If \( K \) is a compact operator, then

\[ \text{s-lim}_{|t| \to \infty} KU(t)^n P^{(a)} = 0, \quad K \in \Theta_{\infty}. \] (15)

PROOF. In view of (2) and (13) relation (4) is a corollary of the Riemann-Lebesgue lemma. Relation (15) is a consequence of (14). \( \square \)

5. We give a convenient criterion for the absolute continuity of the spectrum.

PROPOSITION 2. Suppose for any \( f \in \mathcal{B} \), where \( \mathcal{B} = \mathcal{H} \), for some \( p > 1 \)

\[ \sup_{0 < \varepsilon < 1} \int_a^b (\delta(\lambda, \varepsilon)f, f)^p d\lambda < \infty. \] (16)

Then the spectrum of the operator \( H \) is absolutely continuous on the closed interval \([a, b]\).

PROOF. It suffices to demonstrate the absolute continuity of the measure \((E(X)f, f)\) on \([a, b]\), where \( f \in \mathcal{B} \). It follows from formula (7) that for any interval \( X \)

\[ (E(X)f, f) \leq \lim_{\varepsilon \to 0} \int_X (\delta(\lambda, \varepsilon)f, f) d\lambda. \] (17)

Moreover, according to (7), for closed intervals \( X \) of the form \([a, a + \gamma]\) and \([b - \gamma, b]\) the estimate (17) continues to hold if the right-hand side is multiplied by 2. Of course, the estimate (17) extends to finite unions of nonintersecting intervals. By Hölder's inequality we estimate the right side of (17) by

\[ |X|^s \sup_{0 < \varepsilon < 1} \left( \int_X (\delta(\lambda, \varepsilon)f, f)^p d\lambda \right)^{1/p}, \quad s = 1 - p^{-1} > 0. \]

§5. DECOMPOSITION INTO A DIRECT INTEGRAL

By condition (16) from this it follows that

\[ (E(X)f, f) \leq C|X|^s, \]

and hence (see Part 2 of §1) the measure \((E(X)f, f)\) is absolutely continuous on \([a, b]\). \( \square \)

A simple sufficient condition for (16) to hold is the inequality

\[ \sup_{a < \lambda < b} \left| (R(\lambda + \varepsilon)f, f) \right| < \infty, \quad f \in \mathcal{B}, \quad \mathcal{B} = \mathcal{H}. \] (18)

§5. Decomposition into a direct integral

1. The direct integral of Hilbert spaces

\[ \mathcal{K} = \int_{-\infty}^{\infty} \oplus h(\lambda) d\mu(\lambda) \] (1)

is defined (for details see the book [4]) as the Hilbert space of vector-valued functions \( \mathcal{F}(\lambda) \) that take values in the auxiliary (infinitesimal) Hilbert spaces \( h(\lambda) \) and are measurable and square-integrable with respect to the measure \( \mu \). The scalar product in the space \( \mathcal{K} \) is introduced by

\[ (f, g)_\mathcal{K} = \int_{-\infty}^{\infty} (\mathcal{F}(\lambda), \mathcal{G}(\lambda)) d\mu(\lambda), \] (2)

where \((\cdot, \cdot)\) is the scalar product in \( h(\lambda) \). The norm (including the operator norm) in \( \mathcal{K} \) is denoted by \( \| \cdot \| \). By definition a Hilbert space \( \mathcal{H} \) is decomposed into a direct integral (1) if there is given a unitary mapping \( \mathcal{F} \) of the space \( \mathcal{K} \) onto \( \mathcal{K} \). The existence of such an isomorphism is written in the form \( \mathcal{H} \leftrightarrow \mathcal{K} \). The vector-valued function \( \mathcal{F}(\lambda) = (F(\cdot)(\lambda)) \) is called the representative of the element \( f \in \mathcal{H} \) in the decomposition (1).

Within the framework of abstract operator theory the spaces \( h(\lambda) \) are defined only up to unitary equivalence. In particular, in the case \( dim h(\lambda) = const \) (a.e. \( \lambda \)) all the \( h(\lambda) \) can be identified with a single Hilbert space \( h \) of the same dimension. In the presence of such an identification the direct integral (1) reduces to the space \( L_2(\mathbb{R}; h, dm) \) of vector-valued functions with range in the same auxiliary space \( h \).

One of the formulations of the spectral theorem can be given in terms of decompositions into a direct integral. Namely, for any selfadjoint operator \( H \) in \( \mathcal{K} \) there exists a decomposition (1) in which the operator \( \mathcal{F}H\mathcal{F}^* \) acts as multiplication by \( \lambda \). The quantity \( \mathcal{F}E(X)\mathcal{F}^* \) hereby reduces to multiplication by the indicator \( \chi_X \) of the set \( X \), so that in accordance with (2)

\[ (E(X)f, f) = \int_X (\mathcal{F}(\cdot)(\lambda), \mathcal{F}(\cdot)(\lambda)) d\mu(\lambda). \] (3)

Such a decomposition of \( \mathcal{H} \) into a direct integral is called diagonal for \( H \) or its spectral representation. We remark that in the diagonal decomposition (1) for the operator \( H \) the measure \( \mu \) must have the spectral type of \( H \).
For operators \( H \) with simple spectrum \( \dim h(\lambda) = 1 \). In this case \( H \) can be realized as multiplication by \( \lambda \) in the space \( L^2(\mathbb{R}; dm) \) (see Example 3.9).

If \( A \) is any bounded operator in \( \mathcal{H} \) which commutes with \( H \), then in
the decomposition (1) to it there corresponds a measurable (relative to the 
measure \( m \)) family of operators \( a(\lambda) : h(\lambda) \to h(\lambda) \) such that for any \( f \in \mathcal{H} \)
\[
(\mathcal{F} Af)(\lambda) = a(\lambda)(\mathcal{F} f)(\lambda).
\]
(4)

For such an operator
\[
\|A\| = \text{m-sup}_{\lambda \in \mathbb{R}} |a(\lambda)|,
\]
(5)
where \( \text{m-sup} \) denotes the essential supremum with respect to the measure \( m \).

2. In scattering theory it is necessary to consider a decomposition into 
an integral of the form (1) for the absolutely continuous part \( H^{(a)} \) of 
the operator \( H \). It is constructed on the basis of the absolutely continuous 
component \( m_\sigma \) of the measure \( m \). Since \( m_\sigma \) has the type of restriction 
of Lebesgue measure to the core \( \sigma \) of the spectrum of the operator \( H \), it follows that
\[
H^{(a)} = \int_0^\infty h(\lambda) d\lambda := \mathcal{S}^{(a)}, \quad \sigma = \sigma(H).
\]
(6)

We denote \( \mathcal{F} : \mathcal{H} \to \mathcal{S}^{(a)} \) the unitary mapping of \( \mathcal{H}^{(a)} \) onto \( \mathcal{S}^{(a)} \) 
continued by zero to \( \mathcal{H}^{(a)} \). The role of (3) is now played by the relation
\[
(E^{(a)}(X)f, g) = \int_0^\infty \langle (\mathcal{F} f)(\lambda), (\mathcal{F} g)(\lambda) \rangle d\lambda.
\]
Therefore, from (3,10), (4,11) it follows that for a.e. \( \lambda \in \sigma \)
\[
\lim_{\varepsilon \to 0} (\delta(\lambda, \varepsilon)f, g) = \frac{d(E^{(a)}f, g)}{d\lambda} = \langle \hat{f}(\lambda), \hat{g}(\lambda) \rangle,
\]
(7)
\[
\hat{f} = \mathcal{F} f, \quad \hat{g} = \mathcal{F} g.
\]

We shall need the following almost obvious observation.

**Lemma 1.** Suppose that for a.e. \( \lambda \in \sigma \) there is given an operator \( a(\lambda) \) 
bounded in \( h(\lambda) \) and \( D \) is any dense set in \( \mathcal{H} \). We suppose that for any \( f, g \in D \) on a (dependent on \( f \) and \( g \)) set of full measure in \( \sigma \)
\[
(a(\lambda)f(\lambda), \hat{g}(\lambda)) = 0.
\]
(8)

Then \( a(\lambda) = 0 \) for a.e. \( \lambda \in \sigma \).

**Proof.** Using the separability of \( \mathcal{H} \), we isolate in \( D \) a countable, 
but still dense in \( \mathcal{H} \), subset \( D_0 \). For each pair \( f_m, g_m \) of elements of \( D_0 \) we consider 
the set of full measure in \( \sigma \) on which the functions \( f_m(\lambda), \hat{g}_m(\lambda) \) 
are defined and (8) is satisfied. In view of the countability of \( D_0 \), the 
intersection of all these sets also has full measure. Moreover, by dropping from it a set of 
measure zero, we obtain (see [4]) a set \( X \) of full measure in \( \sigma \) on which the 
elements \( f(\lambda) \) for all possible \( f \in D_0 \) are dense in \( h(\lambda) \). It may be assumed 
that for \( \lambda \in X \) the operators \( a(\lambda) \) are bounded. Therefore, relation (8) 
for all \( f, g \in D_0 \) and some \( \lambda \in X \) implies that \( a(\lambda) = 0 \) for that same \( \lambda \).

It is sometimes convenient to reduce the direct integral (6) to a sum of 
vector spaces of \( L^2 \) type. Namely, suppose the Borel set \( \sigma_k, \lambda = 1, 2, \ldots, \infty, \)
consists of those points \( \lambda \) for which \( \dim h(\lambda) = k \), i.e., the multiplicity of 
the spectrum of \( H \) at the point \( \lambda \) is equal to \( k \). By identifying \( h(\lambda) \) for \( \lambda \in \sigma_k \) 
with one and the same space \( \mathcal{H}_k \) in place of (6) we obtain the decomposition
\[
H^{(a)} = \sum_{1 \leq k \leq \infty} \bigoplus_{\lambda \in \sigma_k} L^2(\sigma_k; \mathcal{H}_k), \quad \dim \mathcal{H}_k = k.
\]
(9)

We further define standard mappings of \( \mathcal{H}_k \) into \( L^2(\sigma_k; \mathcal{H}_k) = L^2(\sigma_k; \mathcal{H}_{\infty}) \).

Let \( X_k \) be some fixed bounded subset of \( \sigma_k \) (for a bounded \( a_k \) it may be 
assumed, in particular, that \( X_k = \sigma_k \)) and let \( \chi_k = \chi_k \) 
be the indicator of \( X_k \). For \( x \in \mathcal{H}_k \) we set \( (\Psi(x)) = \chi_k(x) \).

Then \( \Psi_k \) is a bounded operator from \( \mathcal{H}_k \) to \( L^2(\sigma_k; \mathcal{H}_k) \). It may be assumed that \( \Psi_k \) 
acts from \( \mathcal{H}_k \) into the entire sum on the right-hand side of (9). It is clear that
\[
(E(X)\mathcal{F}^* \Psi_k X, \mathcal{F}^* \Psi_k Y) = \delta_{k_l} \chi_k(\hat{\Psi}_k(\lambda)x, y),
\]
(10)
where \( \delta_{k_l} \) is the Kronecker symbol.

3. We now consider the question of the representation of bounded operators 
\( A \) in the form of integral operators in the decomposition into a direct 
integral. More precisely, in connection with applications to scattering theory 
we speak of the kernels of the operators \( P^{(a)}A P^{(a)} \) in the decomposition (6). 
Of course, everything said remains valid for operators \( A \) defined directly 
in \( \mathcal{H}^{(a)} \). We often set \( P = P^{(a)} \) where there can be no confusion. We make

**Definition 2.** We call the operator \( PAP \) an integral operator in the 
decomposition (6) if for a.e. \( (\mu, \nu) \in \sigma \times \sigma \) there exists a bounded operator 
\( a(\mu, \nu) : h(\nu) \to h(\mu) \) such that for any \( f, g \in \mathcal{H}^{(a)} \)
\[
(\mathcal{F} f, g) = \int_{\sigma} d\mu \int_{\sigma} d\nu (a(\mu, \nu)f(\nu), \hat{g}(\mu)).
\]
(11)

It is here assumed that the integrand in (11) is measurable with respect to 
two-dimensional Lebesgue measure on \( \sigma \times \sigma \) and that the iterated integral 
converges. We call the operator-valued function \( a(\mu, \nu) \) the kernel of the 
operator \( PAP \). Sometimes, admitting some ambiguity, we call the operator \( A \) 
itself an integral operator.

Suppose a representation of the form (11) holds for \( (\mathcal{F} f, g) \) but with the 
order of integration interchanged. Then in accordance with the definition 
adopted, the operator \( A^* \) (or, more precisely, \( P^* A P \)) is an integral operator, 
and its kernel is equal to \( a^*(\nu, \mu) \). In particular, if (11) holds and the 
iterated integral does not depend on the order of integration, then both \( A \) 
and \( A^* \) are integral operators.
From (11) it follows that for any $X \subset \delta$, $Y \subset \delta$

\[
(\mathcal{A}E(X)f, E(Y)g) = \int_Y d\mu \int_X d\nu (a(\mu, \nu)f(\nu), \hat{g}(\mu)).
\] (12)

Therefore, the sesquilinear form of the kernel can be recovered from the equality

\[
\langle a(\mu, \nu)f(\nu), \hat{g}(\mu) \rangle = \frac{\partial}{\partial \nu} \frac{\partial}{\partial \mu}(\mathcal{A}E(\nu)f, E(\mu)g),
\]

\[
f, g \in \mathcal{H}^{(a)}, \quad \text{a.e. } (\mu, \nu) \in \delta \times \delta,
\] (13)

where the mixed derivative on the right exists. Of course, here the following must be born in mind (cf. the proof of Lemma 1). The set of full measure in $\delta \times \delta$ on which (13) holds depends on $f$ and $g$. However, by considering finite linear combinations of elements of any distinguished basis $\{\psi_i\}$ in $\mathcal{H}^{(a)}$, we find that for such $f$ and $g$ (13) is satisfied on a common set of full measure. Moreover, for a set of full measure in $\delta \times \delta$ finite linear combinations of elements $a(\mu, \nu)$ are well defined and dense in $h(\lambda)$. From this it follows that the form $\langle a(\mu, \nu)f(\nu), \hat{g}(\mu) \rangle$ can be recovered by equality (13) on a set dense in $h(\lambda) \times h(\mu)$ if the pairs $(\mu, \nu)$ belong to some set of full measure in $\delta \times \delta$. Since the kernel is assumed to be a bounded operator, this defines it uniquely for a.e. $(\mu, \nu) \in \delta \times \delta$. We thus have

**Lemma 3.** If the kernel $a(\mu, \nu)$ of the operator $\mathcal{PAP}$ exists, then it is uniquely determined up to a set of measure zero in the square $\delta \times \delta$.

Equality (13) takes a more graphic form in terms of the representation (9). On elements $f, g \in \mathcal{S}^{*} \mathcal{Y}_{k}x$, $g = \mathcal{S}^{*} \mathcal{Y}_{k}y$, where $x \in \mathcal{h}_{k}, y \in \mathcal{h}_{k}$, (13) means that for a.e. $(\mu, \nu) \in X_k \times X_k$ (this set may depend on $x$ and $y$)

\[
\langle a(\mu, \nu)x, y \rangle = \frac{\partial}{\partial \nu} \frac{\partial}{\partial \mu}(\mathcal{A}E(\nu)\mathcal{F}^{*} \mathcal{Y}_{k}x, E(\mu)\mathcal{F}^{*} \mathcal{Y}_{k}y).
\] (14)

Since $X_k$ and $X_k$ are arbitrary bounded subsets of $\sigma_k$ and $\sigma_l$, the form $\langle a(\mu, \nu)x, y \rangle$ can be recovered by equality (14) for any $x \in \mathcal{h}_{k}, y \in \mathcal{h}_{k}$ and a.e. $(\mu, \nu) \in \sigma_k \times \sigma_l$.

According to (13), the sesquilinear form of the kernel $a(\mu, \nu)$ does not depend on the choice of the realization of $H^{(a)}$ as a direct integral (of the mapping $\mathcal{F}$). Of course, the kernel $a(\mu, \nu)$ itself depends on the realization. Namely, suppose that together with (6) the space $\mathcal{H}^{(a)}$ is decomposed into a direct integral $\mathcal{S}^{(a)}$ of the same form with "infinitesimal" spaces $h(\lambda)$. Suppose the mapping $\mathcal{F}^{*} : \mathcal{H}^{(a)} \rightarrow \mathcal{S}^{(a)}$ is given by the relation $\mathcal{F}^{*}(\lambda) = u(\lambda)(\mathcal{F}f)(\lambda)$, where $u(\lambda)$ is a unitary mapping of $h(\lambda)$ onto $h(\lambda)$ defined for a.e. $\lambda \in \delta$. In the decomposition $\mathcal{S}^{(a)}$ the kernel of the operator $A$ is then equal to

\[
\hat{a}(\mu, \nu) = u(\mu)a(\mu, \nu)u^{*}(\nu).
\] (12)

We shall not discuss here conditions under which the operator is an integral operator. Sufficient conditions, which also immediately ensure some special properties of its kernel, are indicated in §5.4.

4. Definition 2 extends naturally to a rather broad class of unbounded operators. Moreover, it is possible also to consider integral representations of sesquilinear forms not generated by operators in $\mathcal{H}^{*}$. For a precise description of this class of forms we use the terminology of the theory of dual spaces (see, for example, the book [2]).

Namely, we consider $\mathcal{D} = \mathcal{D}(H)$ as a Hilbert space with the norm

\[
\|f\|_{\mathcal{D}} = \|Hf\| + \|f\|.
\]

The space $\mathcal{D}^{*}$ dual to $\mathcal{D}$ relative to the scalar product in $\mathcal{H}$ is the completion of $\mathcal{H}$ in the norm

\[
\|g\|_{\mathcal{D}^{*}} = \sup_{f \in \mathcal{D}} \|f\|^{-1}_{\mathcal{D}} \langle g, f \rangle.
\]

A bounded operator $A : \mathcal{D} \rightarrow \mathcal{D}^{*}$ is called an integral operator if (11) is satisfied for any $f, g \in \mathcal{H}^{(a)} \cap \mathcal{D}$. We note that in the representation of $\mathcal{H}$ as a direct integral (1) the space $\mathcal{D}^{*}$ consists of vector-valued functions $f(\lambda)$ which are square-integrable with respect to the measure $|\lambda|^2 + 1)^{-1} dm(\lambda)$.

A still broader understanding of Definition 2 is obtained if in it we restrict attention to elements "compactly supported" with respect to $H$, i.e., such that $f = E(X)f$ for some bounded $X$. We denote the (topological) space of compactly supported elements by $\mathcal{E} = \mathcal{E}(H)$, and by $\mathcal{E}^{*}$ the space dual to it relative to the scalar product in $\mathcal{H}$. In the representation in the form of a direct integral $\mathcal{E}^{*}$ consists of vector-valued functions which are locally square-integrable with respect to the measure $dm(\lambda)$. A bounded operator $A : \mathcal{E} \rightarrow \mathcal{E}^{*}$ is called an integral operator if (11) is satisfied for any $f, g \in \mathcal{H}^{(a)} \cap \mathcal{E}$.

We note that the operators $R(z) : \mathcal{D}^{*} \rightarrow \mathcal{H}$ for $z \in \rho(H)$ and $\delta(\lambda, \epsilon) : \mathcal{D} \rightarrow \mathcal{D}$ for $\epsilon > 0$ are bounded. Therefore, for a bounded operator $A : \mathcal{D} \rightarrow \mathcal{D}^{*}$ the product $R(z)AR(z)$, $z \in \rho(H)$, is a bounded operator in $\mathcal{H}$. Similarly, for a bounded operator $A : \mathcal{E} \rightarrow \mathcal{E}^{*}$ the product $E(X)AE(Y)$ is a bounded operator in $\mathcal{H}$ if the sets $X$ and $Y$ are bounded.

Many of the relations of §§3, 4 extend automatically to elements of $\mathcal{D}^{*}$ and even of $\mathcal{E}^{*}$. Thus, relations (3.10), (3.12), and (3.13) are true for any elements $f, g \in \mathcal{E}^{*}$, while relation (4.11) holds for any $f, g \in \mathcal{D}^{*}$.

§6. Classes of compact operators

We here present the facts required below regarding $s$-numbers (singular numbers) of compact operators. A systematic exposition of the corresponding theory can be found in the monographs [7], [36] or in the textbook [4] (for more details, see [31]).
1. Let \( \mathcal{H}_0, \mathcal{H} \) be Hilbert spaces, and let \( A: \mathcal{H}_0 \to \mathcal{H} \) be an arbitrary linear (not necessarily bounded) operator with domain \( \mathcal{D}(A) \) dense in \( \mathcal{H}_0 \). We denote by \( N(A) \) and \( R(A) \) its kernel (null space) and image (range).

Note the orthogonal decomposition

\[
R(A) \oplus N(A) = \mathcal{H}.
\]

If \( A \) is closed, then it admits a so-called polar decomposition

\[
A = FA^*,
\]

where the modulus \( |A| = (A^*A)^{1/2} \) is nonnegative, while the "sign" operator \( F = \text{sgn} A \) is annihilated on \( N(A) \) and unitarily maps \( R(A) = \sigma([|A|]) \) onto \( R(A) \).

The next assertion, valid for (selfadjoint) operator acting in a common Hilbert space, is called the "Heinze inequality."

**Lemma 1.** Suppose \( A \geq 0, B \geq 0, \mathcal{D}(A) \subset \mathcal{D}(B), \) and \( \|Bx\| \leq \|Ax\| \)

for \( x \in \mathcal{D}(A) \). Then for any \( \theta \in (0, 1) \), we have \( \mathcal{D}(\theta A) \subset \mathcal{D}(B^\theta) \)

and \( \|B^\theta x\| \leq \|\theta A\| \) \( x \in \mathcal{D}(\theta A) \).

We denote by \( \mathcal{B} = \mathcal{B}(\mathcal{H}_0, \mathcal{H}) \) and \( \mathcal{B}_\infty = \mathcal{B}_\infty(\mathcal{H}_0, \mathcal{H}) \) the classes of bounded and compact operators acting from \( \mathcal{H}_0 \) to \( \mathcal{H} \). For \( \mathcal{H}_0 = \mathcal{H} \) we write only one "argument"; for example, \( \mathcal{B}(\mathcal{H}) = \mathcal{B}(\mathcal{H}, \mathcal{H}) \). As a rule, the notation for the dependence on the spaces is omitted.

Without specifying it every time, we admit in this section that the spaces \( \mathcal{H}_0 \) and \( \mathcal{H} \) are distinct. If \( A \in \mathcal{B} \) and \( \dim R(A) < \infty \), then the operator \( A \) is called finite-dimensional. The class of finite-dimensional operators is denoted by \( \mathcal{L} \). We recall that in a Hilbert space any compact operator can be obtained as the limit in norm of a sequence of finite-dimensional operators. For \( A \in \mathcal{B}(\mathcal{H}_0) \) (\( A \in \mathcal{H}_0 \)) the adjoint operator \( A^* \) is also compact (finite-dimensional). For any \( B \in \mathcal{B} \) both products \( AB \) and \( BA \) are compact (finite-dimensional) if \( A \in \mathcal{B}_\infty \) (\( A \in \mathcal{E} \)). Thus, \( \mathcal{B}_\infty \) and \( \mathcal{E} \) are two-sided ideals of the algebra \( \mathcal{B} \).

2. The spectrum of an operator \( A = A^* \) is \( \mathcal{B}_\infty \) consists of real eigenvalues which can accumulate only at zero; the nonzero eigenvalues have finite multiplicity. We denote by \( \lambda_n(A) \) \( (-\lambda_n(A)) \) the positive (negative) eigenvalues listed with account of multiplicity in decreasing (increasing) order; \( \varphi_n(A) \) are the corresponding eigenvectors. In the case \( A \geq 0 \) we set \( \lambda_n(A) = \lambda_n(A) \), \( \varphi_n(A) = \varphi_n(A) \).

For any compact operator \( A \) the numbers \( s_n(A) = \lambda_n(A) \) are called its s-numbers or singular numbers. We note that \( s_n(A) = s_n(A^*) \) and \( s_n(A) = \|A\| \). From (2) it is easy to derive an expansion of \( A \in \mathcal{B}_\infty \) in a (Schmidt) series

\[
A = \sum_{n=1}^{\infty} s_n(A) \varphi_n(A), \quad s_n = s_n(A), \quad \varphi_n = \varphi_n(A),
\]

where \( \psi_n = F\varphi_n \). Since \( s_n \to 0 \) as \( n \to \infty \), the series (3) converges in norm.

Another (equivalent) definition of \( s_n(A) \) can be given in terms of the approximation of \( A \) by finite-dimensional operators. Let \( \mathcal{B}_n \) be the set of finite-dimensional operators whose rank does not exceed \( n \). It is clear that \( A \in \mathcal{B}_n \) if and only if the series (3) consists of no more than \( n \) terms. It turns out that

\[
s_n(A) = \min_{L \in \mathcal{B}_n} \|A - L\|, \quad n \geq 2.
\]

Definition (4) makes it possible to ascribe s-numbers to bounded operators which are not compact. We note that the minimum in (4) is achieved for any \( A \in \mathcal{B} \).

The relation (4) is used for the proof of various properties of s-numbers. We present the familiar inequalities (H. Weyl–K. Fan)

\[
s_{n+m-1}(A + B) \leq s_n(A) + s_m(B),
\]

\[
s_{n+m-1}(AB) \leq s_n(A)s_m(B).
\]

In particular,

\[
s_n(B_1A_1B_2) \leq \|B_1\|\|B_2\|s_n(A).
\]

3. We recall that an operator \( T \) is called Fredholm if the ranges of the operators \( T \) and \( T^* \) are closed and

\[
\dim N(T) = \dim N(T^*) < \infty.
\]

For a Fredholm operator \( T \) the inverse \( T^{-1} \) is sometimes conveniently understood in the following generalized sense. By definition we set \( T^{-1}f = 0 \) if \( f \in N(T^*) \) and \( T^{-1}f = g \) if \( f = Tg \) and \( g \in R(T^*) = \mathcal{H} \) \( N(T) \). For a Fredholm \( T \) such an inverse operator \( T^{-1} \) always exists, is bounded, and

\[
T^{-1}T = P_T, \quad TT^{-1} = P_T,
\]

where \( P_T \) and \( P_{T^*} \) are the orthogonal projections onto the subspaces \( R(T) \) and \( R(T^*) \) respectively. The operator \( T^{-1} \) is defined by equalities (7) up to an arbitrary mapping of \( N(T^*) \) into \( N(T) \). Under the additional condition \( N(T^*) \subset N((T^*)^*) \) \( (N(T) \subset N((T^{-1})^*) \) the operator \( T^{-1} \) is fixed uniquely by the first (respectively, second) equality in (7). We note that for the generalized inverse operator the relation \( (T^{-1})^* = (T^*)^{-1} \) is preserved. Moreover, for any two Fredholm operators \( T_j \), \( j = 1, 2 \), there is the identity

\[
T_1^{-1}P_{T_1} - P_{T_2}T_2^{-1} = T_1^{-1}(T_1 - T_2)T_2^{-1}.
\]

We have not found a proof of the next assertion in the textbook literature.

**Proposition 2.** Suppose \( A \in \mathcal{B}_\infty \). Then

\[
s_n((I - A)^{-1}) \leq (1 - s_n(A))^{-1},
\]

provided that \( s_n(A) < 1 \).
PROOF. According to (4), the operator $A$ can be represented in the form $A = L + B$ where $L \in \mathcal{L}_{n-1}$ and $\|B\| = s_n(A) < 1$. The operator $I - B$ is boundedly invertible (in the usual sense), and by the identity (8)

$$(I - A)^{-1} - P_{I - A*}(I - B)^{-1} = (I - A)^{-1}L(I - B)^{-1} \in \mathcal{L}_{n-1}.$$ 

According to (4), from this it follows that

$s_n((I - A)^{-1}) \leq \|P_{I - A*}(I - B)^{-1}\| \leq (1 - \|B\|)^{-1}.$

By construction the right-hand side is equal to $(1 - s_n(A))^{-1}$. □

4. The symmetrically normed ideals of the algebra $\mathcal{B}$ are constructed in terms of the $s$-numbers. We recall that a norm defined on a two-sided ideal $\mathcal{G} \subset \mathcal{B}$ is called symmetric if

$$\|B_1AB_2\|_\mathcal{G} \leq \|B_1\|_\mathcal{G}\|B_2\|_\mathcal{G}\|A\|_\mathcal{G}$$ (9)

for any $A \in \mathcal{G}$ and $B_1, B_2 \in \mathcal{B}$ and also $\|A\|_\mathcal{G} = \|A\|$ for a one-dimensional $A$. The symmetric norm depends only on the $s$-numbers of the operator $A$, i.e., $\|A\|_\mathcal{G} = \|A_2\|_\mathcal{G}$ if $s_n(A_1) = s_n(A_2)$, $n = 1, 2, \ldots$. In particular,

$$\|A\|_\mathcal{G} = \|A^*\|_\mathcal{G} = \|A\|_\mathcal{G}.$$

A two-sided ideal $\mathcal{G}$ is called symmetrically normed if it is a complete (Banach) space relative to the symmetric norm $\|\cdot\|_\mathcal{G}$.

Important symmetrically normed ideals $\mathcal{G}_p$ are formed by operators $A \in \mathcal{B}$ for which the quantity

$$\|A\|_p^p = \sum_{n=1}^\infty s_n^p(A), \quad 1 \leq p < \infty,$$ (10)

is finite. It can be shown that the functional $\|\cdot\|_p$ is a symmetric norm. In particular, inequality (9) follows from (6). We note that $\mathcal{G}_p \subset \mathcal{G}_{p_1}$ for $p_1 \leq p_2$ and $\|A\|_{p_1} \geq \|A\|_{p_2}$. Moreover, we have

PROPOSITION 3. If $A_j \in \mathcal{G}_{p_j}$, $j = 1, 2$, and $p^{-1} = p_1^{-1} + p_2^{-1} \leq 1$, then $A = A_1A_2 \in \mathcal{G}_p$ and

$$\|A\|_p \leq \|A_1\|_{p_1}\|A_2\|_{p_2}.$$ (11)

The set $\mathcal{G}$ of finite-dimensional operators is dense in $\mathcal{G}_p$ with respect to the norm $\|\cdot\|_p$, so that the space $\mathcal{G}_p$ is separable. The most useful ideals are $\mathcal{G}_2$ of Hilbert-Schmidt operators and $\mathcal{G}_1$ of trace class operators. It follows from (11) that the product of two Hilbert-Schmidt operators is a trace class operator. The converse is also true: each trace class operator can be represented (of course, not uniquely) as the product of two Hilbert-Schmidt operators. We note that any symmetrically normed ideal $\mathcal{G}$ lies between $\mathcal{G}_1$ and $\mathcal{G}_\infty$, i.e., $\mathcal{G}_1 \subset \mathcal{G} \subset \mathcal{G}_\infty$.

More general classes of operators generated by symmetric quasinorms are sometimes also considered. The definition of such a quasinorm differs from that of a norm only in that the role of the triangle inequality is played by the more general inequality

$$\|A + B\|_\mathcal{G} \leq C(\|A\|_\mathcal{G} + \|B\|_\mathcal{G}).$$

An important example of quasinormed classes are the ideals $\mathcal{G}_p$ for $p \in (0, 1)$ where the functional $\|\cdot\|_p$ becomes a quasinorm, and $C = 2^{1/p-1}$.

We note that Proposition 3 holds for all $p > 0$.

5. We consider in more detail some properties of Hilbert-Schmidt operators. If $A \in \mathcal{G}_2$ then for any orthonormal basis $v_n$

$$\|A\|_2^2 = \sum_n \|Av_n\|^2 < \infty.$$ (12)

Note that the convergence of the series on the right for at least one such basis ensures that $A \in \mathcal{G}_2$. From (12) it is easy to find that

$$A = \sum_n (\cdot, u_n)v_n, \quad u_n = A^*v_n,$$ (13)

where the series converges in the norm of $\mathcal{G}_2$. In the case where $v_n$ are eigenvectors of the operator $AA^*$ relation (13) reduces to the canonical representation (3).

Suppose now that $\mathcal{F}_0$ and $\mathcal{F}$ are realized as

$$L_2(M_0; dm_0) \text{ and } L_2(M; dm),$$

where $M_0, M$ are abstract spaces with measures $dm_0, dm$. Any operator $A \in \mathcal{G}_2$ then acts as an integral operator, i.e.,

$$(Af)(\mu) = \int_{M_0} a(\mu, \nu) dm_0(\nu),$$ (14)

and

$$\|A\|_2^2 = \int_M \int_{M_0} |a(\mu, \nu)|^2 dm(\mu) dm_0(\nu).$$ (15)

This description carries over directly to Hilbert-Schmidt operators acting in direct integrals of the form (5.1) or (5.6). Namely, any operator $A \in \mathcal{G}_2$ acting in the space $\mathcal{F}(a)$ is an integral operator in the decomposition (5.6) in the sense of Definition 5.2. For a.e. $(\mu, \nu) \in \sigma \times \sigma$ its kernel $a(\mu, \nu)$ belongs to $\mathcal{G}_2(h(\nu), h(\mu))$, and

$$\|A\|_2^2 = \int_{\sigma} \int_{\sigma} |a(\mu, \nu)|^2 d\mu d\nu.$$ (16)

The kernel $a(\mu, \nu)$ can be formed constructively in the following manner. Suppose the operator $A: \mathcal{F}(a) \to \mathcal{F}(a)$ can be represented in the form (13) in some orthonormal basis. We denote by $\tilde{u}_n = \mathcal{F}u_n$ and $\tilde{v}_n = \mathcal{F}v_n$.
1. Preliminary facts

representatives of the elements $u_n$ and $v_n$ in the decomposition (5.6) and set

$$a(\mu, \nu) = \sum_n (\cdot, \tilde{u}_n(\nu))_{b(\nu)} \overline{u}_n(\mu).$$

\hspace{1cm} (17)

Since

$$\sum_n \int_\delta |\tilde{u}_n(\nu)|^2_{b(\nu)} d\nu \leq \sum_n \|A_n u_n\|^2 < \infty,$$

\hspace{1cm} (18)

the series (17) converges in the metric of (16) and hence in the metric of $\Theta_2(b(\nu), b(\mu))$ for a.e. $(\mu, \nu) \in \delta \times \delta$. For the function (17) equality (5.11) holds. Indeed, for any $f, g \in \mathcal{H}(a)$ and $f = \mathcal{F} f$, $g = \mathcal{F} g$

$$\int_\delta \int_\delta (a(\mu, \nu) \tilde{f}(\nu), \tilde{g}(\mu)) d\mu d\nu = \sum_n \int_\delta \langle \tilde{f}(\nu), \tilde{u}_n(\nu) \rangle d\nu \int_\delta \langle \tilde{g}_n(\mu), \tilde{u}_n(\mu) \rangle d\mu.$$  \hspace{1cm} (19)

Under condition (18) the series on the right converges absolutely, so that the interchange of summation and integration on $\mu$ and $\nu$ is legitimate by Fubini’s theorem. The right-hand side of (19) is obviously equal to the form $\mathcal{A}(f, g)$ of the operator (13). The function (17) is thus a kernel of the operator $A$ in the sense of Definition 5.2. In particular, for a.e. $(\mu, \nu)$ in $\delta \times \delta$, this function does not depend on the choice of the decomposition (13).

6. The following assertion is useful for the verification that a bounded operator belongs to the class $\Theta_\gamma$. It is often called the third-line theorem for operator-valued functions residing in the classes $\Theta_\gamma$. The proof is easily obtained (see [7], [8]) by means of Theorem 2.14.

Theorem 4. Suppose that the operator-valued function $A(z)$ is holomorphic in the strip $a_1 < \Re z < a_2$, is continuous up to the boundary, and the following condition holds:

$$\sup_{a_1 < \Re z < a_2} \|A(z)\| < \infty.$$  \hspace{1cm} (20)

Suppose that $A(a_2 + iy) \in \Theta_{p_2}, \ 1 \leq p_2 < \infty,$ and

$$\sup_a \|A(a_2 + iy)\|_{p_2} \leq C_n < \infty, \quad n = 1, 2.$$  \hspace{1cm} (21)

Then for all $x \in (a_1, a_2)$ the values of $A(x + iy)$ belong to the class $\Theta_{p_x}$ where

$$p_x^{-1} = p_1^{-1} + \alpha(p_2^{-1} - p_1^{-1}), \quad \alpha = \alpha(x) = (x - a_1)(a_2 - a_1)^{-1},$$

and

$$\|A(x + iy)\|_{p_x} \leq C_1^{1-x} C_2^{\alpha}.$$  \hspace{1cm} (22)

In Theorem 4 it is admissible that one of the numbers $p_x = \infty$. Moreover, in this case the inclusion $A \in \Theta_\infty$ is not used; only a uniform estimate of the usual norm is required.

7. Below we denote by $\mathbb{R}^d$ (coordinate) Euclidean space of dimension $d$; $\Xi^d$ is the dual (momentum) Euclidean space to $\mathbb{R}^d$; $x$ and $\xi$ are points in $\mathbb{R}^d$ and $\Xi^d$, respectively. We introduce the integral operator

$$(\Phi f)(\xi) = (2\pi)^{-d/2} \int_{\mathbb{R}^d} \exp(-i(x, \xi)) f(x) dx.$$  \hspace{1cm} (23)

The (Fourier) operator $\Phi$ maps $L_2(\mathbb{R}^d)$ unitarily onto $L_2(\Xi^d)$. For details regarding this see, for example, [4].

We now consider the operator $\Phi$ bordered on the left and right by multiplication operators. Namely, we set

$$T = B_1 \Phi B_2,$$  \hspace{1cm} (24)

where $B_1, B_2$ are multiplications by functions $b_1(\xi), b_2(x)$. It is clear that $T \in \Theta_{\beta}$ if $b_1 \in L_2(\Xi^d)$ and $b_2 \in L_2(\mathbb{R}^d)$. By approximating the functions $b_1$ vanishing at infinity by compactly supported functions, we obtain a compactness criterion for the operator $T$.

Lemma 5. Suppose the functions $b_1$ are bounded and tend to zero at infinity. Then the operator (24) is compact.

Criteria for being of trace class are more substantial. The proof of the next assertion can be found in the course [18], Volume 3 or in the survey paper [47].

Theorem 6. Suppose that

$$|b_1(\xi)| \leq C(1 + |\xi|)^{-\beta}, \quad |b_2(x)| \leq C(1 + |x|)^{-\beta},$$

where $\beta > d$. Then the operator (24) belongs to the class $\Theta_\beta$.

§7. The Trace and Determinant

A detailed exposition of the material of this section can be found in the books [4], [7], [31], [36].

1. In this section it is assumed that all operators act in a common Hilbert space $\mathcal{H}$. The spectrum $\sigma(A)$ of an arbitrary closed (and, as always, densely defined) operator $A$ is the set of points $\lambda \in \mathbb{C}$ for which there does not exist a bounded inverse operator $(A - \lambda I)^{-1}$. We note that $\sigma(A^*) = \overline{\sigma(A)}$. For a selfadjoint operator $A$ this definition is equivalent to that given in §3.

The spectrum of a compact operator $A$ is described by the classical Fredholm theorem. Namely, for $A \in \Theta_\infty$, the set $\sigma(A)$ consists of eigenvalues which can accumulate only at zero. Moreover, nonzero eigenvalues have finite algebraic multiplicities. An analogous result holds for compact operators acting in an arbitrary Banach space, and also for operators only a certain power of which is compact (for more details see §8). We mention that in a Banach space an operator is called compact if it takes any bounded set into
a relatively compact set. The class of compact operators, generally speaking, is broader than the closure in norm of the finite-dimensional operators. We suppose that the eigenvalues \( \lambda_n = \lambda_n(A) \) of an operator \( A \in \mathfrak{S}_1 \) (or such that \( A^* \in \mathfrak{S}_1 \), for some natural number \( l \)) are enumerated with account of algebraic multiplicity in nonincreasing order of their moduli.

2. On the class of trace class operators it is possible to define a functional \( \text{Tr} \) which is a natural generalization of the matrix trace in the finite-dimensional case. Namely, for \( A \in \mathfrak{S}_1 \)

\[
\text{Tr} A = \sum_{n=1}^{\infty} \langle Av_n, v_n \rangle,
\]

where the series (1) converges absolutely for any orthonormal basis \( \{v_n\} \), and its sum does not depend on the choice of basis. In view of the spectral representation, for a selfadjoint operator \( A \in \mathfrak{S}_1 \)

\[
\text{Tr} A = \sum_{n=1}^{\infty} \lambda_n(A).
\]

Relation (2) holds also for any \( A \in \mathfrak{S}_1 \). This assertion is called the Lidskii theorem; a proof of it can be found, for example, in [7] or [31]. We further note the inequality

\[
\sum_{n=1}^{\infty} |\lambda_n(A)| \leq \|A\|_1.
\]

In the case \( A \in \mathfrak{S} \) the quantity (1) (or (2)) is, of course, equal to the trace of the operator \( A \) considered in the finite-dimensional space \( R(A) \).

Many properties of the trace known from linear algebra carry over to the infinite-dimensional case.

**Proposition 1.** Suppose \( A, B \in \mathfrak{B} \) and \( AB \in \mathfrak{S}_1 \), \( BA \in \mathfrak{S}_1 \). Then

\[
\text{Tr}(AB) = \text{Tr}(BA).
\]

The proof of this proposition can be obtained in a simple manner by means of (2). It is only necessary to note that the operators \( AB \) and \( BA \) have common nonzero eigenvalues with coinciding algebraic multiplicities.

It follows from (4) that the trace of a trace class operator is a linear invariant:

\[
\text{Tr}(LA^{-1}) = \text{Tr} A,
\]

if the operator \( L \) is bounded and boundedly invertible.

Duality of spaces \( \mathfrak{S}_p \) with conjugate indices can be established in terms of the functional (1). Namely, any continuous linear functional \( l \) on \( \mathfrak{S}_p \), \( 1 < p < \infty \), can be uniquely represented in the form

\[
l(A) = \text{Tr}(AB),
\]

where \( B \in \mathfrak{S}_q, p^{-1} + q^{-1} = 1 \), and \( \|l\| = \|B\|_q \). Thus, \( \mathfrak{S}_p^* = \mathfrak{S}_q \) and for \( 1 < p < \infty \) the spaces \( \mathfrak{S}_p \) are reflexive. The space \( \mathfrak{S}_2 \) is a Hilbert space relative to the scalar product \( \langle A, B \rangle = \text{Tr}(AB^*) \). Moreover, for the duality (6) \( \mathfrak{S}_1 = \mathfrak{B} \) and \( \mathfrak{S}_\infty = \mathfrak{S}_1 \).

3. For an operator \( A \) of trace class another important functional—the determinant \( \text{Det}(I + A) \) is also well defined. Let again \( \{v_n\} \) be an arbitrary orthonormal basis in \( \mathfrak{H} \). Consider the \( N \times N \) matrix with elements \( \delta_{nm} + \langle Av_n, v_m \rangle \) where \( 1 \leq n, m \leq N \). For \( A \in \mathfrak{S}_1 \) the determinant of this matrix has a limit as \( N \to \infty \), and this limit does not depend on the basis \( \{v_n\} \). The limit obtained is called the determinant of the operator \( I + A \) and is denoted by \( \text{Det}(I + A) \).

An equivalent definition is given by the equality

\[
\text{Det}(I + A) = \prod_{n=1}^{\infty} (1 + \lambda_n(A)),
\]

where by (3) this infinite product converges absolutely, and

\[
|\text{Det}(I + A)| \leq \exp \|A\|_1.
\]

Equality (7) generalizes the relation well known for the finite-dimensional case. It is clear that

\[
\text{Det}(I + A^*) = \overline{\text{Det}(I + A)}.
\]

In the case \( A \geq 0 \), comparing (2) and (7), we obtain the inequality

\[
\text{Tr} A \leq \text{Det}(I + A).
\]

From (7) and the Fredholm alternative it follows that the operator \( I + A \) is boundedly invertible if and only if \( \text{Det}(I + A) \neq 0 \).

The function \( \text{Det}(I + A) \) is continuous with respect to \( A \) in the trace class norm, i.e.,

\[
\text{Det}(I + A_n) \to \text{Det}(I + A),
\]

if \( A_n \in \mathfrak{S}_1 \) and \( \|A_n - A\| \to 0 \). In particular, \( \text{Det}(I + A_n) \to 1 \) as \( \|A_n\| \to 0 \). Further, if the operator-valued function \( A(z) \) is holomorphic in some domain and takes values in \( \mathfrak{S}_1 \) there, then \( \text{Det}(I + A(z)) \) is also holomorphic in this domain. There is moreover the differentiation formula

\[
(\text{Det}(I + A(z)))^{-1} \frac{d}{dz} \text{Det}(I + A(z)) = \text{Tr} \left( [I + A(z)]^{-1} \frac{dA(z)}{dz} \right).
\]

Without presenting a rigorous proof of this equality, let us only observe that at least formally \( \ln \text{Det}(I + A) = \text{Tr} \ln(I + A) \). Differentiating the logarithm of the operator-valued function under trace sign by the ordinary scalar rules we obtain the identity (10). The left-hand side of (10) may also be written in the form \( d \ln \text{Det}(I + A(z))/dz \), where the derivative, of course, does not depend on the choice of branch of the logarithm.

Under the conditions of Proposition 1 we can establish in the same way that

\[
\text{Det}(I + AB) = \text{Det}(I + BA).
\]
In analogy to (5) it follows from this that
\[ \text{Det}(I + L A L^{-1}) = \text{Det}(I + A). \] (12)
Moreover, for trace class \( A \) and \( B \)
\[ \text{Det}(I + A)(I + B) = \text{Det}(I + A) \text{Det}(I + B). \] (13)
The last equality can be established by approximating the operators \( A \) and \( B \) in \( \mathcal{S}_1 \) by finite-dimensional operators.

4. The definition of the determinant (7) extends in a natural way to operators \( A \) of any class \( \mathcal{E}_p \), \( p < \infty \). Namely, if \( A \in \mathcal{E}_p \), where \( p = 2, 3, \ldots \), then the regularized determinant \( \text{Det}_p \) is introduced by the equality
\[ \text{Det}_p(I + A) = \prod_{n=1}^{\infty} (1 + \lambda_n) \exp \left( \sum_{k=1}^{p-1} k^{-1} (-1)^k \lambda_n^k \right), \quad \lambda_n = \lambda_n(A). \] (14)
For example,
\[ \text{Det}_2(I + A) = \prod_{n=1}^{\infty} (1 + \lambda_n)(-\lambda_n). \] (15)
According to (14) for \( A \in \mathcal{E}_p \) the quantity \( \text{Det}_p(I + A) \) can be expressed in terms of \( \text{Det}_{p-1}(I + A) \):
\[ \text{Det}_p(I + A) = \text{Det}_{p-1}(I + A) \exp[(p - 1)^{-1} (1 - 1)^{p-1} \text{Tr} A^{p-1}]. \] (16)
In particular, for \( A \in \mathcal{S}_1 \)
\[ \text{Det}_p(I + A) = \text{Det}(I + A) \exp \left( \sum_{k=1}^{p-1} k^{-1} (-1)^k \text{Tr} A^k \right). \] (17)

Regularized determinants retain many of the properties of ordinary determinants. Thus, the function \( \text{Det}_p(I + A) \) is continuous with respect to \( A \) as \( A \) varies in the metric of \( \mathcal{E}_p \). If an operator-valued function \( A(z) \) is holomorphic in some domain and takes values in \( \mathcal{E}_p \) there, then the function \( \text{Det}_p(I + A) \) is also holomorphic in the same domain. The generalization of (10) has the form
\[ \frac{d}{dz} \ln \text{Det}_p(I + A(z)) = (-1)^{p-1} \text{Tr}[(I + A(z))^{-1} A^{p-1}(z) A^p(z)]. \] (18)
For \( A \in \mathcal{S}_1 \), this formula is, of course, a consequence of (10) and (17).

Further, according to (14), for \( A \in \mathcal{E}_p \), the operator \( I + A \) is boundedly invertible if and only if \( \text{Det}_p(I + A) \neq 0 \). For \( AB \in \mathcal{E}_p \), and \( BA \in \mathcal{E}_p \), the identities (11), and therefore also (12), carry over to the regularized determinants. The generalization of (13) we give only for \( p = 2 \):
\[ \text{Det}_2(I + A)(I + B) = \text{Det}_2(I + A) \text{Det}_2(I + B) \exp(-\text{Tr}(AB)). \] (19)

\section{8. The analytic Fredholm alternative}

In this section we consider the problem of inversion of a holomorphic operator-valued function whose values differ from the identity by compact operators. For this, standard results of linear algebra are combined with various uniqueness theorems for scalar analytic functions. The simplest possibility is the use of the theorem on the absence of the accumulation of zeros inside a domain of an analytic function not identically equal to zero. In this way we obtain a generalization (see Theorem 2) of the classical Fredholm theorem formulated in Part 1 of \S 7. We further apply the uniqueness theorems in terms of boundary values of analytic functions presented in Part 1 of \S 2. As a result, it is possible to invert an operator-valued function (see Theorems 3 and 5) up to the boundary of its domain of analyticity.

1. For a finite-dimensional operator \( A \) the construction of the inverse operator \( (I - A)^{-1} \) reduces to a problem of linear algebra. Indeed, denoting by \( A_R \) the restriction of \( A \) to its range \( R(A) \), we find that
\[ (I - A)^{-1} = I + (I - A_R)^{-1} A. \] (1)
In the construction of \( (I - A)^{-1} \) we allow not the operator \( A \) itself but only some power of it is compact. In connection with this we recall (see, for example, the book [10]) that for any \( A \in \mathcal{B} \) and any positive integer \( I \)
\[ \sigma(A^I) = \sigma(A)^I. \] (2)
By definition, the right-hand side here consists of points \( \mu \) such that \( \mu = \lambda^I \) for some \( \lambda \in \sigma(A) \).

In what follows, the analyticity (holomorphicity) of operator-valued functions may always be understood in the weak sense. The following almost obvious assertion has local character.

**Lemma 1.** Suppose the operator-valued function \( A(z) \) is holomorphic in a neighborhood of some point \( z_0 \) and \( 1 \notin \sigma(A(z_0)) \). Then in a sufficiently small neighborhood of \( z_0 \) the operator \( (I - A(z))^{-1} \) exists, is bounded, and depends holomorphically on \( z \).

**Proof.** Since the operator \( (I - A(z_0))^{-1} = F(z_0) \) exists and is bounded, it follows that
\[ I - A(z) = [I - (A(z) - A(z_0))F(z_0)](I - A(z_0)). \] (3)
For sufficiently small \( |z - z_0| \) the norm of the operator \( (A(z) - A(z_0))F(z_0) \) is less than 1. Therefore, for such \( z \) an operator inverse to the first factor on the right-hand side exists and can be represented by a series of successive approximations. \( \square \)

2. Let \( \Omega \subset \mathbb{C} \) be an open connected set. We call a function (including an operator-valued function) \( F(z) \) meromorphic in \( \Omega \) if it is holomorphic there
with the exception of the discrete set $\mathfrak{N}$ of its poles. Thus, a meromorphic function $F(z)$ is holomorphic with respect to $z \in \Omega \setminus \mathfrak{N}$, where the "singular set" $\mathfrak{N}$ has no accumulation points in $\Omega$. Moreover, the expansions of $F(z)$ in Laurent series in neighborhoods of the points of $\mathfrak{N}$ must contain only a finite number of singular terms.

The next assertion is often called the analytic Fredholm alternative.

**Theorem 2.** Suppose the operator-valued function $A(z)$ is holomorphic in $\Omega$; $A(z) \in \mathcal{S}_0$, for some positive integer $l$, and for some $z_1 \in \Omega$ there exists the bounded operator $(I - A(z_1))^{-1}$. Then the operator-valued function $(I - A(z))^{-1} = F(z)$ is meromorphic with respect to $z \in \Omega$, and the set of its poles is

$$\mathfrak{N} = \{ z \in \Omega : 1 \not\in \sigma(A(z)) \}. \quad (4)$$

Moreover, in the expansion of $F(z)$ in a Laurent series in a neighborhood of any point $z_0 \in \mathfrak{N}$ the coefficients of negative powers of $z - z_0$ are finite-dimensional operators.

**Proof.** According to Lemma 1, the operator-valued function $F(z)$ is holomorphic in $z \in \Omega \setminus \mathfrak{N}$, where $\mathfrak{N}$ is defined by equality (4). It is therefore only necessary to establish the discreteness of $\mathfrak{N}$ and also to verify the properties of the expansion of $F(z)$ in a Laurent series in a neighborhood of $z_0 \in \mathfrak{N}$.

We suppose first that the operator $A(z)$ itself is compact. In a neighborhood of each point $z_0 \in \Omega$ the problem reduces to a finite-dimensional problem. Namely, suppose that $A(z_0) = K_0 + B_0$ where $K_0 = \text{dim} R(K_0) < \infty$, and $\|B_0\| = c_1 < 1$. We consider a disk

$$D_{z_0}(r) = \{ z : |z - z_0| < r \} \subset \Omega, \quad r = r(z_0),$$

such that $\|A(z) - A(z_0)\| \leq c_2$ for $z \in D_{z_0}(r)$ and $c = c_1 + c_2 < 1$. We set

$$B(z) = B_0 + A(z) - A(z_0).$$

Then $\|B(z)\| \leq c < 1$, and

$$I - A(z) = I - B(z) - K_0 = (I - K(z))(I - B(z)), \quad (5)$$

where the operator-valued function

$$K(z) = K_0(I - B(z))^{-1} \quad (6)$$

is holomorphic in $D_{z_0}(r)$ and $R(K(z)) \subset R(K_0)$. Hence, the set $\mathfrak{N} \cap D_{z_0}(r)$ consists of those and only those $z$ where the operator $I - K(z)$ is not invertible or, equivalently, where $\text{Det}(I - K(z)) = 0$. Since the function $\text{Det}(I - K(z))$ is analytic in $D_{z_0}(r)$, its zeros either have no accumulation points in $D_{z_0}(r)$ or it is identically equal to zero in $D_{z_0}(r)$. Thus, either the set $\mathfrak{N}$ has no accumulation points in $D_{z_0}(r)$ or $\mathfrak{N}$ fills out the entire disk $D_{z_0}(r)$.

Assume that $\mathfrak{N}$ accumulates at some point $z_0 \in \Omega$. Because of the connectedness of $\Omega$ the points $z_1$ and $z_0$ can be joined by a continuous curve. By the Heine-Borel theorem on this curve there is a finite set of points $z_1, z_2, \ldots, z_{n-1}, z_n = z_0$ such that the corresponding disks $D_{z_i}(r_i), r_i = r_i(z_0)$, intersect pairwise, i.e., $D_{z_i}(r_i) \cap D_{z_{i+1}}(r_{i+1}) \neq \emptyset$, $n = 1, \ldots, N - 1$. Since, as already verified, $D_{z_i}(r_i) \subset \mathfrak{N}$, we now see successively that all the disks $D_{z_{n-1}}, \ldots, D_{z_1}(r_1)$ must belong to the set $\mathfrak{N}$. At the same time, by hypothesis $z_j \notin \mathfrak{N}$. This implies the discreteness of $\mathfrak{N}$.

We consider the expansion of $F(z)$ in a Laurent series in powers of $z - z_0$ for $z_0 \in \mathfrak{N}$. From equality (5) it follows that

$$F(z) = (I - B(z))^{-1}(I - K(z))^{-1}, \quad (7)$$

where the factor $(I - B(z))^{-1}$ is holomorphic in a neighborhood of the point $z_0$ and $K(z)$ is defined by equality (6). We restrict the operator $I - K(z)$ to the subspace $R(K_0)$ of dimension $k_0$. The determinant of this operator-valued function cannot have zeros of order higher than $k_0$. By constructing the inverse operator according to the usual recipe of linear algebra, we find that on the subspace $R(K_0)$, invariant for $K(z)$,

$$(I - K(z))^{-1} = I + \sum_{n = n_0}^\infty \mathcal{H}_n(z - z_0)^n, \quad (8)$$

where $n_0 \leq k_0$ and $\dim \mathcal{H}_n \leq k_0$. By (1) this implies that an analogous decomposition holds for the operator $I - K(z)$, considered on the whole space. Therefore, according to (7), (8), $F(z)$ can also be expanded in a Laurent series with only a finite number of negative powers of $z - z_0$ and with finite-dimensional coefficients there.

We proceed to consider the general case. Suppose $A'(z_1) \in \mathcal{S}_0$, and $1 \not\in \sigma(A(z_1))$. We shall show that there is a positive integer $m \geq l$ such that $1 \not\in \sigma(A^m(z_1))$. Indeed, the spectrum of the compact operator $A'(z_1)$ consists of eigenvalues accumulating only at zero. By (2) the spectrum of the operator $A(z_1)$ possesses the same property. In particular, on the unit circle there are only a finite number of eigenvalues $\nu_1, \ldots, \nu_s$ of the operator $A(z_1)$. Again by (2), the spectrum of $A^m(z_1)$ on the unit circle consists of the numbers $\nu_1^m, \ldots, \nu_s^m$. Since by hypothesis $\nu_j \neq 1, j = 1, \ldots, s$, it is possible to choose $m$ so large that $\nu_j^m \neq 1, j = 1, \ldots, s$.

The operator-valued function $\tilde{A}(z) = A^m(z)$ thus assumes compact values and $(I - \tilde{A}(z))^{-1}$ exists. The part of the theorem proved (for the case $l = 1$) we apply to $\tilde{A}(z)$. From this it follows that the set

$$\tilde{\mathfrak{N}} = \{ z \in \Omega : 1 \not\in \sigma(\tilde{A}(z)) \}$$

is discrete. The discreteness of $\tilde{\mathfrak{N}}$ now follows from the obvious inclusion $\mathfrak{N} \subset \tilde{\mathfrak{N}}$. Finally, the assertions regarding the expansion of $F(z)$ in a Laurent
series follow from the equality

$$(I - A(z)^{-1}) = (I + A(z) + \cdots + A^{m-1}(z))(I - A(z)^{-1})$$

and the corresponding assertions regarding \((I - \tilde{A}(z))^{-1}\).

In the special case \(A(z) = zA\), \(\Omega = \mathbb{C}\), \(z_1 = 0\), Theorem 2 reduces to the Fredholm theorem regarding the spectrum of an operator, some power of which is compact.

3. In scattering theory we need information regarding the structure of the singular set of an operator-valued function on the boundary of its domain of analyticity. The next results can be established in full analogy to Theorem 2 by means of Theorem 2.1 or rather the corollary of it valid for functions continuous up to the boundary.

**Theorem 3.** Suppose \(\Lambda\) is an interval of the real axis and the operator-valued function \(A(z)\) satisfies the conditions of Theorem 2 in the rectangle

$$\Omega = \Omega^{(a)} = \{z \in \mathbb{C} : \text{Re} z \in \Lambda, \pm \text{Im} z \in (0, \varepsilon_0)\}. \tag{10}$$

Suppose, moreover, that \(A(z)\) is continuous in norm up to \(\overline{\Lambda}\). Then the set

$$\mathcal{N} = \mathcal{M}^{(a)} = \{\lambda \in \overline{\Lambda} : 1 \in \sigma(A(\lambda \pm i0))\}$$

is closed and has Lebesgue measure zero, while the operator-valued function \(F(z) = (I - A(z)^{-1})\) is meromorphic in \(\Omega\) and is continuous in norm as \(z\) approaches points of the set \(\overline{\Lambda} \setminus \mathcal{N}\).

**Proof.** The meromorphicity of \(F(z)\) follows from Theorem 2. For \(\lambda_0 \in \overline{\Lambda} \setminus \mathcal{N}\) and \(z_0 = \lambda_0 \pm i0\) the operator \(F(z_0)\) exists and is bounded. In analogy to the proof of Lemma 1, it is easy to see that the bounded operator \(F(z)\), \(z \in \Omega\), exists in a sufficiently small neighborhood of the point \(z_0\) and \(\|F(z) - F(z_0)\| \to 0\) as \(|z - z_0| \to 0\). From this it follows that the poles of \(F(z)\) cannot accumulate at points of \(\overline{\Lambda} \setminus \mathcal{N}\) and the set \(\overline{\Lambda} \setminus \mathcal{N}\) is open (in \(\overline{\Lambda}\)). Hence, the complementary set \(\mathcal{N}\) is closed.

It remains to show that \(|\mathcal{N}| = 0\). We use the construction presented in the proof of Theorem 2. Suppose first that \(l = 1\). For any \(\lambda_0 \in \overline{\Lambda}\) we consider the disk \(D_{\lambda_0}(r), r = r(\lambda_0)\), such that for \(z \in D_{\lambda_0}(r) \cap \Omega\) the representations (5), (6) hold, where \(\|B(z)\| \leq c \leq 1\) and \(\dim K_{\lambda_0} < \infty\). The scalar function \(\text{Det}(I - K(z))\) is holomorphic in \(z \in D_{\lambda_0}(r) \cap \Omega\) and is continuous up to \(\overline{\Lambda}\), while the relations \(1 \in \sigma(A(z))\) and \(\text{Det}(I - K(z)) = 0\) are equivalent. The function \(\text{Det}(I - K(z))\) is not identically zero in \(D_{\lambda_0}(r) \cap \Omega\). Therefore, by Theorem 2.1 the set of its zeros on \((\lambda_0 - r, \lambda_0 + r)\) has measure zero. From this it follows that \(|\mathcal{N} \cap (\lambda_0 - r, \lambda_0 + r)| = 0\). Since by the Heine-Borel lemma \(\overline{\Lambda}\) can be covered by a finite number of the intervals \((\lambda_0 - r, \lambda_0 + r)\) constructed, the measure of the whole of \(\mathcal{N}\) is also equal to zero.

To pass to the general case it is again necessary (cf. the proof of Theorem 2) to consider the auxiliary operator-valued function \(A(z) = A^m(z)\). It satisfies the conditions of the theorem to be proved for the case \(l = 1\). Hence, the set

$$\mathcal{M} = \{\lambda \in \overline{\Lambda} : 1 \in \sigma(A(\lambda \pm i0))\} \tag{11}$$

has measure zero. It remains to note that \(\mathcal{N} \subset \mathcal{M}\).

**Remark.** As is evident from the proof, Theorems 2 and 3 extend to Banach spaces if \(\Theta_{\infty}\) is understood as the set of bounded operators approximable in norm by finite-dimensional operators.

We shall need a modification of Theorem 3 for the case where in place of continuity up to the boundary of the operator-valued function \(A(z)\) we have only angular limit values, but the convergence is understood in the sense of \(\Theta_p\).

**Theorem 5.** Suppose in the domain (10) the operator-valued function \(A(z)\) satisfies the conditions of Theorem 2 and has angular limit values in \(\mathcal{B}\) for a.e. \(\lambda \in \Lambda\). Suppose also that for some positive integers \(l\) and \(p, p A(z)\) takes values in \(\Theta_p\) and has angular limit values in this class for a.e. \(\lambda \in \Lambda\). Then the meromorphic \(\Omega\) operator-valued function \(F(z) = (I - A(z))^{-1}\) has, in the sense of convergence in norm, angular limit values for a.e. \(\lambda \in \Lambda\).

**Proof.** We choose \(m \geq l\) as in Theorems 2 and 3 and set \(A(z) = A^m(z)\). In analogy to the proof of Theorem 3 it suffices to show that the operator \(I - A(\lambda \pm i0)\) is invertible for a.e. \(\lambda \in \Lambda\). To this end it is necessary to verify that the measure of the set (11) is equal to zero. The regularized (see Part 4 of §7) determinant \(\text{Det}_p(I - A(z))\) is holomorphic with respect to \(z \in \Omega\), is not identically zero, and has angular limit values for a.e. \(\lambda \in \Lambda\). Moreover, the inclusion \(\lambda \in \mathcal{M}\) is equivalent to the equality \(\text{Det}_p(I - A(\lambda \pm i0)) = 0\). At the same time, by Theorem 2.1 this equality may be satisfied only on a set of measure zero.

**§9. The resolvent equation**

In the stationary construction of scattering theory for a pair of selfadjoint operators \(H_x\) and \(H\) in one way or another the equation connecting their resolvents plays an important role. This equation can be applied (see §10) also for a correct definition of the full Hamiltonian \(H\).

1. Let \(H_x, H\) be selfadjoint operators in the Hilbert spaces \(X, X\) respectively; let \(R_x = (H_x - z)^{-1}, R = (H - z)^{-1}\) be their resolvents; let \(2: \mathcal{R} \rightarrow \mathbf{r}\) be as a bounded (identification) operator; let \(\Phi\) be an auxiliary Hilbert space. We recall that an operator \(G: \mathcal{R} \rightarrow \Phi\) is called \(H\)-bounded if \(D(H) \subset D(G)\) and the operator \(GR(z)\) is bounded for \(z\) in the resolvent set \(\rho = \rho(H)\). For such \(G\) we systematically assume that by definition

$$R(z)G^* := (GR(z))^*. \tag{1}$$
If the operator $G$ is closable, and hence $G^*$ is densely defined, then

$$(GR(z))^* = R(z)^{G^*}.$$  

In terms of the dual spaces $\mathcal{D} = \mathcal{D}(H)$ and $\mathcal{D}^*$ (See Part 4 of §5) the relation (1) means that $G^*$ can be considered as a bounded operator from $\mathcal{D}$ to $\mathcal{D}^*$. Below, without special mention, for an $H$-bounded operator $G$ we write $G^*$, understanding that its precise meaning is given by relation (1).

Similarly, if $G$ is defined only on compactly supported elements in $\mathcal{E} = \mathcal{E}(H)$ and the operators $GE(\Lambda) = G_\Lambda$ are bounded for bounded sets $\Lambda$, then the operator $E(\Lambda)G^* := G_\Lambda^*$ is meaningful. In this case $G^*$ may be considered as a bounded operator from $\mathcal{E}$ to $\mathcal{E}^*$.

We shall often use the factorization of the perturbation

$$V := H^3 - 3H_0 = G^*G_0, \quad G_0: \mathcal{H}_0 \rightarrow \mathcal{E}, \quad G: \mathcal{H} \rightarrow \mathcal{E}. \tag{2}$$

In a precise sense relation (2) is understood as equality of the corresponding sesquilinear forms

$$\psi[f_0, f] := (f_0, Hf) - (3H_0f_0, f) = (G_0f_0, Gf). \tag{3}$$

As a rule, we suppose that the operators $G_0$ and $G$ are bounded relative to $H_0$ and $H$, respectively, while equality (3) is considered on elements $f_0 \in \mathcal{D}(H_0)$, $f \in \mathcal{D}(H)$. Since the operator $V$, considered as a mapping of $\mathcal{D}_0 = \mathcal{D}(H_0)$ into $\mathcal{D}^*$, is bounded, the factorization (2) always exists but is, of course, not uniquely determined. Sometimes, on the other hand, it suffices to consider (3) only on some subsets of $\mathcal{D}(H_0)$ and $\mathcal{D}(H)$. The collections $\mathcal{D}_0 = \mathcal{E}(H_0)$ and $\mathcal{E}$ of "compactly supported" elements $f_0$ and $f$ give the most restricted natural choice of such subsets. The broadest possible interpretation of (3) is hereby attained.

We shall now obtain an equation connecting the resolvents of the operators $H_0$ and $H$. To this end we consider (3) on elements $f_0 = R_0(z)\psi_0$, $f = R(\overline{z})\psi$ where $\psi_0 \in \mathcal{H}_0$, $\psi \in \mathcal{H}$ are arbitrary, and $z \in \rho_0 \cap \rho$, $\rho_0 = \rho(H_0)$. Then

$$(3R_0(z)\psi_0, \psi) - (3\psi_0, R(z)\psi) = (G_0R_0(z)\psi_0, GR(z)\psi).$$

All the operators here are well defined and bounded, and hence

$$3R_0(z) - R(z)3 = (GR(z))^*G_0R_0(z). \tag{4}$$

Relation (4) is called the resolvent identity or equation. We thus have

**Relation (4) is called the resolvent identity or equation. We thus have**

**Proposition 1.** Suppose in (2) the factors $G_0$ and $G$ are bounded relative to the operators $H_0$ and $H$, respectively. Then for all $z \in \rho_0 \cap \rho$ the identity (4) holds.

Equality (4) can formally be written in the form

$$3R_0(z) - R(z)3 = R(z)V R_0(z). \tag{5}$$

This relation becomes meaningful if $V$ is considered as an operator from $\mathcal{D}_0$ to $\mathcal{D}^*$. We distinguish a special case, when the perturbation (2) is a bounded operator. In a precise sense this means that the form (3) satisfies the estimate

$$|\psi[f_0, f]| \leq C\|f_0\| \|f\|.$$  

Under this condition to the form (3) there corresponds a bounded operator $V: \mathcal{H}_0 \rightarrow \mathcal{E}$ such that

$$(V f_0, f) = \psi[f_0, f], \quad f_0 \in \mathcal{D}(H_0), \quad f \in \mathcal{D}(H).$$

The operator $V$ defined like this gives a correct interpretation to the perturbation (2). On the other hand, the estimate for $\psi[f_0, f]$ ensures that $\mathcal{D}(H_0) \rightarrow \mathcal{D}(H)$ (and $\mathcal{D}^*(H) \rightarrow \mathcal{D}(H_0)$) and that the operator $H^3 - 3H_0$ extends from $\mathcal{D}(H_0)$ to a bounded operator on all of the space $\mathcal{H}_0$.

2. The relation (4) may be considered as an equation for the resolvent of the full Hamiltonian. Solution of this equation for $R(z)$, however, requires additional assumptions. In subsequent considerations of this section the operators $H_0$ and $H$ lose equal footing. The first of them is considered given, and assumptions regarding the perturbation $V = G^*G_0$ are made in terms of it. The full Hamiltonian is defined on the basis of the unperturbed operator $H_0$ and the perturbation $V$ only with the help of an equality of the form (3) for its sesquilinear form.

Below in this section we assume that $\mathcal{H}_0 = \mathcal{H}$, $\mathcal{J} = I$, and that the operators $G_0$ and $G$ are bounded relative to the operators $|H_0|^\theta_0$ and $|H_0|^\theta$ respectively, where $\theta_0$ is some number of the interval $[1/2, 1]$ and $\theta = 1 - \theta_0 \in [0, 1/2]$. Thus,

$$G_0(|H_0| + 1)^{-\theta_0} \in \mathcal{B}(\mathcal{H}, \mathcal{E}), \quad G(|H_0| + 1)^{-\theta} \in \mathcal{B}(\mathcal{H}, \mathcal{E}). \tag{6}$$

The perturbation $V$ is assumed to be symmetric:

$$(G_0f, Gg) = (Gf, G_0g), \quad f, g \in \mathcal{D}(|H_0|^{\theta_0}). \tag{7}$$

Under these conditions we adopt

**Definition 2.** We say that a selfadjoint operator $H$ corresponds to the sum $H_0 + G^*G_0$ if the following two conditions are satisfied. First, for any regular point $z \in \rho$ its resolvent $R(z)$ admits the representation

$$R(z) = (|H_0| + 1)^{-\theta_0}(1 - 1)^{-\theta},$$

where the operator $\Gamma(z)$ is bounded. In particular, $\mathcal{D}(H) \subset \mathcal{D}(|H_0|^{\theta_0})$. Second, for all $f_0 \in \mathcal{D}(H_0)$ and $f \in \mathcal{D}(H)$

$$(Hf, f_0) = (f, H_0f_0) + (Gf, G_0f_0). \tag{9}$$

In Definition 2 the operator $H$ is assumed to be selfadjoint a priori. However, we have
REMARK 3. Under conditions (6), (7) an operator $H$, for which $\mathcal{D}(H) = \mathcal{H}$, $\mathcal{D}(H) \subseteq \mathcal{D}(\langle H_0 \rangle^\delta)$ and (9) is satisfied, is necessarily symmetric. Indeed, setting $H_0^\delta = \text{sgn} H_0 |H_0|^\delta$, equality (9) in the form

$$(H f, g) = (H_0^\delta f, |H_0| g) + (G_0 f, G g), \quad f, g \in \mathcal{D}(H),$$

(10)
can be extended to all $g \in \mathcal{D}(\langle H_0 \rangle^\delta)$. In particular, (10) is applicable to any $f, g \in \mathcal{D}(H)$. It remains to consider the selfadjointness of the operator $H_0$ and equality (7).

If under the conditions of this remark for some two points $z_\pm + \pm \text{Im} z_\pm > 0$, (or one real point) the range $R(H - z I)$ coincides with $\mathcal{H}$, then the operator $H$ is selfadjoint. Below we shall see (see Theorem 5) that only one selfadjoint operator can satisfy Definition 2. We shall also need

REMARK 4. By the Hilbert identity

$$R(z) = R(z_1) - (z - z_1)^2 R(z_1) + (z - z_1)^2 R(z) R(z_1) R(z) R(z_1)$$

it suffices to verify the representation (8) for any one regular point $z_1 \in \rho(H)$.

We note also that under condition (8) the operator

$$G_0 R(z) = [G_0 (|H_0| I + I)^{-\theta_0}] \Gamma(z) [G(|H_0| I + I)^{-\theta_0}]^*$$

(11)
is well defined, bounded, and holomorphic for $z \in \rho(H)$.

A sufficient condition for the validity of the representation (8) is the inclusion

$$\mathcal{D}(\langle H \rangle^\delta) \subseteq \mathcal{D}(\langle H_0 \rangle^\delta).$$

Indeed, (12) implies that the operator

$$L_\theta = (|H_0| I + I)^{\theta_0}[|H_0| I + I]^{-\theta_0}$$

is bounded. By the Heinz inequality (see Lemma 6.1) the operator $L_\theta$ is also bounded. It is clear that (8) holds if we set

$$\Gamma(z) = L_\theta [R(z) I + R(z)] L_\theta^*.$$

Under condition (12) this operator is a product of three bounded factors. The relation $\mathcal{D}(H) \subseteq \mathcal{D}(H_0)$ is in turn sufficient for the validity of (12).

3. We return to a consideration of the resolvent identity. For $\mathcal{H} = I$ and under condition (7) we have, together with (4), the identity symmetric to it. Thus, for $z \in \rho(H) \cap \rho$

$$R(z) = R_0(z) - (GR(z))^* G_0 R_0(z) = -(GR(z))^* (G_0 R(z)).$$

(13)

If $R(z)$ satisfies relation (8), then we can apply the operator $G_0$ on the left and the operator $G^*$ on the right to both sides of (13). This leads to the identity

$$[I + G_0 R_0(z) G^*] [I - G_0 R(z) G^*] = [I - G_0 R(z) G^*] [I + G_0 R_0(z) G^*] = I.$$

(14)

From this it follows that both factors in (14) are boundedly invertible for $z \in \rho_0 \cap \rho$. Iterating (13), we find that

$$R(z) = R_0(z) - R_0(z) G^* [I - G_0 R(z) G^*] G_0 R_0(z).$$

By (14) from this we obtain the representation

$$R(z) = R_0(z) - R_0(z) G^* [I + G_0 R_0(z) G^*]^{-1} G_0 R_0(z).$$

(15)

Conversely, it is possible to express in a similar way $R_0(z)$ in terms of $R(z)$:

$$R_0(z) = R(z) + R(z) G^* [I - G_0 R(z) G^*]^{-1} G_0 R(z).$$

(16)

According to (15), the resolvent of a selfadjoint operator $H$ satisfying the conditions of Definition 2 can be uniquely recovered from $H_0$, $G_0$, and $G$. From this it follows that the operator $H$ itself is uniquely determined by these conditions. We have thus established

THEOREM 5. Only one selfadjoint operator $H$ can satisfy Definition 2. If such an $H$ exists, then for $z \in \rho(H_0) \cap \rho$ the identities (13), (14) hold for its resolvent. Moreover, $R(z)$ can be expressed in terms of $R_0(z)$ by equality (15), and the inverse operator on the right-hand side of (15) exists for all $z \in \rho(H_0) \cap \rho$.

As a rule, only the cases $\theta_0 = 1$ and $\theta_0 = 1/2$ are encountered in applications. In the first of them $\mathcal{D}(H) \subseteq \mathcal{D}(H_0)$ by Definition 2. Actually, we must here have the equality $\mathcal{D}(H) = \mathcal{D}(H_0)$. The opposite inclusion $\mathcal{D}(H_0) \subseteq \mathcal{D}(H)$ follows from the relation $R_0(z) = R(z) \Gamma$, where $\Gamma \in \mathfrak{B}$, which is a corollary of equality (16).

For $\theta_0 = 1/2$ we consider specially the case of (lower) semibounded operators. By making a shift, if necessary, it may be assumed that $H_0 \geq c I > 0$ and $H \geq c I > 0$. We shall show that then

$$\mathcal{D}(H^{1/2}) = \mathcal{D}(H_0^{1/2}).$$

(17)

Indeed, equality (8), i.e.,

$$(H^{-1} f, f) = (H_0^{-1/2} H_0^{-1/2} f, f), \quad \Gamma \in \mathfrak{B},$$

shows that $\|H^{-1/2} f\| \leq C\|H_0^{-1/2} f\|$. This implies that $\mathcal{D}(H^{1/2}) \subseteq \mathcal{D}(H_0^{1/2})$ and $H_0^{1/2} H^{-1} H_0^{1/2} \in \mathfrak{B}$. From the representation (16) it now follows that

$$(H_0^{-1} = H^{1/2} H^{-1}) \Gamma H_0^{-1/2} \Gamma$$

for a bounded $\Gamma$, and hence $\mathcal{D}(H_0^{1/2}) \subseteq \mathcal{D}(H^{1/2})$ and $H^{-1} H_0^{1/2} \in \mathfrak{B}$. From this we obtain (17).

4. We shall establish the connection of the spectrum $\sigma$ of the operator $H$ with the set $\mathfrak{R} \subseteq \rho_0$ of points $z$ where the operator

$$(I + G_0 R_0^* G_{0}^*)^{-1}$$

(18)
does not exist. We recall that the operator-valued function $G_0 R_0^* G_{0}^*$ is holomorphic on the set $\rho_0$. According to Lemma 8.1, if the operator (18) exists at some point, then it is holomorphic in a neighborhood of that point.
PROPOSITION 6. Suppose the operator $H$ satisfies Definition 2. Then
\[ \mathcal{R} = \sigma(H) \cap \rho(H_0), \] (19)

PROOF. Let $z \in \rho_p$. For $z \in \rho$, the existence and boundedness of the operator (18) follows from equalities (14). Conversely, if at some point $z$ the operator (18) exists and is bounded, then the operator (15) is holomorphic in a neighborhood of that point. This means that $z \in \rho_p$. \[ \square \]

COROLLARY 7. Under the conditions of Proposition 6 the operator (18) exists and is bounded for all nonreal points $z$.

5. In all the constructions of this section condition (6) can be relaxed somewhat. Namely, it suffices to assume that the operators $G_0 h^{-1}$ and $G h^{-1}$ are bounded for some auxiliary positive definite operator $h$. In addition, it is necessary to suppose that
\[ R_0(z) = h^{-1}\gamma_0(z)h^{-1}, \quad \gamma_0(z) \in \mathcal{B}, \] (20)
and, consequently, $\mathcal{D}(H_0) \subset \mathcal{D}(h^0)$. Definition 2 is then preserved if in place of (8) we require that
\[ R(z) = h^{-1}\gamma(z)h^{-1}, \quad \gamma(z) \in \mathcal{B}, \] (21)
whence $\mathcal{D}(H) \subset \mathcal{D}(h^0)$.

Under the assumptions made the operators $G_0 R_0(z) G^* = G_0 R(z) G^*$ are well defined as before. This makes it possible to literally carry over Theorem 5 to the more general case in question. It reduces to the case already considered for $h = H_0 + I$. The representation (20) is thereby trivial, while (21) goes over into (8).

This generalization may be more convenient due to the greater symmetry of the operators $H_0$ and $H$. For example, it follows directly from it that for a pair of operators $H_1$ and $H_2$ satisfying Definition 2 (with the same operator $H_0$) the resolvent identity holds. Indeed, suppose $H_1 = H_0 + V_1$, $H_2 = H_0 + V_2$, where $V_1 = G_1 G_0^{-1}$, $V_2 = G_2 G_0^{-1}$. Then $H_1 = H_2 = V$ for $V = V_1 - V_2 = G G_0$, where $\Theta = \Theta_0 \Theta_0^2$, $G = G_1 G_2$, $G_0 = (-G_0^0)^{-1}$. It follows that the pair of operators $H_1, H_2$ satisfies Definition 2 with $h = (H_0 + I)$. Therefore, relations of the form (13)–(16) are valid for the estimates of the resolvents of the operators $H_1, H_2$ according to Theorem 5 (in the generalized form discussed in this part).

\[ \S 10. \text{Conditions for selfadjointness} \]

In studying the resolvent of the full Hamiltonian $H = H_0 + V$ by means of the resolvent $R_0(z)$ of the unperturbed operator $H_0$, it is natural to formulate the condition for selfadjointness of $H$ also in terms of $R_0(z)$. Here we shall show that if the operator (9.18) exists and is bounded, then there exists a selfadjoint operator $H$ satisfying Definition 9.2. From this it is not hard to extract a number of often used selfadjointness criteria. In this section, as previously, it is assumed that $\mathcal{R}^* = \mathcal{R}$, $\delta = I$, that the operator $H_0$ is selfadjoint, and that conditions (9.6), (9.7) are satisfied.

1. Below the selfadjoint operator $H$ is constructed in terms of its resolvent, which is, in turn, determined by equality (9.15). Properties of operators of the form (9.15) are described in the next two elementary assertions. For brevity we set
\[ B_0(z) = -G_0 R_0(z) G^*. \] (1)

LEMMMA 1. Suppose the operator (9.18) exists and is bounded. Then the operator
\[ \mathcal{R} = \mathcal{R}(z) = R_0(z) - R_0(z) G^* [I + G_0 R_0(z) G^*]^{-1} G_0 R_0(z) \] (2)
and its adjoint have trivial kernels.

PROOF. Suppose for some $f$ that
\[ R_0 f - R_0 G^* [I + G_0 R_0(z) G^*]^{-1} G_0 R_0(z) f = 0, \quad R_0 R_0(z) f = 0. \] (3)
By condition (9.6) the operator $G_0$ can be applied to this equality. Then for $g = G_0 R_0(z)$ it follows from (3) that $g + B_0(I - B_0)^{-1} g = (I - B_0)^{-1} g = 0$. Hence, $g = 0$. Returning to (3), we find that $R_0 f = 0$, and hence $f = 0$.

We note that, together with (9.18), also the inverse to the adjoint operator must exist, i.e., $(I + G R_0 G_0^{-1})^{-1} (R_0^* R_0(z))$ since the operators
\[ \mathcal{R}^* = R_0^* - R_0 G^* [I + G R_0 G_0^{-1}]^{-1} G R_0 \]
and $\mathcal{R}$ have the same structure, the proofs of the equalities $N(\mathcal{R}) = \{0\}$ and $N(\mathcal{R}^*) = \{0\}$ are altogether similar. \[ \square \]

LEMMMA 2. Suppose the operators (9.18) exist and are bounded for $z = z_1$ and $z = z_2$. Then for the operators (2) we have $R(\mathcal{R}(z_1)) = R(\mathcal{R}(z_2))$.

PROOF. Let us verify, for example, the inclusion $R(\mathcal{R}(z_1)) \subset R(\mathcal{R}(z_2))$. To this end it is sufficient to show that the equation $\mathcal{R}(z_1) f_1 = \mathcal{R}(z_2) f_2$ can be solved for $f_2$. For brevity we set $f_1 = R_0(z_1) f_1 \in \mathcal{D}(H_0)$. We determine the desired element $f_2$ by the equalities
\[ f_2 = f_2 - (R_0(z_1) - R_0(z_2)) G^* (I - B_0(z_1))^{-1} G f_1 \]
and $f_2 = (H_0 - z_2) f_2$. Applying the operator $\mathcal{R}(z_2)$ to $f_2$, according to (2) we find that
\[ \mathcal{R}(z_2) f_2 = f_2 - R_0(z_1) G^* (I - B_0(z_1))^{-1} G f_1 \]
\[ + R_0(z_1) G^* (I - B_0(z_1))^{-1} (I - B_0(z_2))^{-1} \]
\[ - (I - B_0(z_2))^{-1} (B_0(z_1) - B_0(z_2)) (I - B_0(z_1))^{-1} G f_1. \]
It is clear that the operator in square brackets is here equal to zero. Therefore, again according to (2), the right-hand side is equal to \( A(z) f \). \( \square \)

Under the conditions ofLemma 1 we construct an operator \( H = H_z \) satisfying Definition 9.2 but, perhaps, not selfadjoint. To this end we remark that by Lemma 1 the operator \( A^{-1}(z) \) is well defined on \( \mathcal{R}(A(z)) \) and, according to (6.1), its domain is dense in \( \mathcal{D} \). Moreover, the operator \( A^{-1}(z) \) is necessarily closed. Hence, the operator

\[
H_z := A^{-1}(z) + zI
\]

is also closed on the dense domain \( \mathcal{D}(H_z) := \mathcal{R}(A(z)) \). The point \( z \) is regular for the operator (4), and its resolvent \( R_z(\zeta) = (H_z - \zeta)^{-1} \) for \( \zeta = z \) coincides with the operator \( A(z) \).

By definition (2) of the operator \( A(z) \) under conditions (9.6) the representation (9.8) holds for \( R_z(\zeta) \) at the point \( \zeta = z \). By Remark 9.4 this representation is preserved for all regular \( \zeta \). From (9.8) it follows, in particular, that \( \mathcal{D}(H_z) \subset \mathcal{D}(H_0) \). Further, from (2) we obtain directly the equation

\[
A(z) - R_0(z) = - (GR_0(z))^* G_0 A(z).
\]

We apply both sides of it to the vector \( \psi = (H_z - z)f, f \in \mathcal{D}(H_z) \), and we take the scalar product of the element obtained with the vector \( \psi_0 = (H_0 - z)f_0, f_0 \in \mathcal{D}(H_0) \). Since \( A(z) \psi = f \), this leads to equality (9.9) for the operator \( H_z \). Thus, the operator \( H_z \) satisfies all the conditions of Definition 9.2. Moreover, by Remark 9.3 under condition (9.7) this operator is symmetric. Finally, since the point \( z \) is regular for \( H_z \), its defect number at this point is equal to zero.

We now suppose that the operators (9.18) exist and are bounded at two points \( z_1 \) and \( z_2 \). We construct the corresponding operators \( H_{z_1} \) and \( H_{z_2} \). Each of them satisfies equality (9.9) whence it follows that \( H_{z_1} f = H_{z_2} f \) for \( f \in \mathcal{D}(H_{z_1}) \cap \mathcal{D}(H_{z_2}) \). At the same time, according to Lemma 2, the domains of the operators \( H_{z_1} \) and \( H_{z_2} \) coincide, and hence \( H_{z_1} = H_{z_2} \). Thus, under the conditions of Lemma 2 the defect numbers of this operator are equal to zero at the two points \( z_1 \) and \( z_2 \). Finally, we note that a symmetric operator is selfadjoint if its defect numbers are equal to zero at least for one point of the upper half-plane and one point of the lower half-plane. We have thus established

**Theorem 3.** Suppose conditions (9.6) and (9.7) are satisfied. Suppose that for some two points \( z_{\pm}, \pm \text{Im} z_{\pm} > 0 \), (or one real point \( z_{\pm} = z_{\pm} \in \rho(H_0) \)) the operator (9.18) exists and is bounded. Then there exists a selfadjoint operator \( H \) satisfying the conditions of Definition 9.2.

Of course, under the conditions of Theorem 3 the operator \( H \) is defined by the equality \( H = H_{z_{\pm}} \), where \( H_{z_{\pm}} \) are constructed according to formulas (2), (4). We emphasize that existence of the operator (9.18) only at a single point \( z \) with \( \text{Im} z \neq 0 \) is not sufficient for the selfadjointness of the corresponding operator \( H = H_z \). This can easily be seen by considering a symmetric operator \( H \) having only one defect index equal to zero.

2. We shall discuss sufficient criteria for the existence of a bounded operator (9.18). The simplest of them is given by the following assertion.

**Proposition 4.** Suppose that

\[
\|(G_0 R_0(z_{\pm}) G^*)\| < 1, \quad \pm \text{Im} z_{\pm} \geq 0
\]

for some positive integer 1. Then there exists a selfadjoint operator \( H \) satisfying Definition 9.2.

**Proof.** By Theorem 3 it suffices to verify the existence and boundedness of the operator (9.18) for \( z = z_{\pm} \). To this end it is only necessary to use the identity

\[
(I - B_0) = (I - B_0(-1))^* (I - B_0 + \cdots + B_0(-1)), \quad B_0 = B_0(z_{\pm}),
\]

for the operator (1) and remark that under condition (5) the operator \( (I - B_0)^{-1} \) clearly exists and is bounded. \( \square \)

The next result is often very useful in the proof of the existence of the operator (9.18).

**Lemma 5.** Suppose \( \text{Im} z \neq 0 \). Then

\[
N(I + G_0 R_0(z) G^*) = \{0\}.
\]

**Proof.** Assume that

\[
f + G_0 R_0(z) G^* f = 0.
\]

We take the scalar product of (6) with the element \( G_0 R_0(z) G^* f \):

\[
(f, G_0 R_0(z) G^* f) + (G_0 R_0(z) G^* f, G_0 R_0(z) G^* f) = 0
\]

and we take the imaginary part of this equality. Then by (9.7)

\[
\text{Im}(f, G_0 R_0(z) G^* f) = - \text{Im} z \cdot \|R_0(z) G^* f\|^2 = 0.
\]

According to (6), the equality \( R_0(z) G^* f = 0 \) implies that also \( f = 0 \). \( \square \)

**Corollary 6.** Suppose that for some positive integer 1

\[
(G_0 R_0(z_{\pm}) G^*)^l \in \mathcal{S}_\infty, \quad \pm \text{Im} z_{\pm} > 0.
\]

Then there exists a selfadjoint operator \( H \) satisfying Definition 9.2.

**Proof.** By Lemma 5 and the Fredholm alternative (see Part 1 of §7) the operators (9.18) exist at the points \( z = z_{\pm} \). It remains to use Theorem 3. \( \square \)

Condition (7) automatically imposes stringent conditions on the essential spectrum \( \sigma_{\text{ess}}(H) \) of the operator \( H \) constructed in the proof of Theorem 3. Namely, we have
Theorem 7. Suppose conditions (9.6), (9.7) are satisfied, and the operator $(G_0 R_0(z)G^*)^{-1}$ is compact for $z \in \rho(H_0)$. Then $\sigma^{(\text{ess})}(H) = \sigma^{(\text{ess})}(H_0)$.

Proof. According to Lemma 5 the operator-valued function (1) satisfies the conditions of Theorem 8.2 on the set $\rho_0$ (or on its two connected components). Therefore, the corresponding set of "singular" points is discrete. Because of equality (9.19), from this it follows that the spectrum of $H$ in $\rho_0$ may consist solely of eigenvalues accumulating only at points of $\sigma(H_0)$. Moreover, by Theorem 8.2 the singularities of the operator (9.15) at singular points have the character of poles with finite-dimensional residues. This means that the eigenvalues of $H$ have finite multiplicity, and hence $\sigma^{(\text{ess})}(H) \subset \sigma^{(\text{ess})}(H_0)$.

We now change the roles of the operators $H_0$ and $H$. By equality (9.14) for the operator-valued function $\tilde{B}(z) = G_0 R_0(z)G^*$, holomorphic for $z \in \rho$, the bounded operator $(I - \tilde{B}(z))^{-1}$ exists for $\text{Im} z \neq 0$. Moreover, $\tilde{B}(z) = -B_0(z)(I - B_0(z))^{-1}$, and hence $\tilde{B}'(z) \in \mathcal{S}_\infty$. Therefore, by Theorem 8.2 the operator-valued function (9.16) is holomorphic in $\rho$ with the exception of a discrete set of poles with finite-dimensional residues. Thus, $\sigma^{(\text{ess})}(H_0) \subset \sigma^{(\text{ess})}(H)$. \(\square\)

It is not hard to see that under the conditions of this theorem the solutions of equation (6) and the eigenfunctions $\psi$ of the operator $H$, $H\psi = \lambda \psi$, are connected by the equalities

$$\psi = (GR_0(z))^*f, \quad f = -G_0 \psi.$$  

We further note that the equality $\sigma^{(\text{ess})}(H) = \sigma^{(\text{ess})}(H_0)$ is satisfied also in the cases when one of the operators $G_0 R_0(z)$ or $GR_0(z)$ is compact. Indeed, according to (9.15) in these cases the difference of the resolvents $R(z) - R_0(z)$ is also compact. So it suffices to appeal to the generalized theorem of H. Weyl (see the book [4]).

3. Under the conditions of Theorem 3 for $\theta_0 = 1$ the operator $H$ is selfadjoint (see Part 3 of 39) on $\mathcal{D}(H) = \mathcal{D}(H_0)$. On the other hand, it can be defined on this domain by the equality $Hf = H_0f + Vf$, $V = G^*G_0$. The selfadjointness of this operator follows from Theorem 3, but it can also be verified directly. Namely, for any $h \in \mathcal{K}$ the element

$$f = R_0(z)h - R_0(z)G^*[I + G_0 R_0(z)G^*]^{-1}G_0 R_0(z)h$$

belongs to $\mathcal{D}(H_0)$ and $(H_0 - z)f + G^*G_0f = h$. This implies that $R(H - zI) = \mathcal{K}$.

A sufficient (Rellich-Kato) condition for selfadjointness of the operator $H = H_0 + V$ on $\mathcal{D}(H_0)$ is the inequality

$$\|Vf\| \leq \gamma\|H_0f\| + C\|f\|, \quad \gamma < 1, \quad f \in \mathcal{D}(H_0).$$  

§10. Conditions for Selfadjointness

In this case the estimate (5) is satisfied for $G_0 = V$, $G = I$, $l = 1$, and $z_\pm = \pm iy$ where $y$ is sufficiently large. For $\theta_0 = 1/2$ the analogous condition

$$\|G_0 f\| \leq \gamma\|H_0^{1/2}f\| + C\|f\|, \quad \gamma < 1, \quad f \in \mathcal{D}(H_0^{1/2})$$

and the same sort of estimate for $\|Gf\|$ ensure the existence of a selfadjoint operator $H$ satisfying Definition 9.2. As before, the estimate (5) is satisfied for $l = 1$ and $z_\pm = \pm iy$ with a sufficiently large $y$.

If the conditions of Theorem 3 hold for $\theta_0 = 1/2$ and both operators $H_0$, $H$ are lower semibounded, then necessarily $\mathcal{D}(H^{1/2}) = \mathcal{D}(H_0^{1/2})$. However, we observe that for $H_0 \geq c > 0$ the existence of the operator (9.18) for some $z < 0$ still does not imply that the operator $H$ is also lower semibounded. Thus, for example, for $H = -H_0$ and $G = H_0^{1/2}$, $G_0 = -2H_0^{1/2}$ the operator (9.18) exists at the point $z = 0$ (and is equal to $-I$). At the same time the operator $H$ is not lower bounded if $H_0$ is not upper bounded. The case is also possible where for a semibounded $H_0$ the operator $H$ is not bounded on either side.

Theorem 3 makes it possible to give conditions for the selfadjointness of $H$ based on the sign of the perturbation.

Proposition 8. Suppose that $H_0 \geq 0$ and for $G_0 = G$ (9.6) is satisfied for $\theta_0 = \theta = 1/2$. Then there exists an operator $H \geq 0$ satisfying the conditions of Definition 9.2.

Proof. It suffices to note that $I + G_0 R_0(z)G^* \geq I$ for $z < 0$, and hence the operator (9.18) exists and is bounded. \(\square\)

4. In the (lower) semiboundedness case it is convenient to use Friederichs’ method (see the book [4]) for constructing a selfadjoint operator in terms of the corresponding quadratic form. We recall that to each positive definite quadratic form $h[u, u]$, closed on its domain of definition $\mathcal{D}[h]$ dense in $\mathcal{F}$, there corresponds a selfadjoint operator $H$ such that $\mathcal{D}(H) \subset \mathcal{D}[h]$, and

$$h[f, g] = (Hf, g), \quad f, g \in \mathcal{D}[h].$$  

Here the lower bounds of the operator $H$ and of the form $h$ coincide, and $\mathcal{D}[h] = \mathcal{D}(H^{1/2})$. It is important that the quadratic form $(\tilde{H} f_0, f')$ of any symmetric positive definite operator $\tilde{H}$ admits closure in the original Hilbert space $\mathcal{F}$. The selfadjoint operator $H$ corresponding to this closed, quadratic form is called the Friederichs’ completion of the operator $\tilde{H}$. The lower semiboundedness case reduces to the positive definite one by a shift by an operator $c f$, $c = \text{const}$. From what has been said we obtain

Theorem 9. Suppose $H_0$ is a nonnegative, selfadjoint operator, the real quadratic form $v[f, f]$ is defined on $\mathcal{D}(H_0^{1/2})$ and

$$-\gamma\|H_0^{1/2}f\|^2 + C_1\|f\|^2 \leq v[f, f] \leq C_2\|H_0^{1/2}f\|^2 + \|f\|^2, \quad f \in \mathcal{D}(H_0^{1/2}),$$

(11)
for some positive numbers $C_1$, $C_2$ and $\gamma < 1$. Then for sufficiently large $\delta$ the form

$$h_3(f, g) = \|H_0^{1/2} f\|^2 + \nu(f, g) + \delta\|f\|^2$$

is positive definite and closed on the domain $\mathscr{D}[h] = \mathscr{D}(H_0^{1/2})$, and hence to it there corresponds a selfadjoint operator taken to be $H + \delta I$.

**Proof.** Under condition (11) the metrics of $\|H_0^{1/2} f\|^2$ and (12) are equivalent. Since the first of them is closed, the second is also closed. \( \square \)

The sesquilinear form of the perturbation $V = G^* G_0 = v(f, g) = (G_0 f, G f)$. Under condition (9.6), with $\theta_0 = \theta = 1/2$, it is bounded on $\mathscr{D}(H_0^{1/2})$, while under condition (9.7) it is symmetric. Under the assumptions of Theorem 9 the operator $H$ satisfies Definition 9.2. Indeed, the equality $\mathscr{D}((H + \delta I)^{1/2}) = \mathscr{D}(H_0^{1/2})$ implies that $(H + \delta I)^{1/2} = H_0^{1/2} \Gamma$, where the operators $\Gamma_i$ are bounded. The representation (9.8) therefore also holds. Finally, equality (9.9) follows directly from the definition (10). Thus, Theorem 9 has the following

**Corollary 10.** Suppose $H_0 \geq 0$, conditions (9.6), (9.7) are satisfied for $\theta_0 = \theta = 1/2$, and for $f \in \mathscr{D}(H_0^{1/2})$

$$-(G_0 f, G f) \leq \gamma \|H_0^{1/2} f\|^2 + C\|f\|^2$$

for some $\gamma < 1$. Then there exists a lower semibounded operator $H$ satisfying Definition 9.2.

Of course, the estimate (13) is similar to condition (5) for $l = 1$ and $z_1 = z_2 < 0$, but it is formulated in somewhat different terms.

In contrast to Proposition 8 the Friedrichs' method makes it possible to consider nonnegative perturbations of the operator $H_0 \geq 0$ without any subordinate conditions. In the general case there is only the inclusion $\mathscr{D}(H^{1/2}) \subset \mathscr{D}(H_0^{1/2})$, whereas under the conditions of Proposition 8 these domains coincide.

**§11. Unitary operators. Perturbation theory**

All of the propositions of the selfadjoint theory carry over more or less automatically to the unitary case. Here we collect the results we need.

1. We first consider modifications of the results in §§3–5. The spectral measure $E_\nu(X)$ of the unitary operator $U$ is defined on Borel sets $X$ of the unit circle $T \subset \mathbb{C}$. As a rule, we denote points of $T$ by the letter $\mu$ (or $\nu$), $|\mu| = 1$; we denote by $|X|$ the Lebesgue measure of a set $X \subset T$, $|T| = 2\pi$. Absolutely continuous and singular elements relative to this measure are defined in a complete analogy to the selfadjoint case. This makes it possible to literally carry over the classification of the spectrum, presented in §3, to the unitary case.

By the spectral theorem the resolvent $R_0(\xi) = R_0^{-1}(\xi)$ of a unitary operator $U$ admits the representation

$$\langle R(\xi)f, g \rangle = \int_T (\mu - \xi)^{-1} d(E(\mu)f, g), \quad E = E_U.$$

This function is holomorphic for $|\xi| < 1$ and $|\xi| > 1$ and has radial (and angular) limits from within and without for a.e. $\mu \in T$.

We note that

$$R^*(\xi) = -\zeta U R(\zeta), \quad \zeta' = \zeta^{-1},$$

where the point $\zeta'$ is symmetric to $\zeta$ relative to $T$. We set

$$\delta(\mu, \eta) = \pi^{-1} |\eta| \zeta(\xi) R(\xi) R^*(\zeta) = \pi^{-1} |\eta| \zeta(\xi) R(\xi) R^*(\zeta), \quad \zeta = (1 - \eta)\mu.$$ (3)

Applying relation (2.13) to the function (1) and taking (2) into account, we find that

$$\lim_{\eta \to 0} \delta(\mu, \eta) f, g) = i \mu d(E(\mu)f, g)/d\mu, \quad \text{a.e. } \mu \in T.$$ (4)

This relation is, of course, analogous to (4.11).

The decomposition of $\mathcal{H}$ into a direct integral diagonalizing $U$ (cf. (5.1)) is constructed relative to a measure defined on $T$. In particular, on the absolutely continuous subspace $\mathcal{H}^{(a)}$ the role of (5.3) is played by the equality

$$E(X)f, g) = \int_{\delta \times \delta} \langle \hat{f}(\mu), \hat{g}(\mu) \rangle |i\mu|^{-1} d\mu.$$ (5)

Here at least one of the elements $f, g$ belongs to $\mathcal{H}^{(a)}$, and $\hat{f}$ and $\hat{g}$ are representatives of $P f$ and $P g$ ($P = P_U^{(a)}$) in the direct integral; $\delta$ is the core of the spectrum of the operator $U$, and $X$ is an arbitrary Borel set on $T$. From (5) it follows that for any $f, g \in \mathcal{H}$

$$i \mu d(E(\mu)f, g)/d\mu = \langle \hat{f}(\mu), \hat{g}(\mu) \rangle, \quad \text{a.e. } \mu \in \delta.$$ (6)

As in the selfadjoint case, a bounded operator $A$ commuting with $U$ acts as multiplication by $a(\mu)$ in the decomposition of $\mathcal{H}^{(a)}$ into a direct integral, and for all $f, g \in \mathcal{H}$

$$\langle a(\mu) \hat{f}(\mu), \hat{g}(\mu) \rangle = i \mu d(AE(\mu)f, g)/d\mu, \quad \text{a.e. } \mu \in \delta.$$ (7)

For the sesquilinear form of a kernel of the bounded operator $A$ (more precisely, of $PAP$), which is an integral operator in the direct decomposition,

$$\langle a(\mu, \nu) \hat{f}(\nu), \hat{g}(\mu) \rangle = (i\mu)(i\nu) \frac{\partial}{\partial \nu} \frac{\partial}{\partial \mu} (AE(\nu)f, E(\mu)g),$$ (8)

$$f, g \in \mathcal{H}, \quad \text{a.e. } (\mu, \nu) \in \delta \times \delta.$$
2. In contrast to the selfadjoint theory, where the perturbation is usually additive, in the theory of unitary operators the perturbation is naturally introduced multiplicatively. Actually, if the unperturbed operator $U_0$ and the perturbation $M$ are unitary the product

$$U = MU_0$$

is automatically unitary. The additive perturbation $V = U - U_0$ is connected with the multiplicative perturbation $M$ by the equality $V = (M - I)U_0$. It is clear that (9) is equivalent to the relation $U = U_0M^*$ where the operator $M^* = U_0^{-1}MU_0$ is unitarily equivalent to $M$ but is written to the right of $U_0$.

The assertions presented here are well known for the selfadjoint case (see, for example, the textbook [4]) and are easily extended to unitary operators.

The proof of Weyl’s theorem on the invariance of the essential spectrum in the unitary case is different than that in the selfadjoint case.

**Proposition 1.** Suppose the difference of the unitary operators $U$ and $U_0$ is compact. Then $\sigma^{(ess)}(U) = \sigma^{(ess)}(U_0)$.

**Corollary 2.** If the operator $U$ is unitary and $U - I \in \Theta_{\infty}$, then its spectrum consists of eigenvalues accumulating possibly only at the point 1. Eigenvalues distinct from 1 have finite multiplicity.

In the selfadjoint case the eigenvalues can be displaced by no more than the norm of the perturbation. We shall discuss the corresponding results for the unitary case. The point $1 \in \mathbb{T}$ now plays the role of the spectral point 0 $\in \mathbb{R}$, a shift along $\mathbb{R}$ is replaced by a rotation of $\mathbb{T}$, while the role of selfadjoint operators with small norm is played by unitary operators whose spectrum lies on a small arc with center at the point 1. We denote by $[\mu_1, \mu_2]$ and $[\bar{\mu}_1, \bar{\mu}_2]$, where $|\mu| = 1$, the corresponding open and closed arcs of $\mathbb{T}$ swept out as $\mu$ moves from $\mu_1$ to $\mu_2$ in the positive direction (counterclockwise).

**Proposition 3.** Suppose $\tau_0$ is an eigenvalue of the operator $U_0$ of multiplicity $k$ and the spectrum of $M$ lies on the arc $[\mu_1, \mu_2]$ of the unit circle. Then for the product $U = MU_0$

$$\dim E_U(\{\tau_0 \mu_1, \tau_0 \mu_2\}) \geq k.$$  

**Proof.** Suppose first that the arc $[\mu_1, \mu_2]$ is symmetric relative to the point 1, i.e., $\mu_1 = \bar{\mu}_2$, $\mu_2 = \bar{\mu}_1$, $|\mu_1| > 1$. In this case $|F| = |M - I| = |\mu - 1|$. Assume that (10) is violated. Then there exists an element $f \neq 0$ orthogonal to the subspace $E_U(\{\tau_0 \mu_1, \tau_0 \mu_2\})$ and such that $U_0f = \tau_0f$. From this equation it follows that

$$\|Uf - \tau_0f\| = \|F\| \leq |\mu - 1||f||,$$

At the same time by the spectral theorem

$$\|Uf - \tau_0f\|^2 = \int_{|\nu - \tau_0| < |\mu - 1|} |\nu - \tau_0|^2 d(E_U(\nu)f, f) > |\mu - 1|^2 \|f\|^2.$$  

(12)

on the orthogonal complement to $E_U(\{\tau_0 \mu_1, \tau_0 \mu_2\})$. Comparing (11) and (12), we find that $f = 0$. This proves (10) in the symmetric case.

In the general case we denote by $\mu_0$, $|\mu_0| = 1$, the center of the arc $[\mu_1, \mu_2]$ and represent $M$ in the form $M = \mu_0M'$. The spectrum of the operator $M'$ lies on the arc $[\bar{\mu}_0 \mu_1, \bar{\mu}_0 \mu_2]$ which is symmetric relative to the point 1. As already verified, for the operator $U' = M'U_0$

$$\dim E_{U'}(\{\tau_0 \bar{\mu}_0 \mu_1, \tau_0 \bar{\mu}_0 \mu_2\}) \geq k.$$

Since $U = \mu_0U'$, this is equivalent to (10).

We shall also need a version of this theorem pertaining to the case where $\tau_0$ is an arbitrary point of the spectrum of the operator $U_0$.

**Proposition 4.** Suppose $\tau_0 \in \sigma(U_0)$, $\sigma(M) \subset [\mu_1, \mu_2]$, and $U = MU_0$. Then $\sigma(U) \cap [\tau_0 \mu_1, \tau_0 \mu_2] \neq \emptyset$.

**Proof.** The proof is almost the same as for Proposition 3. The matter again reduces to the symmetric case $\mu_1 = \bar{\mu}_2$, $\mu_2 = \bar{\mu}_1$, $|\mu_1| > 1$. It is further noted that if $\sigma(U) \cap [\tau_0 \mu_1, \tau_0 \mu_2] = \emptyset$, then the closed set $\sigma(U)$ does not intersect some extended arc $[e^{-i\delta} \tau_0 \mu_1, e^{i\delta} \tau_0 \mu_2]$, where $\delta > 0$ is sufficiently small. Therefore, analogously to (12), for each $f \in F$ the quantity $\|Uf - \tau_0f\|$ is bounded below by $|\nu - \mu - 1||\|f\||$. On the other hand, since $\tau_0 \in \sigma(U_0)$, for any $\epsilon > 0$ there exists an element $f = f_{\epsilon}$, $\|f\| = 1$, such that $\|Uf_{\epsilon} - \tau_0f_{\epsilon}\| < \epsilon$. Analogously to (11) this implies that $\|Uf - \tau_0f\|$ is bounded above by $|\mu - 1| + \epsilon$. The lower and upper bounds together show that for sufficiently small $\epsilon$ we must have $f_{\epsilon} = 0$.

The next assertion gives conditions for the parts of the spectrum of the operator $U$ on the arc $[\tau_0 \mu_1, \tau_0 \mu_2]$ and off it to be separated from one another by a positive distance.

**Proposition 5.** Suppose that under the conditions of Proposition 3 on some arc $(\tau_1, \tau_2) \subset \mathbb{T}$ the eigenvalue $\tau_0$ is the only point of the spectrum of the operator $U_0$. Let

$$|(\mu_1, \mu_2)| < \inf_{j=1,2} |(\tau_j, \tau_0)|.$$  

(13)

Then the spectrum of the operator $U = MU_0$ has no points on the arcs $(\tau_1 \mu_2, \tau_0 \mu_1)$ and $(\tau_0 \mu_2, \tau_2 \mu_1)$.

**Proof.** Suppose, for example, that there exists a point $\tau \in \sigma(U)$ on the arc $(\tau_1 \mu_1, \tau_0 \mu_1)$. We use Proposition 4, changing the roles of the operators $U_0$ and $U$ there. Since $U_0 = M^*U$ and $\sigma(M^*) \subset [\bar{\mu}_2, \bar{\mu}_1]$, we have that

$$\sigma(U_0) \cap [\tau \bar{\mu}_2, \tau \bar{\mu}_1] \neq \emptyset.$$  

(14)

At the same time under the condition $|(\mu_1, \mu_2)| < |(\tau_1, \tau_0)|$ the arc $[\tau \bar{\mu}_2, \tau \bar{\mu}_1]$ is contained in $(\tau_1, \tau_0)$ for any $\tau \in (\tau_1 \mu_2, \tau_0 \mu_1)$. Therefore, it follows from (14) that $\sigma(U_0) \cap (\tau_1, \tau_0) \neq \emptyset$. This contradicts the fact that by hypothesis the arc $(\tau_1, \tau_0)$ consists of regular points of the operator $U_0$. □
We further note that under the conditions of Proposition 5 inequality (10) becomes an equality, i.e., there are exactly $k$ eigenvalues (counting multiplicity) of $U$ on the arc $[\tau_0 \mu_1, \tau_0 \mu_2]$. However, we shall not need this fact.

We apply Proposition 5 in the special case where one of the numbers $\mu_1$ or $\mu_2$ is equal to one. Such an operator $M$ plays the role of a sign-definite perturbation in the selfadjoint case.

**Corollary 6.** Suppose that $\tau_0$ is an isolated eigenvalue of the operator $U_0$, separated from the remaining spectrum of $U_0$ by a distance $d$, $d \in (0, 2)$. Suppose that $\sigma(M) \subset [\mu, \mu]$, or $\sigma(M) \subset [\mu, 1]$, where $\Im \mu \geq 0$ and $|\mu - 1| < d$. Then for some $\theta > 0$, the spectrum of the operator $U$ has no points on the arc $(\tau_0 e^{i\theta}, \tau_0)$ (respectively, on the arc $(\tau_0, \tau_0 e^{i\theta})$).

Together with Proposition 3, Corollary 6 shows that for “sign-definite” perturbations the isolated eigenvalues turn in the “direction” of the perturbation.

### 3. Here we present an exhaustive result on preservation of the total multiplicity of the spectrum in the perturbation theory of unitary operators. Theorem 8 and Corollary 9 generalize assertions of the preceding part, but are not explicitly used in this book. Along with the spectral theorem, we now need an elementary assertion of geometric character.

**Lemma 7.** Suppose that $f_n \in \mathcal{H}$ and $\|f_n\| = 1$, $n = 1, 2, 3$. We set $\cos \alpha_n = \text{Re}(f_m, f_n)$, $n \neq m \neq i$, $\alpha_n \in [0, \pi]$. Then $\cos \alpha_n \leq \cos(\alpha_m - \alpha_i)$, and hence $\alpha_i \leq \alpha_n + \alpha_m$ for any $n \neq m \neq i$.

**Proof.** We consider the nonnegative quadratic form

$$
\| \xi_1 f_1 + \xi_2 f_2 + \xi_3 f_3 \|^2 = \xi_1^2 + \xi_2^2 + \xi_3^2 + 2 \cos \alpha_1 \xi_1 \xi_2 + 2 \cos \alpha_2 \xi_2 \xi_3 + 2 \cos \alpha_3 \xi_3 \xi_1.
$$

The determinant

$$
1 - \cos^2 \alpha_1 - \cos^2 \alpha_2 - \cos^2 \alpha_3 + 2 \cos \alpha_1 \cos \alpha_2 \cos \alpha_3
$$

corresponding to it must also be nonnegative. We shall establish, for example, that

$$
\cos \alpha_2 \leq \cos(\alpha_3 - \alpha_1).
$$

To this end we write (15) in the form

$$
[\cos(\alpha_3 - \alpha_1) - \cos \alpha_2][\cos \alpha_2 - \cos(\alpha_3 + \alpha_1)].
$$

Suppose (16) is violated and, for example, $\alpha_3 \geq \alpha_1$. Then $\alpha_2 < \alpha_3 - \alpha_1$. Together with the first factor, the second factor in (17) must also be nonpositive. This means that either $\alpha_2 \geq \alpha_3 + \alpha_1$ or $\alpha_1 + \alpha_3 \geq 2\pi - \alpha_2$. Since $\alpha_2 < \alpha_3 - \alpha_1$, the first of these relations contradicts the condition $\alpha_2 \geq \pi$. Similarly, the second relation contradicts the condition $\alpha_3 \leq \pi$. □

### 8. Then for $U = MU_0$

$$
\dim E_V((\tau_1 \mu_2, \tau_2 \mu_1)) \leq \dim E_V((\tau_1, \tau_2)) \mathcal{H}.
$$

**Proof.** Analogously to the proof of Proposition 3, the matter reduces to consideration of the case of “centered” arcs where $\mu_1 = \mu_2 = \mu$, $\text{Im} \mu \geq 0$, $\tau_1 = \tau$, $\tau_2 = \tau$, $\text{Im} \tau \geq 0$. For this it is only necessary to go over to the new operators $M' = \mu_0 M$, $U' = \tau_0 U_0$, where $\mu_0$ and $\tau_0$ are the centers of the arcs $(\mu_1, \mu_2)$ and $(\tau_1, \tau_2)$ respectively.

If (19) is violated, then there exists an element $f$ such that $\|f\| = 1$ and $f \in E_V((\tau_0, \tau_0)) \mathcal{H}$, $f \perp E_V((\tau_0, \tau_0)) \mathcal{H}$.

We set $\beta = \text{arctan} \tau \in (0, \pi)$, $\gamma = \text{arctan} \mu \in [0, \pi]$. By the spectral theorem it follows from the first relation of (20) that

$$
\text{Re}(Uf, f) = \int_{\cos \nu \geq \cos(\beta - \gamma)} \cos \nu \, d(E_V(\nu)f, f) > \cos(\beta - \gamma),
$$

and from the second it follows that

$$
\text{Re}(U_0 f, f) = \int_{\cos \nu \leq \cos \beta} \cos \nu \, d(E_V(\nu)f, f) \leq \cos \beta.
$$

Moreover, by hypothesis $\sigma(M) \subset [\mu, \mu]$, $\text{Re}(Uf, U_0 f) = \text{Re}(MU_0 f, U_0 f) = \int_{\cos \nu \geq \cos \beta} \cos \nu \, d(E_M(\nu)U_0 f, U_0 f) \geq \cos \gamma \|U_0 f\|^2 = \cos \gamma.

We now set $f_1 = f$, $f_2 = U_0 f$, $f_2 = Uf$, and, as in Lemma 7, denote by $\alpha_n$, $\alpha_n \in [0, \pi]$ the corresponding angles $\cos \alpha_n = \text{Re}(f_m, f_i)$, $n \neq i \neq l$. Then

$$
\cos \alpha_2 > \cos(\beta - \gamma), \quad \cos \alpha_3 \leq \cos \beta, \quad \cos \alpha_1 \geq \cos \gamma.
$$

This means that $\alpha_2 < \beta - \gamma$, $\alpha_3 \geq \beta$, $\alpha_1 \geq \gamma$.

The inequality $\alpha_1 + \alpha_2 \leq \alpha_3$ following from this contradicts Lemma 7. □

**Corollary 9.** Suppose $\sigma(M) \subset [\mu, \mu]$ and $\|f_1\| + \|f_2\| < 2\pi$.

Then for $U = MU_0$

$$
\dim E_V((\tau_1 \mu_2, \tau_2 \mu_1)) \geq \dim E_V((\tau_1, \tau_2)) \mathcal{H}.
$$

**Proof.** Since $U_0 = M^* U$ and $\sigma(M^*) \subset [\mu_1, \mu_1]$, it suffices to apply Theorem 8, changing the roles of the operators $U_0$ and $U$. The role of the arc $(\tau_1, \tau_2)$ is hereby played by the arc $(\tau_1 \mu_1, \tau_2 \mu_2)$. □
CHAPTER 2

Basic Concepts of Scattering Theory

As is evident from the title, the material of this chapter is the basis for all subsequent considerations. All assertions here can be proved in an elementary way. The term "theorem" refers only to the importance of an assertion but not the substance of its proof. The last two sections differ in style from the exposition of the preceding sections. In them stationary representations for the basic objects of scattering theory are derived at a heuristic level, justification of these representations is postponed to Chapter 5.

From the start we allow that the "free" Hamiltonian $H_0$ and "full" Hamiltonians $H$ act in different Hilbert spaces denoted respectively by $\mathcal{H}_0$ and $\mathcal{H}$. Construction of scattering theory requires that in this case there is given a bounded operator $\mathcal{J}: \mathcal{H}_0 \rightarrow \mathcal{H}$ called the identification operator. Results pertaining to the important special case $\mathcal{H}_0 = \mathcal{H}$, $\mathcal{J} = I$ are formulated separately as corollaries of more general assertions.

§1. The wave operators (WO)

1. The wave operator is the basic object of scattering theory. It is constructed with respect to a pair of selfadjoint operators $H_0$ and $H$ acting in Hilbert spaces $\mathcal{H}_0$ and $\mathcal{H}$ and a bounded operator $\mathcal{J}: \mathcal{H}_0 \rightarrow \mathcal{H}$.

In the notation of various objects (for example, the unitary group $U_H(t) = \exp(-iHt)$, the projection $P_H^{(a)}$ onto the absolutely continuous subspace $\mathcal{H}_H^{(a)}$ (see §1.3) of the operator $H$, etc.) the dependence on $H$ is, as a rule, omitted. The similar objects related to the "unperturbed" operator $H_0$ are equipped with the index "0"; for example, $U_0(t) = U_{H_0}(t)$. We set $P_0 = P_0^{(a)}$, $P = P^{(a)}$ where no confusion is possible.

The following definition was motivated in detail in the Introduction.

**Definition 1.** The wave operator (WO) for a pair of selfadjoint operators $H_0$, $H$ and an identification $\mathcal{J}$ is the operator

$$W_\pm(H, H_0; \mathcal{J}) = \lim_{t \rightarrow \infty} U(-t)\mathcal{J}U_0(t)P_0,$$

provided that the corresponding strong limit exists.

We recall that everywhere in the book any relation containing the signs "$\pm$" is understood as two independent equalities.
In order to emphasize that in (1) the strong limit is considered such a WO is sometimes called strong. For \( \mathcal{K}_0 = \mathcal{H} \), \( \mathcal{I} = \mathcal{I} \) the WO (1) is denoted by \( W_\pm(H, H_0) \). Where no confusion arises the dependence of the WO \( W_\pm(H, H_0; \mathcal{I}) \) on \( H, H_0 \), and \( \mathcal{I} \) may be omitted.

It is clear that even for \( \mathcal{K}_0 = \mathcal{H} \) the inclusion in (1) of an operator \( \mathcal{I} \neq \mathcal{I} \) contributes new features to the WO. However, for a unitary mapping \( \mathcal{I}: \mathcal{K}_0 \to \mathcal{H} \) the general case reduces to \( \mathcal{K}_0 = \mathcal{H} \) and \( \mathcal{I} = \mathcal{I} \).

Indeed, for \( \mathcal{K}_0 = \mathcal{H} \).

\[
W_\pm(H, H_0; \mathcal{I}) = W_\pm(\mathcal{H}, \tilde{H}_0),
\]

and the WO on both sides exist simultaneously.

We set

\[
W(t) = W_\pm(t) = U(-t)3U_0(t).
\]

In view of the estimate \( \|W_\pm(t)\| \leq \|\mathcal{I}\| \) to prove the existence of the WO \( W_\pm(H, H_0; \mathcal{I}) \) it suffices to verify the convergence of \( W_\pm(t)f \) as \( t \to \pm \infty \) on a set of elements \( f \) dense in \( \mathcal{K}_0^{(a)} \).

By Lemma 1.4.1 for a compact \( \mathcal{I} \) both limits (1) clearly exist and are equal to zero. However, for an arbitrary bounded \( \mathcal{I} \) the WO (1) exist, of course, not for any pair of selfadjoint operators \( H_0, H \). For example, in the case \( \mathcal{I} = \mathcal{I} \) the existence of the limit (1) automatically imposes strong restrictions on the pair \( H_0, H \).

2. Assuming the existence of the limit (1), we study the properties of the WO \( W_\pm(H, H_0; \mathcal{I}) \) and its relation to the operators \( H_0, H \).

It is clear that the kernel \( N(W_\pm) \) of the WO \( W_\pm \) always contains the singular subspaces of \( H_0 \), i.e.,

\[
\mathcal{K}_0^{(a)} := \mathcal{K}_0^{(a)} \cap \mathcal{K}_0^{(a)} \subset N(W_\pm).
\]

Moreover, we have

**Lemma 2.** The WO \( W_\pm \) is bounded, and

\[
\|W_\pm f\| \leq \|\mathcal{I}\| \|P_0 f\|.
\]

For the isometricity of \( W_\pm \) on \( \mathcal{K}_0^{(a)} \) it is necessary and sufficient that for any \( f \in \mathcal{K}_0^{(a)} \)

\[
\lim_{t \to \pm \infty} \|3U_0(t)f\| = \|f\|.
\]

**Proof.** With respect to strong convergence the norm is a continuous functional. Therefore, by the unitarity of \( U(t) \)

\[
\|W_\pm f\| = \lim_{t \to \pm \infty} \|W(t)P_0 f\| = \lim_{t \to \pm \infty} \|3U_0(t)P_0 f\|.
\]

From this it follows that for \( f \in \mathcal{K}_0^{(a)} \) the equalities \( \|W_\pm f\| = \|f\| \) and (5) are equivalent. To prove (4) it remains to use the unitarity of \( U_0(t) \).

For isometricity of \( W_\pm \) on \( \mathcal{K}_0^{(a)} \) it suffices that (5) be satisfied only for some sequence \( t_n \to \pm \infty \). Moreover, this sequence may depend on the element on which the limit (5) is computed. We note that the isometricity of \( W_\pm \) on \( \mathcal{K}_0^{(a)} \) is equivalent to the equality

\[
W_\pm(H, H_0; \mathcal{I})W_\pm(H, H_0; \mathcal{I}) = P_0.
\]

We now give a convenient sufficient criterion for the validity of (5).

**Proposition 3.** The WO \( W_\pm \) is isometric on \( \mathcal{K}_0^{(a)} \) if

\[
s\lim_{t \to \pm \infty} (3\mathcal{I} - I)U_0(t)P_0 = 0.
\]

It also suffices that the operator \( 3\mathcal{I} - I \) be compact or even that the relation

\[
(3\mathcal{I} - I)E_0(\Lambda) \in \mathcal{S}_\infty
\]

be satisfied for any bounded interval \( \Lambda \).

**Proof.** Since

\[
\|3U_0(t)f\| = \|3\mathcal{I}U_0(t)f\| + \|f\|,
\]

relation (5) follows from (8) and the unitarity of \( U_0(t) \). Further, condition (8) can be verified on a dense set of finite elements of the form \( f = E_0(\Lambda)f \).

In other words, in place of (8) it suffices to establish that for an arbitrary bounded \( \Lambda \)

\[
s\lim_{t \to \pm \infty} (3\mathcal{I} - I)E_0(\Lambda)U_0(t)P_0 = 0.
\]

This relation follows directly from (9) by Lemma 1.4.1. □

For \( \|\mathcal{I}\| \leq 1 \) condition (8) is also necessary for the WO (1) to be an isometry on \( \mathcal{K}_0^{(a)} \). Indeed, according to (5), for \( f \in \mathcal{K}_0^{(a)} \) and \( t \to \pm \infty \)

\[
((3\mathcal{I} - I)U_0(t)f, U_0(t)f) = \|U_0(t)f\|^2 \to 0.
\]

From this we obtain (8).

Condition (5) for the isometricity of the WO on \( \mathcal{K}_0^{(a)} \) (and also the sufficient conditions (8) and (9)) is satisfied if the operator \( \mathcal{I} \) itself is isometric. In particular, in the case \( \mathcal{K}_0 = \mathcal{H} \), \( \mathcal{I} = \mathcal{I} \), the WO \( W_\pm(H, H_0) \) is automatically isometric on \( \mathcal{K}_0^{(a)} \) when it exists. It is important to emphasize that even in this case (and for absolutely continuous \( H_0 \)) the WO \( W_\pm(H, H_0) \), generally speaking, is only isometric but not unitary. The following property of the WO is called its intertwining property.

**Theorem 4.** For any bounded Borel function \( \varphi \)

\[
\varphi(H)W_\pm(H, H_0; \mathcal{I}) = W_\pm(H, H_0; \mathcal{I})\varphi(H_0).
\]

In particular, for any Borel set \( \Lambda \subset \mathbb{R} \)

\[
E(\Lambda)W_\pm(H, H_0; \mathcal{I}) = W_\pm(H, H_0; \mathcal{I})E_0(\Lambda).
\]
Proof. By definition (1)
\[ U(s)W_\pm = \lim_{r \to \pm \infty} U(s - t)U_0(t)P_0 \]
\[ = \lim_{r \to \pm \infty} U(-\tau)U_0(\tau + s)P_0 = W_\pm U_0(s), \]
which immediately proves (10) for \( \varphi(\lambda) = e^{-i\lambda t} \). In view of the representation (1.4.2) this means that for any \( f_0 \in \mathcal{H}_0 \), \( f \in \mathcal{H} \) and \( s \in \mathbb{R} \)
\[ \int_{-\infty}^{\infty} e^{-i\lambda t} d(E(\lambda)f_0, f) = \int_{-\infty}^{\infty} e^{-i\lambda t} d(E_0(\lambda)f_0, W_\pm^* f). \]
Applying the uniqueness theorem for Fourier-Stieltjes integrals to this equality (see, for example, Volume 5 of the course [21]), we establish (11). Finally, according to (1.4.1) relation (10) for an arbitrary function \( \varphi \) can be derived from (11) by integration. □

By (10) the operator \( W_\pm \) maps \( D(H_0) \) into \( D(H) \), and for any \( f \in D(H_0) \)
\[ HW_\pm f = W_\pm H_0 f. \]
Passing to adjoint operators in (10), we further obtain the equality
\[ W_\pm^* \varphi(H) = \varphi(H_0)W_\pm^*. \tag{12} \]
The next two assertions are not specific to scattering theory. Their derivation requires only the intertwining property (10), or (11), and also the equality
\[ \overline{\mathcal{R}(W_\pm)} \oplus N(W_\pm^*) = \mathcal{H}, \tag{13} \]
which follows from (1.6.1).

Lemma 5. The subspaces
\[ \mathcal{H}_0 \ominus N(W_\pm) =: \mathcal{H}_0^{(\pm)} \quad \text{and} \quad \overline{\mathcal{R}(W_\pm)} =: \mathcal{H}^{(\pm)} \tag{14} \]
reduce the operators \( H_0 \) and \( H \), respectively.

Proof. The subspace \( N(W_\pm) \) is invariant with respect to the spectral measure \( E_0(\cdot) \), since for \( W_\pm f = 0 \) the equality \( W_\pm E_0(\Lambda)f = 0 \) is a consequence of (11). Thus, \( \mathcal{H}_0^{(\pm)} \) reduces \( H_0 \). By equality (13), the second assertion is equivalent to the invariance of \( N(W_\pm^*) \) with respect to the family of projections \( E(\cdot) \). Therefore, it follows directly from (12). □

We denote by \( H_0^{(\pm)} \) and \( H^{(\pm)} \) the restrictions of \( H_0 \) and \( H \) to the subspaces \( \mathcal{H}_0^{(\pm)} \) and \( \mathcal{H}^{(\pm)} \), i.e.,
\[ H_0^{(\pm)} = H_0|_{\mathcal{H}_0^{(\pm)}}, \quad H^{(\pm)} = H|_{\mathcal{H}^{(\pm)}}. \]
According to (3), \( \mathcal{H}_0^{(\pm)} \subseteq \mathcal{H}_0^{(a)} \), i.e., the operator \( H_0^{(\pm)} \) is absolutely continuous. By (11) for an isometric WO \( W_\pm \) on \( \mathcal{H}_0^{(a)} \) the operators \( H_0^{(\pm)} \) and \( H^{(\pm)} \) are unitarily equivalent. Under the additional condition
\[ N(W_\pm) = \mathcal{H}_0^{(a)} \tag{15} \]
this leads to the unitary equivalence of \( H^{(\pm)} \) to the entire absolutely continuous part \( H_0^{(\pm)} \) of the operator \( H_0 \). Actually, similar results hold also without the assumption of partial isometry of \( W_\pm \). For the proof we consider the polar representation (see Part 1 of §1.6) of the WO (1):
\[ W_\pm = F_\pm W_\pm^*, \quad |W_\pm| = (W_\pm W_\pm^*)^{1/2}, \tag{16} \]
where \( F_\pm = \text{sgn} W_\pm ; \mathcal{H}_0^{(a)} \to \mathcal{H} \) is a partial isometry mapping \( \mathcal{H}_0^{(a)} \) unitarily onto \( \mathcal{H}^{(\pm)} \) and vanishing on \( N(W_\pm) \).

Theorem 6. We have the equality
\[ \varphi(H)F_\pm = F_\pm \varphi(H_0), \tag{17} \]
and, in particular, the operators \( H_0^{(\pm)} \) and \( H^{(\pm)} \) are unitarily equivalent.

Proof. According to (10) and (12),
\[ W_\pm^* W_\pm \varphi(H_0) = W_\pm^* \varphi(H)W_\pm = \varphi(H_0)W_\pm^* W_\pm. \]
From this it follows that any function of the nonnegative operator \( W_\pm^* W_\pm \) commutes with \( \varphi(H_0) \). In particular,
\[ |W_\pm| \varphi(H_0) = \varphi(H_0)|W_\pm|. \]
Comparing this equality with (10) and (16), we find that
\[ \varphi(H)F_\pm |W_\pm| = F_\pm \varphi(H_0)|W_\pm|. \]
Therefore, relation (17) is satisfied at least on the elements of \( \mathcal{R}(W_\pm) \). It remains to show that on the elements of \( \mathcal{H}_0 \ominus \mathcal{R}(W_\pm) = N(W_\pm^*) \) the operators on both sides of (17) are equal to zero. The left-hand side is zero since, by construction, \( F_\pm \) vanishes on \( N(W_\pm) \). As for the right-hand side it is necessary to recall additionally that, according to Lemma 5, the subspace \( N(W_\pm) \) reduces the operator \( H_0 \).

By Theorem 6 the operator \( H^{(\pm)} \) is absolutely continuous, i.e.,
\[ \mathcal{R}(W_\pm) \subset \mathcal{H}^{(a)}, \quad \text{or} \quad PW_\pm = W_\pm. \tag{18} \]
The operator \( H^{(\pm)} \) is unitarily equivalent to \( H_0^{(\pm)} \) if (15) is satisfied.

We note that under condition (15) the operator \( W_\pm \), considered as a mapping of \( \mathcal{H}_0^{(a)} \) onto \( \mathcal{H}^{(a)} \), has trivial kernel. If, in addition, its range is closed, i.e., \( \mathcal{R}(W_\pm) = \mathcal{R}(W_\pm^*) \), then the mapping \( W_\pm : \mathcal{H}_0^{(a)} \to \mathcal{H}(W_\pm) \) is boundedly invertible. In other words, there exists a bounded operator \( T : \mathcal{H} \to \mathcal{H}_0^{(a)} \) for which \( T W_\pm = P_0 \). Such operators \( W_\pm \) are called left invertible on \( \mathcal{H}_0^{(a)} \).

Example. Isometries on \( \mathcal{H}_0^{(a)} \) are left invertible on this subspace.

4. The next assertion is called the chain rule or multiplication theorem of WO. In the proof of it we use the following fact: if families of operators \( A_t \) and \( B_t \) converge strongly to \( A \) and \( B \) as \( t \to \infty \), their product \( A_t B_t \) converges strongly to \( AB \).
2. Basic Concepts of Scattering Theory

**Theorem 7.** If the WO $W_{+}(H_{1}, H_{0}; \mathcal{J}_{0})$ and $W_{-}(H_{1}, H_{0}; \mathcal{J}_{1})$ exist, then for $\mathcal{J} = \mathcal{J}_{0}$, the WO $W_{\pm}(H_{1}, H_{0}; \mathcal{J})$ also exists and

$$W_{\pm}(H_{1}, H_{0}; \mathcal{J}) = W_{\pm}(H_{1}, H_{0}; \mathcal{J}_{0}) W_{\mp}(H_{1}, H_{0}; \mathcal{J}_{1}).$$

(19)

**Proof.** For $P_{i} = P^{(a)}_{H_{i}}$ and $U_{i}(t) = U_{H_{i}}(t)$ we write the equality

$$U(-t)3U_{0}(t)_{P_{0}} = U(-t)3U_{t}(t)_{P_{1}} + (I - P_{1})U(-t)3U_{0}(t)_{P_{0}}.$$  

(20)

According to (18), for any $f_{0} \in \mathcal{H}^{(a)}_{H_{0}}$,

$$||(I - P_{1})U_{(-t)}3U_{t}(0)_{P_{0}}|| \rightarrow 0$$

as $t \rightarrow \pm \infty$. Therefore, on the right-hand side of (20) the term corresponding to the term $I - P_{1}$ in square brackets tends strongly to zero. The remaining term is the product of two factors converging to $W_{\mp}(H_{1}, H_{0}; \mathcal{J})$ and $W_{\mp}(H_{1}, H_{0}; \mathcal{J}_{1})$, respectively. Thus the right- and therefore also the left-hand side, of (20) converges to the right-hand side of (19). □

From equalities (18) and (19) it follows that isomority of the WO $W_{\pm}(H_{1}, H_{0}; \mathcal{J}_{0})$ and $W_{\pm}(H_{1}, H_{0}; \mathcal{J}_{1})$ on $\mathcal{H}^{(a)}_{H_{0}}$ and $\mathcal{H}^{(a)}_{H_{1}}$, respectively, induces isomority of the WO $W_{\pm}(H_{1}, H_{0}; \mathcal{J}, \mathcal{J}_{0})$ on $\mathcal{H}^{(a)}_{H_{0}}$. Similarly, left invertibility of these operators on $\mathcal{H}^{(a)}_{H_{0}}$ and $\mathcal{H}^{(a)}_{H_{1}}$ induces left invertibility of $W_{\pm}(H_{1}, H_{0}; \mathcal{J}, \mathcal{J}_{0})$ on $\mathcal{H}^{(a)}_{H_{0}}$.

5. For a given pair $H_{0}, H$ different identifications $\mathcal{J}$ can lead to the same WO $W_{\pm}(H_{1}, H_{0}; \mathcal{J})$. In this respect we introduce the following important

**Definition 8.** Two identifications $\mathcal{J}_{1}$ and $\mathcal{J}_{2}$ are called equivalent (relative to the operator $H_{0}$ and the sign "±") if

$$s\lim_{t \rightarrow \pm \infty} (\mathcal{J}_{1} - \mathcal{J}_{2}) U_{0}(t)_{P_{0}} = 0.$$  

(21)

The relation (21) obviously implies that for any operator $H$ the WO $W_{\pm}(H_{1}, H_{0}; \mathcal{J})$ and $W_{\pm}(H_{1}, H_{0}; \mathcal{J}_{2})$ exist simultaneously and coincide.

It is clear that if the WO $W_{\pm}(H_{1}, H_{0}; \mathcal{J})$ exists and $\mathcal{J}$ is equivalent to an isometry $\mathcal{J}$, then the WO $W_{\pm}(H_{1}, H_{0}; \mathcal{J})$ is an isometry on $\mathcal{H}^{(a)}_{H_{0}}$.

We present a simple sufficiency test for the equivalence of $\mathcal{J}_{1}$ and $\mathcal{J}_{2}$.

The next assertion follows directly from Lemma 1.4.1 (cf. the proof of Proposition 3).

**Proposition 9.** Two identifications $\mathcal{J}_{1}$ and $\mathcal{J}_{2}$ are $H_{0}$-equivalent (for both signs of the time) if for any bounded interval $\Lambda$

$$\{\mathcal{J}_{1} - \mathcal{J}_{2}\} E_{0}(\Lambda) \in \mathcal{S}_{\infty}.$$  

(22)

In particular, if the difference $\mathcal{J}_{1} - \mathcal{J}_{2}$ is compact, then these identifications are equivalent relative to any selfadjoint operator $H_{0}$.

By relation (21) all bounded operators $\mathcal{J} : \mathcal{H}^{(a)}_{0} \rightarrow \mathcal{H}^{(a)}_{0}$ are partitioned into classes (equivalence classes) of identifications $H_{0}$-equivalent to one another.

§2. Modifications of the Concept of WO

Suppose the WO $W_{\pm}(H_{1}, H_{0}; \mathcal{J})$ exists; then

$$s\lim_{t \rightarrow \pm \infty} [W_{\pm}(H_{1}, H_{0}; \mathcal{J}) - \mathcal{J}] U_{0}(t)_{P_{0}} = 0.$$  

(23)

by equality (10) for $\varphi(\lambda) = \exp(-i\lambda t)$. If the WO $W_{\pm}(H_{1}, H_{0}; \mathcal{J})$ is itself considered as a new identification, then it is $H_{0}$-equivalent to the original identification $\mathcal{J}$. Thus, for given $H_{0}$ and $H$ a judicious choice of $\mathcal{J}$ (i.e., one for which the WO $W_{\pm}(H_{1}, H_{0}; \mathcal{J})$ exists) reduces to finding the WO up to $H_{0}$-equivalence. This choice may be different for different signs "±", i.e., $\mathcal{J} = \mathcal{J}_{\pm}$. From this viewpoint the WO $W_{\pm}(H_{1}, H_{0})$ corresponds to the case where the identity operator is already a sufficiently good approximation to the WO.

Introduction of an identification $\mathcal{J}$ may also be useful when considering the WO $W_{\pm}(H_{1}, H_{0})$. Namely, for a suitably chosen $\mathcal{J}$ the existence of the WO $W_{\pm}(H_{1}, H_{0}; \mathcal{J})$ may be verified more simply than the existence of $W_{\pm}(H_{1}, H_{0})$. If that operator $\mathcal{J}$ turns out to be equivalent to the identity operator, then the WO $W_{\pm}(H_{1}, H_{0})$ also exists. This technique is often used in application of the Crook criterion for the existence of WO (see §5). In particular, it is applied in the example considered in §3.1.

§2. Various modifications of the concept of WO

(weak, local, Abelian WO, etc.)

1. Along with the strong WO, defined by relation (1.1), it is often convenient to also consider so-called weak WO

$$\tilde{W}_{\pm}(H_{1}, H_{0}; \mathcal{J}) = \lim_{t \rightarrow \pm \infty} P(U(-t)3U_{0}(t)_{P_{0}}.$$  

(1)

If the WO (1.1) exists, then, of course, the weak WO $\tilde{W}_{\pm} := \tilde{W}_{\pm}(H_{1}, H_{0}; \mathcal{J})$ also exists and $W_{\pm} = \tilde{W}_{\pm}$. However, the WO (1) has a "better chance" of existing. Just as the strong WO, the operator $\tilde{W}_{\pm}$ is bounded and satisfies the estimate (1.4). Since, however, the norm is not a continuous functional with respect to weak convergence, relation (1.6) fails for $\tilde{W}_{\pm}$. Thus, even for $\mathcal{J} = I$ the WO $\tilde{W}_{\pm}(H_{1}, H_{0})$ may not be an isometry on $\mathcal{H}^{(a)}_{H_{0}}$. The intertwining property (1.10) is preserved for weak WO. From this, in particular, it follows that $R(\tilde{W}_{\pm}) \subset \mathcal{H}^{(a)}_{H_{0}}$. In contrast, the chain rule (1.19) is violated, since the product of two weakly convergent families of operators is, generally speaking, not even weakly convergent. On the other hand, weak convergence of operators withstands conjugation, in contrast to strong convergence. This means that

$$\tilde{W}_{\pm}(H_{0}, H; \mathcal{J}) = \tilde{W}_{\pm}(H_{1}, H_{0}; \mathcal{J})$$

(2)

necessarily exists alongside with WO $\tilde{W}_{\pm}(H_{1}, H_{0}; \mathcal{J})$.

The connection of the existence of the strong (1.1) and weak (1) WO can be established in terms of auxiliary WO defined only in terms of $H_{0}$ and $\mathcal{J}$. These wave operators correspond to the "triple" $H_{0}, H_{0}, \mathcal{J}$.3.
THEOREM 1. The existence of the WO $W_\pm(H, H_0; J)$ is equivalent to the existence of the weak WO $\tilde{W}_\pm(H, H_0; J)$ and the equality
\[ \tilde{W}_\pm(H, H_0; J) \tilde{W}_\pm(H, H_0; J) = \tilde{W}_\pm(H, H_0; J') \tilde{W}_\pm(H, H_0; J') \] (3).

PROOF. Suppose the weak WO $\tilde{W}_\pm(H, H_0; J) =: \tilde{W}_\pm$ exists. Consider
\[ \| U(-t) \mathcal{U}_0(t) P_g f - \tilde{W}_\pm f \| = (\mathcal{P}_0 U_\tau(t) \tilde{W}_\pm f, f) - 2 \text{Re}(U(-t) \mathcal{U}_0(t) P_g f, \tilde{W}_\pm f) + \| \tilde{W}_\pm f \|^2. \] (4)

Since $\tilde{W}_\pm = P \tilde{W}_\pm$, the second term on the right-hand side of (4) tends to $-2\| \tilde{W}_\pm f \|^2$. Therefore, convergence to zero of the left-hand side of (4) is equivalent to the existence of a limit of the first term on the right (i.e., the existence of the WO $\tilde{W}_\pm(H, H_0; J')$) and equality to zero of the limit of the right-hand side. The latter equality has the form
\[ (\tilde{W}_\pm(H, H_0; J') f, f) = \| \tilde{W}_\pm(H, H_0; J) f \|^2 \]
and is hence equivalent to (3). □

COROLLARY 2. Under condition (1.8) (in particular in the case $\mathcal{P}_0 = \mathcal{P}_1$, $J = I$) the existence of the strong WO $W_\pm(H, H_0; J)$ is equivalent to the existence of the weak WO $\tilde{W}_\pm(H, H_0; J)$ and its isometricity on $\mathcal{F}_0^{(a)}$.

PROOF. Under condition (1.8) $W_\pm(H_0, H_0; J') = \mathcal{P}_0$. □

REMARK 3. The existence and isometricity on $\mathcal{F}_0^{(a)}$ of the weak WO $\tilde{W}_\pm(H, H_0; J)$ are sufficient for the existence of the strong WO $W_\pm(H, H_0; J)$ under the additional condition $\|J\| \leq 1$.

PROOF. For isometric $\tilde{W}_\pm$ the right-hand side of (4) is equal to
\[ (J^* J - I) U_0(t) P_g f, U_0(t) P_g f + o(1), \quad t \to \pm \infty. \]
In the case $\|J\| \leq 1$ the first term here is not positive. Since the left-hand side of (4) is not negative, it must tend to zero as $t \to \pm \infty$. □

2. In the case where the WO (1.1) does not exist there may nevertheless exist so-called local WO whose definition is connected with Borel sets $\Lambda$ on the real (spectral) axis. We set $E_0^{(a)}(\Lambda) = E(\Lambda) P_0^{(a)}$.

The local WO for a triple $H_0$, $H$, $J$ and a Borel set $\Lambda \subset \mathbb{R}$ is the operator $W_\pm(H, H_0; \Lambda) = \lim_{t \to \pm \infty} U(-t) \mathcal{U}_0(t) E_0^{(a)}(\Lambda).$ (5)

if the strong limit in (5) exists. The concept of a local weak WO is connected with a pair of Borel sets $\Lambda_0, \Lambda \subset \mathbb{R}$. By definition,
\[ W_\pm(H, H_0; \Lambda, \Lambda_0) = \lim_{t \to \pm \infty} E_0^{(a)}(\Lambda) U(-t) \mathcal{U}_0(t) E_0^{(a)}(\Lambda_0), \] (6)
and $W_\pm(H, H_0; \Lambda, \Lambda_0) =: \tilde{W}_\pm(H, H_0; \Lambda, \Lambda_0)$.

It is clear that if the limit (5) exists and $\Lambda_1 \subset \Lambda$ then the WO
\[ W_\pm(H, H_0; J, \Lambda_1) = W_\pm(H, H_0; J, \Lambda_1) E_0(\Lambda_1) \]
also exists. Moreover, if $|\Lambda_1 \cap \Lambda_2| = 0$, then
\[ W_\pm(H, H_0; J, \Lambda_1 \cap \Lambda_2) = W_\pm(H, H_0; J, \Lambda_1) + W_\pm(H, H_0; J, \Lambda_2), \]
where the left- and right-hand sides exist simultaneously.

Formally speaking, definitions (5) and (6) reduce respectively to (1.1) and (1.2). For this it is necessary to go over to new identifications $J^* = E_0^{(a)}(\Lambda)$ (for strong WO) and $J^* = E(\Lambda) E_0(\Lambda)$ (for weak WO), i.e.,
\[ W_\pm(H, H_0; J, \Lambda) = W_\pm(H, H_0; J, \Lambda') E_0(\Lambda). \]
(7)

Thus, the results of §1 and of Part 1 of this section, established for "global" WO, carry over automatically to the local WO (5), (6). In particular, local WO, even weak ones, possess the intertwining property (1.10). From this it follows that
\[ \tilde{W}_\pm(H, H_0; J, \Lambda, \Lambda_0) = \tilde{W}_\pm(H, H_0; J, \Lambda) \Lambda_0, \] (8)

whence the existence of the WO on the left implies the existence of the WO on the right. Theorem 1 is also preserved, assuming that all the WO correspond to the same set $\Lambda$. Specific features of locality are, however, lost under the reduction (7). Thus, the operator (5) can be isometric only on the subspace $E_0^{(a)}(\Lambda) \mathcal{F}_0^{(a)}$, and its range is always contained in the subspace $E_0^{(a)}(\Lambda) \mathcal{F}_0^{(a)}$. A sufficient condition for isometricity of $W_\pm(H, H_0; J, \Lambda)$ on $E_0^{(a)}(\Lambda) \mathcal{F}_0^{(a)}$ is altogether analogous to (1.8):
\[ \lim_{t \to \pm \infty} \| J^* J - I \| U_0(t) E_0^{(a)}(\Lambda) = 0. \]

Local WO are also useful when considering "global" WO. Namely, in place of (1.1) it is often possible to establish the existence of the local WO (5) for a rather broad class of sets $\Lambda$. Since the existence of the limit (1.1) can be verified only on a dense set, the WO $W_\pm(H, H_0; J)$ extends provided $W_\pm(H, H_0; J, \Lambda_0)$ exist and the union of the sets $\Lambda_0$ exhausts $\mathbb{R}$ up to a set of Lebesgue measure zero. It suffices, by the way, that the union of $\Lambda_0$ coincide with the core of the spectrum (see Definition 1.3.8) of the operator $H_0$. Similarly, the local WO $W_\pm(H, H_0; J, \Lambda)$ exists if $W_\pm(H, H_0; J, \Lambda_0)$ exist and $\bigcup \Lambda_0 = \Lambda \pmod{0}$.

3. Sometimes the limits in (1.1) and (1.2) are understood in an extended sense—the Abel sense. Namely, suppose that a nonnegative function $\omega(t)$, $t \geq 0$, is normalized by the condition
\[ \int_0^\infty \omega(t) dt = 1, \]
and that the Fourier transform \( \tilde{\psi}(\lambda) \) of the function \( \psi(t) = e^{-\lambda \omega}e^{-\lambda t} \) has no real zeros. We introduce the averaging kernel \( \omega(t) = \omega \epsilon(t), \quad \epsilon > 0 \).

An Abelian WO is an operator \( \mathfrak{A}_\pm = \mathfrak{A}_\pm(H, H_0; \mathfrak{J}) \) such that for any \( f \in \mathfrak{A}_0^\prime \)

\[
\lim_{\epsilon \to 0} \int_0^\infty \omega(t)\left\| W_\pm(\pm t)P_0f - \mathfrak{A}_\pm f \right\|^p dt = 0, \quad p > 0, \quad (9)
\]

where we have used the notation (1.2). Since \( \| W_\pm(t) \| \leq 1 \), relations (9) for different functions \( \omega(t) \) and different \( p \) are equivalent to one another. This is a consequence of a general Tauberian theorem (see the book [5]).

Thus, under the conditions formulated the Abelian WO do not depend on the choice of \( \omega(t) \) and \( p \). In view of this equivalence, the dependence on \( \omega \) and \( p \) is reflected in the notation, and are chosen in (9) according to convenience. Strictly speaking, to the Abelian limit corresponds the function \( \omega(t) = \exp(-t) \), and to the Cesàro limit—the indicator of the interval \((0, 1)\). Setting \( \epsilon^{-1} = T \), it is possible to write the Cesàro definition of the WO in the standard form

\[
\lim_{T \to \infty} T^{-1} \int_0^T \omega(t)\left\| W_\pm(\pm t)P_0f - \mathfrak{A}_\pm f \right\|^p dt = 0, \quad p > 0. \quad (10)
\]

It is not hard to see that all the assertions of §1 hold for the WO \( \mathfrak{A}_\pm \). If the WO \( W_\pm \) exists, then (9) is also satisfied for \( \mathfrak{A}_\pm = W_\pm \). Of course, the converse assertion is not true.

Weak Abelian WO \( \mathfrak{A}_\pm = \mathfrak{A}_\pm(H, H_0; \mathfrak{J}) \) are defined by the equality

\[
\mathfrak{A}_\pm = \omega \lim_{t \to 0} \int_0^\infty \omega(t)PW_\pm(\pm t)P dt. \quad (11)
\]

The remarks just made pertaining to \( \mathfrak{A}_\pm \) remain valid for the WO \( \mathfrak{A}_\pm \).

Namely, the existence of the limits (11) for different \( \omega_\epsilon \) are equivalent and follows from (1), whereby in this case \( \mathfrak{A}_\pm = \overline{W}_\pm \). Of course, the converse assertion is not true. The WO \( \mathfrak{A}_\pm \) retain all the properties of the WO \( \overline{W}_\pm \) listed in Part I, Theorem 1 on the connection of strong and weak WO carries over to the Abelian WO; the role of the operator on the right in (3) is, of course, played by the weak Abelian WO \( \mathfrak{A}_{\pm}(H_0, H_0; \mathfrak{J}) \).

Local Abelian WO \( \mathfrak{A}_{\pm}(H, H_0; \mathfrak{J}, L, \Lambda) \) and \( \mathfrak{A}_{\pm}(H, H_0; \mathfrak{J}, L, \Lambda, \Lambda_0) \) are also considered. These operators can be defined by equalities of the form (9) and (11) where \( \mathfrak{J} \) is replaced, respectively, by \( \mathfrak{J}E_0(\Lambda) \) and \( E(\Lambda)\mathfrak{J}E_0(\Lambda_0) \).

In general, scattering theory for the usual and Abelian WO can be constructed in strict parallel. We therefore omit, as a rule, formulations in "Abelian terms."

4. For a pair of unitary operators \( U_0, U \) the WO are introduced in analogy to (1.1). The definition of the WO involves the discrete groups generated by them:

\[
W_\pm(U, U_0; \mathfrak{J}) = \mathfrak{J}\lim_{n \to \infty, n \in \mathbb{Z}} U^n U_0^{-n}P_0, \quad P_0 = P_{U_0}^a. \quad (12)
\]

The results of §1 on WO in the selfadjoint case are preserved for the WO (12). Equally the notions of weak, local, Abelian WO, etc. extend to the WO for a pair of unitary operators, and the interrelations between these objects are preserved.

The construction of scattering theory for unitary operators is essentially similar to the selfadjoint case. For this reason we usually do not make special remarks regarding the unitary case.

We recall that the Cayley transform of a selfadjoint operator \( \mathfrak{H}_a = (H - \imath)(H + \imath)^{-1} \).

Sometimes in place of the WO \( W_\pm(H, H_0; \mathfrak{J}) \) it is convenient to consider the WO (12) for the Cayley transforms \( U_0 = \mathfrak{H}_a, \quad U = \mathfrak{H}_a \).

\[
\mathfrak{H}_a \mathfrak{J} - 2\mathfrak{H}_a = 2[(R_\mu(-\imath)\mathfrak{J} - 2R_\mu(-\imath)), \quad (13)
\]

the conditions for the existence of such WO, formulated in terms of the difference \( U_0 - \mathfrak{H}_a \), coincide with the conditions for the existence of WO, formulated in terms of the difference of resolvents in the selfadjoint case. Conditions of this kind are discussed, for example, in the trace class theory (see Part 3 of §6.5).

\section{3. completeness of the WO}

As explained in the Introduction, scattering theory is concerned with the investigation of the asymptotics of \( U(t)f \) as \( t \to \pm \infty \) (in terms of the function \( U_0(t)f_0 \)) and the problem of unitary equivalence of the operators \( H_0 \) and \( H \). In the case \( \mathfrak{J} = \mathfrak{J} \) both these problems have a definitive solution if the WO \( W_\pm(H, H_0) \) exist and are complete, i.e., their ranges coincide with the absolutely continuous subspace \( \mathfrak{J}^{(a)} \) of the operator \( H \). The completeness of \( W_\pm(H, H_0) \) is hereby equivalent to the existence of the WO \( W_\pm(H, H_0) \). In practical verification one usually establishes the existence of both WO \( W_\pm(H, H_0) \) and \( W_\pm(H_0, H) \).

We shall here discuss the corresponding concepts and assertions directly for the case of a pair of spaces.

1. The existence of the WO (1.1) will be assumed in this subsection. The asymptotics of \( U(t)f \) can be computed for any element \( f \) from the range \( R(W_\pm) \) of the WO \( W_\pm = W_\pm(H, H_0; \mathfrak{J}) \). Namely, if \( f \in R(W_\pm) \), then for some \( f_0 \in \mathfrak{J}^{(a)} \) of the operator \( H_0 \), under the additional condition (1.15) this choice is unique. In the case

\[
R(W_\pm) = \mathfrak{J}^{(a)} \quad (2)
\]
the asymptotics (1) holds for any \( f \in \mathscr{F}(a) \). In connection with this we adopt the

**Definition 1.** The operator \( W_\pm = W_\pm(H, H_0; \mathcal{J}) \) is called complete if equalities (1.15) and (2) hold.

As concerns the second problem, the restrictions \( H_0^{(a)} \) and \( H^{(a)} \) of the operators \( H_0 \) and \( H \) to the subspaces \( \mathscr{A}_0^{(a)} \) and \( \mathscr{A}^{(a)} \) (see (1.14)) contained in \( \mathscr{A}_0^{(a)} \) and \( \mathscr{A}^{(a)} \), respectively, are unitarily equivalent.

Under the conditions \( \mathscr{A}_0^{(a)} = \mathscr{A}_0^{(a)} \) and \( \mathscr{A}^{(a)} = \mathscr{A}^{(a)} \), the operators \( H_0^{(a)} \) and \( H^{(a)} \) themselves are unitarily equivalent. From Theorem 1.6 we thus obtain

**Proposition 2.** Under the conditions (1.15) and

\[
\overline{\mathcal{R}(W_\pm)} = \mathscr{A}^{(a)}
\]  

the absolutely continuous parts of the operators \( H_0 \) and \( H \) are unitarily equivalent.

Thus, completeness of \( W_\pm \) guarantees unitary equivalence of the operators \( H_0^{(a)} \) and \( H^{(a)} \). For a WO, left-invertible on \( \mathscr{A}_0^{(a)} \), the definition of completeness reduces to the single equality (3) (or (2)). For a WO \( W_\pm \) isometric on \( \mathscr{A}_0^{(a)} \) the product \( W_\pm W_\mp^* \) is the orthogonal projection onto \( \mathcal{R}(W_\pm) \). Therefore, equality (2) is equivalent to

\[
W_\pm(H, H_0; \mathcal{J})W_\pm^*(H, H_0; \mathcal{J}) = P.
\]  

If the WO \( W_\pm \) is an isometry on \( \mathscr{A}_0^{(a)} \) and is complete, then it itself realizes a unitary equivalence of \( H_0^{(a)} \) and \( H^{(a)} \).

By means of Theorem 1.7 and relation (1.18) we can verify in an elementary way the following

**Theorem 3.** Suppose the WO \( W_\pm(H_0, H_0; \mathcal{J}_0) \) and \( W_\pm(H, H; \mathcal{J}) \) exist and are complete. Then the WO \( W_\pm(H, H_0; \mathcal{J}_0; \mathcal{J}) \) also exists and is complete.

2. We shall now discuss the connection of the completeness of the “direct” WO \( W_\pm(H, H_0; \mathcal{J}) \) with the existence of the “inverse” WO \( W_\pm(H_0, H; \mathcal{J}) \) for a suitable \( \mathcal{J}_0: \mathcal{F} \to \mathcal{F}_0 \). It is hereby required that in a certain sense \( \mathcal{J}_0 \) \( \mathcal{J} \) be a “quasi-inverse” to \( \mathcal{J} \). The next definition gives the most flexible relation between the identifications \( \mathcal{J} \) and \( \mathcal{J}_0 \).

**Definition 4.** A bounded operator \( \mathcal{J}_0: \mathcal{F} \to \mathcal{F}_0 \) is called an \( H_0 \)-asymptotically (left) inverse to \( \mathcal{J} \) if

\[
\mathbf{s-lim}_{t \to \infty} (\mathcal{J}_0 \mathcal{J} - I) U_0(t) P_0 = 0.
\]  

Of course, relation (5) means that the product \( \mathcal{J}_0 \mathcal{J} \) is \( H_0 \)-equivalent to the identity operator \( I \) (for the sign “\( \pm \)”) in the sense of Definition 1.8.

**Lemma 5.** If \( \mathcal{J} \) has an \( H_0 \)-asymptotically inverse \( \mathcal{J}_0 \), then

\[
\| P_0 f \| \leq \| \mathcal{J}_0 \| \| W_\pm f \|, \quad W_\pm = W_\pm(H, H_0; \mathcal{J}),
\]  

so that the WO \( W_\pm \) is left invertible on \( \mathcal{A}_0^{(a)} \).

**Proof.** We use the estimate

\[
\| \mathcal{J}_0 U_0(t) P_0 f \| \leq \| \mathcal{J}_0 \| \| U_0(t) P_0 f \|.
\]

As \( t \to \pm \infty \) the left-hand side here tends to \( \| P_0 f \| \) by (5), while the right-hand side tends to \( \| \mathcal{J}_0 \| \| W_\pm f \| \) by (1.6). The left invertibility of \( W_\pm \) is a consequence of (6). \( \square \)

**Theorem 6.** Suppose the WO \( W_\pm(H, H_0; \mathcal{J}) \) exists and for some \( \mathcal{J}_0: \mathcal{F} \to \mathcal{F}_0 \) \( \mathcal{J} \)-condition (5) is satisfied. Then the completeness of \( W_\pm(H, H_0; \mathcal{J}) \) is equivalent to the existence of \( W_\pm(H_0, H; \mathcal{J}_0) \) and the condition

\[
\mathbf{s-lim}_{t \to \pm \infty} (\mathcal{J}_0 \mathcal{J} - I) U(t) P_0 = 0.
\]

In this case the WO \( W_\pm(H_0, H; \mathcal{J}_0) \) is also complete, and \( W_\pm(H, H_0; \mathcal{J}) \) and \( W_\pm(H_0, H; \mathcal{J}_0) \) are mutually invertible mappings of \( \mathcal{A}_0^{(a)} \) onto \( \mathcal{A}^{(a)} \), i.e.,

\[
W_\pm(H_0, H; \mathcal{J}_0) W_\pm(H, H_0; \mathcal{J}) = P_0, \quad W_\pm(H, H_0; \mathcal{J}) W_\pm(H_0, H; \mathcal{J}_0) = P_0.
\]

**Proof.** Suppose the WO \( W_\pm(H_0, H; \mathcal{J}_0) = W_\pm \) exists and (7) is satisfied. By Theorem 1.7 the left-hand sides in (8) are equal to \( W_\pm(H_0, H_0; \mathcal{J}, \mathcal{J}_0) \) and \( W_\pm(H, H; \mathcal{J}, \mathcal{J}_0) \), respectively. From conditions (5) and (7) it follows that the first of these operators is equal to \( P_0 \) and the second is equal to \( P_0 \). According to Lemma 5, to prove the completeness of \( W_\pm = W_\pm(H, H_0; \mathcal{J}) \) it suffices to demonstrate equality (3) or, according to (1.13) and (1.18), the inclusion \( N(W_\pm) \subset \mathcal{A}_0^{(a)} \). By passing to adjoint operators in the second relation (8), we find that \( W_\pm^* W_\pm^* = P_0 \). Therefore, the equality \( W_\pm^* = 0 \) implies that \( P f = 0 \), i.e., \( f \in \mathcal{A}_0^{(a)} \). Finally, the completeness of \( W_\pm \) follows from the assertion proved applied to the triple \( H, H_0, \mathcal{J}_0 \).

Suppose that (2) is satisfied. Then for any \( f \in \mathcal{A}_0^{(a)} \) relation (1) holds. This ensures that

\[
\mathbf{s-lim}_{t \to \infty} \| U_0(t) \mathcal{J}_0 U(t) f - U_0(t) \mathcal{J}_0 U_0(t) P_0 f \| = 0.
\]

Thus, according to (5), \( U_0(t) \mathcal{J}_0 U(t) f \) has a limit (equal to \( P_0 f_0 \)) as \( t \to \pm \infty \). This implies the existence of the WO \( W_\pm(H_0, H; \mathcal{J}_0) \). To prove (7) we note that by (5)

\[
\mathbf{s-lim}_{t \to \infty} (\mathcal{J}_0 \mathcal{J} - I) U_0(t) P_0 = 0.
\]

In view of (1) under the condition (2) this equality is equivalent to (7). \( \square \)
Remark 7. Suppose the WO $\mathcal{W}_\pm(H, H_0; 3)$ exists and is complete. Then by Theorem 6 WO $\mathcal{W}_\pm(H_0, H; \mathcal{J})$ also exists and is complete for any identification $\mathcal{J}$, satisfying (5). It turns out that all such identifications are $H$-equivalent, and hence for different identifications $\mathcal{J}_1, H_0$-asymptotically inverse to $\mathcal{J}$, the WO $\mathcal{W}_\pm(H_0, H; \mathcal{J}_1)$ actually coincide.

Proof. If $\mathcal{J}_1$ and $\mathcal{J}_1'$ satisfy (5), then the operators $\mathcal{J}_1, \mathcal{J}$ and $\mathcal{J}_1', \mathcal{J}$ are $H_0$-equivalent. The $H$-equivalence of $\mathcal{J}_1$ and $\mathcal{J}_1'$ now follows from (1) by the completeness of $\mathcal{W}_\pm(H, H_0; 3)$. □

We shall discuss various corollaries of Theorem 6. If $\mathcal{J}$ has a bounded inverse operator $\mathcal{J}^{-1}$, then, of course, it is asymptotically inverse to $\mathcal{J}$ relative to any operator $H_0$.

Corollary 8. Suppose $\mathcal{J}$ is boundedly invertible and that the WO $\mathcal{W}_\pm(H, H_0; 3)$ exists. Then the completeness of $\mathcal{W}_\pm(H, H_0; 3)$ is equivalent to the existence of the "inverse" $\mathcal{W}^-\pm(H_0, H; \mathcal{J}^{-1})$.

Proof. For $\mathcal{J}_1 = \mathcal{J}^{-1}$ relations (5) and (7) are trivially satisfied. □

In applications the most important case is that of a single space, when $\mathcal{J} = \mathcal{J}_1 = I$.

Corollary 9. Suppose that $\mathcal{J}_0 = \mathcal{J} = I$, and the WO $\mathcal{W}_\pm(H, H_0)$ exists. Then the completeness of $\mathcal{W}_\pm(H, H_0)$ is equivalent to the existence of the WO $\mathcal{W}_\pm(H_0, H)$.

We now consider the case where it is possible to set $\mathcal{J}_1 = \mathcal{J}^*$. Then condition (5) coincides with (1.8).

Corollary 10. Suppose the WO $\mathcal{W}_\pm(H, H_0; 3)$ exists and condition (1.8) is satisfied. Then the completeness of $\mathcal{W}_\pm(H, H_0; 3)$ is equivalent to the existence of the WO $\mathcal{W}_\pm(H_0, H; \mathcal{J}^*)$ and the condition

\[ \lim_{t \to \pm \infty} (\mathcal{J}^* - I) U(t) P = 0. \]  

(9)

By Proposition 1.3, under the conditions of Corollary 10 the WO $\mathcal{W}_\pm(H, H_0; 3)$ and $\mathcal{W}_\pm(H_0, H; \mathcal{J}^*)$ are isometries on the subspaces $\mathcal{J}_0(a)$ and $\mathcal{J}_0(a)$, respectively. We further note the following relation between these operators.

Proposition 11. Suppose both WO $\mathcal{W}_\pm = \mathcal{W}_\pm(H, H_0; 3)$ and $\mathcal{W}_\pm = \mathcal{W}_\pm(H_0, H; \mathcal{J}^*)$ exist. Then

(1) They are adjoint to one another, i.e.,

$\mathcal{W}_\pm H_0, H; \mathcal{J}^* = \mathcal{W}_\pm H, H_0; 3)$.  

(10)

(2) Equalities (1.5) and (3) are equivalent to the equalities $R(\mathcal{W}_\pm) = \mathcal{J}_0(a)$ and $N(\mathcal{W}_\pm) = \mathcal{J}_0(a)$, respectively. If the WO $\mathcal{W}_\pm$ is left invertible on $\mathcal{J}_0(a)$,

then its completeness is equivalent to the equality $N(\mathcal{W}_\pm) = \mathcal{J}_0(a)$. In particular, if the WO $\mathcal{W}_\pm$ and $\mathcal{W}_\pm$ are left invertible on $\mathcal{J}_0(a)$ and $\mathcal{J}_0(a)$, then they are both complete.

(3) If the WO $\mathcal{W}_\pm$ is isometric on $\mathcal{J}_0(a)$, then its completeness is equivalent to the isometry of $\mathcal{W}_\pm$ on $\mathcal{J}_0(a)$. Under these conditions $\mathcal{W}_\pm$ and $\mathcal{W}_\pm$ are mutually inverse mappings.

Proof. Equality (10) is a direct consequence of equality (22) for the corresponding weak WO. By (10) the equivalence of equalities (3) and $N(\mathcal{W}_\pm) = \mathcal{J}_0(a)$ follows from relation (1.13), while the equivalence of (1.15) and $R(\mathcal{W}_\pm) = \mathcal{J}_0(a)$ follows from the analogous relation for $\mathcal{W}_\pm$. In particular, for a left invertible $\mathcal{W}_\pm$ the equality $N(\mathcal{W}_\pm) = \mathcal{J}_0(a)$ is equivalent to (2). For WO $\mathcal{W}_\pm$ isometric on $\mathcal{J}_0(a)$ the definition of completeness can be written in the form (4). With (10) taken into account equality (4) means that $\mathcal{W}_\pm$ is an isometry of $\mathcal{J}_0(a)$. Finally, replacing $\mathcal{W}_\pm H, H_0; 3)$ by $\mathcal{W}_\pm(H_0, H; \mathcal{J}^*)$ in (1.7) and (4), we establish that $\mathcal{W}_\pm$ and $\mathcal{W}_\pm$ are inverse to one another. □

Corollary 12. If $\mathcal{J}$ is boundedly invertible and the WO $\mathcal{W}_\pm(H, H_0; 3)$ and $\mathcal{W}_\pm(H_0, H; \mathcal{J}^*)$ exist, then they are complete.

Proof. By Lemma 5 these WO are left invertible on $\mathcal{J}_0(a)$ and $\mathcal{J}_0(a)$, respectively. Therefore, it is possible to use the second assertion of Proposition 11. □

3. The local WO $\mathcal{W}_\pm = \mathcal{W}_\pm(H, H_0; 3, \Lambda)$ annihilates on the subspace $\mathcal{J}_0(a) \oplus \mathcal{J}_0(a)(\Lambda) \mathcal{J}_0(a)$, and its range is necessarily contained in $E(\Lambda) \mathcal{J}_0(a)$. Therefore, a local WO is naturally called complete if in place of (1.15), (3) the equalities

\[ N(\mathcal{W}_\pm) = \mathcal{J}_0(a) \oplus E(\Lambda) \mathcal{J}_0(a), \quad R(\mathcal{W}_\pm) = E(\Lambda) \mathcal{J}_0(a) \mathcal{J}_0(a) \]  

(11)

hold. In this case the operators $E(\Lambda) H_0$ and $E(\Lambda) H$ are unitarily equivalent. The last assertion remains in force in the case where in the second equality of (11) $R(\mathcal{W}_\pm)$ is replaced by $R(\mathcal{W}_\pm)$. Theorem 6 and Proposition 11 carry over to local WO. Here the roles of the "inverse" and "adjoint" are played by the local WO $\mathcal{W}_\pm(H_0, H; \mathcal{J}_0; H_0, H; \mathcal{J}^*, H_0, H; \mathcal{J}^*)$, while relations (5) and (7) are considered only on the subspaces $E(\Lambda) \mathcal{J}_0(a)$ and $E(\Lambda) \mathcal{J}_0(a)$. The concept of completeness and all the assertions of this section carry over without change to Abelian WO. In the same way all the concepts and assertions introduced remain meaningful for WO (including Abelian WO) of the form (2.12) in the unitary case.
§4. The scattering operator and matrix. An elementary example

1. In the context of abstract scattering theory the scattering operator \( S = S(H, H_0; J) \), also called the \( S \)-operator, is defined in terms of the WO as follows

\[
S = W^*_+ W_-, \quad W_\pm = W_\pm(H, H_0; J).
\]

(1)

By (1.3) the operator \( S \) vanishes on \( \mathcal{H}_0^{(a)} \), and its range is in \( \mathcal{H}_0^{(a)} \). We therefore consider the scattering operator only on the absolutely continuous subspace \( \mathcal{H}_0^{(a)} \).

We shall list some basic properties of the scattering operator. It follows from Lemma 1.2 that the operator \( S \) is bounded and \( ||S|| \leq ||\lambda||^2 \), while from Theorem 1.4 it follows that \( S \) commutes with \( H_0 \). Conditions for the unitarity of the \( S \)-operator are given by the following assertion.

**Proposition 1.** Suppose the WO \( W_{\pm} \) is isometries on \( \mathcal{H}_0^{(a)} \). Then isometry of \( S \) is equivalent to the inclusion \( R(W_-) \subset R(W_+) \), while unitarity of \( S \) is equivalent to the equality \( R(W_-) = R(W_+) \).

**Proof.** For an isometric operator \( W_- \) the equality \( ||Sf|| = ||f|| \) is equivalent to \( ||W_+^* g|| = ||g|| \) for any \( g \in R(W_-) \). In turn, for an isometric \( W_+ \) the last equality is satisfied if and only if \( g \in R(W_+) \).

Further, the relation \( R(S) = \mathcal{H}_0^{(a)} \) implies that any \( g \in \mathcal{H}_0^{(a)} \) can be represented in the form \( g = W_+^* W_- f \) or \( W_+ g = P_+ W_- f \) where \( P_+ = W_+^* W_-^* \) is the orthogonal projection onto \( R(W_-) \). Therefore, \( R(S) = \mathcal{H}_0^{(a)} \) if and only if for any \( h_+ \in R(W_+) \) there exists \( h_- \in R(W_-) \) such that \( h_+ = P_+ h_- \). Since \( R(W_-) \subset R(W_+) \), the equality \( R(W_-) = R(W_+) \) is necessary and sufficient for this. \( \square \)

**Corollary 2.** The scattering operator is unitary if the WO \( W_{\pm} \) is isometric on \( \mathcal{H}_0^{(a)} \) and are complete.

2. We now consider the decomposition of the subspace \( \mathcal{H}_0^{(a)} \) into a direct integral of the form (1.5.6)

\[
\mathcal{H}_0^{(a)} = \bigoplus_{\lambda \in \hat{\sigma}_0} h_0(\lambda) d\lambda =: \mathcal{H}_0^{(a)}(\lambda), \quad \hat{\sigma}_0 = \hat{\sigma}(H_0),
\]

(2)

which diagonalizes the operator \( H_0^{(a)} \). Here \( \hat{\sigma}_0 \) is a core of the spectrum (see Definition 1.3.8) of the operator \( H_0 ; \) i.e., any set of full \( E_0 \)-measure (and hence of full \( E_0^{(a)} \)-measure) having "minimum" Lebesgue measure. Since the scattering operator (1) commutes with \( H_0^{(a)} \), under the correspondence (2) it goes over into multiplication by the operator-valued function \( S(\lambda) : h_0(\lambda) \to h_0(\lambda) \). The operator \( S(\lambda) = S(\lambda; H, H_0; J) \) is called the scattering matrix.

In this definition the scattering matrix is determined only for a.e. \( \lambda \) of \( \hat{\sigma}_0 \) and up to unitary equivalence in \( h_0(\lambda) \). Of course, all the properties of \( S(\lambda) \) discussed below are also satisfied only for a.e. \( \lambda \in \hat{\sigma}_0 \), which is not mentioned each time.

Various assertions regarding the \( S \)-operator can be reformulated in terms of the scattering matrix. For example, \( ||S(\lambda)|| \leq ||\lambda||^2 \) because of the analogous inequality for the \( S \)-operator. From Proposition 1 it follows that for WO \( W_\pm \) isometric on \( \mathcal{H}_0^{(a)} \), the isometry of \( S(\lambda) \) for a.e. \( \lambda \in \hat{\sigma}_0 \) is equivalent to the inclusion \( R(W_-) \subset R(W_+) \), while unitarity of \( S(\lambda) \) is equivalent to the equality \( R(W_-) = R(W_+) \).

In analogy to (1) by means of the local WO (2.5), it is possible to define the scattering operator \( S \) on the subspace \( E_0^{(a)}(\lambda) \mathcal{H}_0^{(a)} \). If the WO (1.1) exist this "local" scattering operator coincides with the restriction to \( E_0^{(a)}(\lambda) \mathcal{H}_0^{(a)} \) of the operator (1). Existence of the WO (2.5) is sufficient for defining the scattering matrix \( S(\lambda) \) for a.e. \( \lambda \in \Lambda \cap \hat{\sigma}_0 \). As before, isometry on \( E_0^{(a)}(\Lambda) \mathcal{H}_0^{(a)} \) of the local WO \( W_{\pm}(H, H_0; \Lambda) \) and their completeness ensure the unitarity of \( S(\lambda) \).

3. We shall now consider a very simple example in which scattering theory can be constructed explicitly. Let \( \mathcal{H} = L_2(\mathbb{R}) \), and let \( H_0 = -i d/dx \), where \( D(H_0) \) consists of all absolutely continuous functions which together with their derivative belong to \( L_2(\mathbb{R}) \). The operator \( H_0 \) is unitarily equivalent to the operator \( A \) of multiplication by the independent variable in the "dual" space \( L_2^*(\mathbb{R}) \). Namely, \( H_0 = \Phi^* A \Phi \), where \( \Phi \) is the Fourier transform (see Remark 7 of §1.6). Thus, the spectrum of \( H_0 \) is absolutely continuous, covers the entire spectral axis, and is simple. Moreover, \( (U_0(t)f)(x) = f(x - t) \).

Suppose now that \( q \) is any locally summable real function. We denote by \( \mathcal{D}(H) \) the set of absolutely continuous functions which together with the function \( -if' + qf \) belong to the space \( L_2(\mathbb{R}) \). The operator \( H \) is defined by the equality

\[
Hf = -if' + qf
\]

on the domain \( \mathcal{D}(H) \). We set

\[
(Yf)(x) = \exp \left( -i \int_0^x q(y) dy \right) f(x).
\]

(4)

It is clear that the operator \( Y \) is unitary in \( \mathcal{H} \), takes \( \mathcal{D}(H_0) \) onto \( \mathcal{D}(H) \), and

\[
HYf = YH_0f, \quad f \in \mathcal{D}(H_0).
\]

(5)

From this, in particular, it follows that the operator \( H \) is self-adjoint on \( \mathcal{D}(H) \).

In the construction for the pair \( H_0, H \) of the WO we assume that \( J = I \).

Using equalities (3)–(5), we find that the operator (1.2) acts according to the formula

\[
(W(t)f)(x) = (YY_0(-t))^{-1}U_0(t)f(x)
\]

\[
= \exp \left( -i \int_x^{x+t} q(y) dy \right) f(x).
\]
Thus, we have

**Proposition 3.** Suppose the integral of $q$ over the semiaxis $\mathbb{R}_+$ converges. Then the WO $W_\pm(H, H_0)$ exists and acts as multiplication by the function

$$
\exp \left( i \int_0^\infty q(y) \, dy \right).
$$

In this example the WO turn out to be unitary and are, in particular, complete in the sense of Definition 3.1. The scattering operator $S$ acts as multiplication by the constant function

$$
\exp \left( -i \int_{-\infty}^\infty q(x) \, dx \right). \tag{6}
$$

The scattering matrix $S(\lambda)$ does not depend on $\lambda$ and also reduces to multiplication by the number (6).

§5. Existence of the WO. The Cook criterion

1. In the proof of the existence of WO it is systematically considered that it suffices to verify convergence in (1.1) for a set of elements dense in $\mathcal{H}$ or at least in $\mathfrak{H}_0^{(a)}$. The following simple, but effective sufficient condition, for the existence of the WO is usually called Cook's criterion.

**Theorem 1.** Suppose the operator $\mathfrak{A}$ takes the domain $\mathfrak{D}(H_0)$ of the operator $H_0$ into $\mathfrak{D}(H)$. Suppose that for some set $D_0 \subset \mathfrak{D}(H_0) \cap \mathfrak{H}_0^{(a)}$ dense in $\mathfrak{H}_0^{(a)}$ for any $f \in D_0$

$$
\int_{-\infty}^{\infty} \| (H - \mathfrak{H}_0)f \| \, dt < \infty. \tag{1}
$$

Then the WO $W_\pm(H, H_0; \mathfrak{A})$ exists.

**Proof.** Since $\mathfrak{D}_0 = \mathfrak{H}_0^{(a)}$, it suffices to verify strong convergence of the vector-valued function $U(t) = U(t)f$ for all $f \in \mathbb{C}$. From the conditions $f \in \mathfrak{D}(H_0)$ and $\mathfrak{D}(H_0) \rightarrow \mathfrak{D}(H)$ it follows that the vector-valued function $w(t)$ is differentiable, and

$$
w'(t) = iU(t)(H - \mathfrak{H}_0)f.
$$

By condition (1) the function $w'(t)$ is absolutely integrable on $\mathbb{R}_+$, and hence $w(t)$ has a limit as $t \rightarrow \pm \infty$.

**Remark 2.** The condition $\mathfrak{D}(H_0) \rightarrow \mathfrak{D}(H)$ can be replaced by the assumption $\mathfrak{H}_0(t)f \in \mathfrak{D}(H)$ for all $t \in \mathbb{R}$ and $f \in D_0$.

Theorem 1 requires a rather explicit representation for the group $U_0(t)$. Such a representation can often be obtained for the free dynamics which makes it possible to verify the existence of the direct WO under rather general assumptions regarding the perturbation. On the other hand, there is, as a rule, no explicit representation for the unitary group $U(t)$ of the full Hamiltonian. It is just for this reason that the proof of the existence of the inverse WO $W_\pm(H_0, H; \mathfrak{A})$ or of the completeness of the direct operator is a considerably more difficult problem. In the theory of differential operators, when $H_0$ has constant coefficients, a representation for $U_0(t)$ can be constructed by means of the Fourier transform. A simple example of this type is presented in §3.1.

Sometimes condition (1) for the existence of the WO is not optimal. Thus, in the example in Part 3 of §4 to satisfy (1) it is required that $\| q \|_{L_2(-1, +1)} \in L_2(\mathbb{R})$. At the same time, as we have seen, the WO $W_\pm(H, H_0)$ exists if the integral of $q$ over $\mathbb{R}_+$ converges only conditionally.

2. In the verification of the existence of WO in abstract scattering theory an important role is played by the set $\mathfrak{H} = \mathfrak{H}_0^{(a)}$ for which

$$
r_H^2(f) := \text{ess sup} \frac{d(E_H(f), f)}{d\lambda} < \infty. \tag{2}
$$

Its utility is based on the inequality

$$
(2\pi)^{-1} \int_{-\infty}^{\infty} \| (U(t), f, g) \|^2 \, dt = \int_{-\infty}^{\infty} \left( \frac{d(E_H(f), g)}{d\lambda} \right)^2 \, d\lambda \leq r_H^2(f) \| P g \|^2, \tag{3}
$$

where $f \in \mathfrak{H}$, $g \in \mathfrak{H}$, $r_H = r_H$.

Here the equality on the left is a consequence of the Parseval equality applied to relation (1.4.2) with an absolutely continuous function $(E(\lambda)f, g)$.

To prove inequality (3) it is only necessary to use the estimate (1.3.9) and consider the representation (1.3.11).

From (1.3.9) it follows also that the set $\mathfrak{H}$ is linear. Moreover, we have

**Lemma 3.** The set $\mathfrak{H}$ is dense in the absolutely continuous subspace $\mathfrak{H}_0^{(a)}$.

**Proof.** For $f \in \mathfrak{H}_0^{(a)}$ we set $X_N = \{ \lambda : d(E(\lambda), f)/d\lambda \leq N \}$, $Y_N = \mathbb{R} \setminus X_N$. Then $Y_N = \cap Y_N$ is the set of those $\lambda$ at which $d(E(\lambda)f, f)/d\lambda$ does not exist or is equal to $\infty$. Since the function $d(E(\lambda)f, f)/d\lambda$ is integrable, it follows that $|Y_\infty| = 0$, and hence $|Y_N| \rightarrow 0$ as $N \rightarrow \infty$. By (1.3.13) for the elements $f_N = E(X_N)f$

$$
\frac{d(E(\lambda)f_N, f_N)}{d\lambda} = Y_N(\lambda) \frac{d(E(\lambda)f, f)}{d\lambda} \leq N,
$$

so that $f_N \in \mathfrak{H}$. Moreover, $\| f - f_N \| = \| E(Y_N)f \| \rightarrow 0$ as $|Y_N| \rightarrow 0$ in view of the absolute continuity of the measure $(E(\cdot)f, f)$.

The definition of the set $\mathfrak{H}$ becomes especially transparent in terms of the decomposition of the space $\mathfrak{H}_0^{(a)}$ into a direct integral (1.5.6). Namely, by (1.5.7) $\mathfrak{H}$ consists of elements whose representatives $f(\lambda)$ are essentially bounded on $\delta$. Lemma 3 implies the assertion regarding the density of $L_\infty$ in $L_2$, and $L_\infty$ can be extended to vector functions with values in direct integrals.
§6. Birman's invariance principle (IP)

1. The invariance principle (IP) is a general hypothesis that under particular conditions on a real function \( \varphi \) (including the assumption \( \varphi'(\lambda) > 0 \)) the WO for the pair \( H_\emptyset, H \) and \( h = \varphi(H_\emptyset) \), \( h = \varphi(H) \) exist simultaneously and coincide with one another, i.e.,

\[
W_\pm := W_{\pm}(\varphi(H), \varphi(H_\emptyset); \emptyset) = W_{\pm}(H, H_\emptyset; \emptyset) =: W_\pm.
\]

The IP is satisfied in practically all concrete criteria for the existence of the WO although its "absolute" formulation just presented is not true.

As soon as the IP turns out to be true under the conditions of some criterion for the existence of the WO, the range of the applicability of the latter is automatically extended. Indeed, by verifying a criterion for the pair \( H, H_\emptyset \) we automatically obtain the existence of the WO also for another pair \( H_\emptyset, H \).

As concerns the invariance equality (1), it is useful for carrying over information regarding the WO \( W_\pm \) to the WO \( W_{\pm} \). In concrete applications, on the contrary, the WO \( W_{\pm} \) is usually studied by means of \( W_\pm \) for a suitable invertible function \( \varphi \). This technique is especially convenient in the theory of trace class perturbations (see Part 2 of §6.5).

2. Of course, in considering the operator \( \varphi(H) \) the function \( \varphi \) is assumed to be measurable and finite a.e. relative to the spectral measure \( E \) of the operator \( H \). We recall that the spectral measure of a function of an operator can be constructed by the formula

\[
E_{\varphi(H)}(\Lambda) = E_H(\varphi^{-1}(\Lambda)),
\]

where \( \varphi^{-1}(\Lambda) \) is the full preimage of the Borel set \( \Lambda \) under the mapping \( \varphi \).

In justifying the IP it suffices to make the assumptions concerning \( \varphi \) on any open set of full \( E \)-measure. Such a set can be restricted even further due to the presence in the definition of the WO of the absolutely continuous projection \( P \). We suppose first that there exists a Borel set \( M \) of full \( E \)-measure, i.e.,

\[
E(\mathbb{R} \setminus M) = 0,
\]

on which the function \( \varphi \) is defined and assumes finite values. Actual assumptions on \( \varphi \) will be made on an open set \( \Omega \) such that

\[
|\Omega| = 0, \quad |\varphi(\Omega)| = 0.
\]

For example, conditions (4), (5) are satisfied if the set \( M \setminus \Omega \) is countable.

We set \( \mathcal{R}_\emptyset = M \setminus \Omega \) and consider the representation of the open set

\[
\Omega = \bigcup_{n=1}^{\infty} \Omega_n.
\]
DEFINITION 2. We call a real function \( \varphi \) admissible on an open set \( \Omega \subset \mathbb{R} \) if on \( \Omega \) \( \varphi \) has a derivative \( \varphi' \), which is absolutely continuous and of constant sign on each of the component intervals \( \Omega_{n} \), \( n = 1, 2, \ldots \), of \( \Omega \) (i.e., \( \varphi'(\lambda) > 0 \) or \( \varphi'(\lambda) < 0 \) for \( \lambda \in \Omega_{n} \)). If, moreover, this function is \( E \)-measurable, is \( E \)-finite a.e., and for some Borel set \( M \) equalities (3)–(5) hold, then \( \varphi \) is called admissible with respect to the operator \( H \).

3. We refine the formulation of the IP. We set \( \Omega_{+} = \{ \lambda \in \Omega : \pm \varphi'(\lambda) > 0 \} \); then \( \Omega_{+} \cap \Omega = \emptyset \). The formulation of the IP (1) pertains to the case where \( \varphi'(\lambda) > 0 \) for \( \lambda \in \Omega \). In the general case the role of (1) is played by the relation

\[
W_{\pm} (\varphi(H), \varphi(H_{0}); \mathfrak{I}) = W_{\pm}(H, H_{0}; \mathfrak{I}, \Omega_{n}) \quad \text{formulated in terms of the local WO (2.5). By (8) and (9) this relation is equivalent to the two equalities}
\]

\[
W_{\pm} (\varphi(H), \varphi(H_{0}); \mathfrak{I} \mathfrak{E}_{n}(\Omega_{n})) = W_{\pm}(H, H_{0}; \mathfrak{I}, \Omega_{n}) \quad \text{(12)}
\]

\[
W_{\pm} (\varphi(H), \varphi(H_{0}); \mathfrak{I} \mathfrak{E}_{n}(\Omega_{n})) = W_{\pm}(H, H_{0}; \mathfrak{I}, \Omega_{n}) \quad \text{(12)_-}
\]

Strictly speaking, one calls IP the assertion that the existence of the local WO on the right in (12) or (12)_- implies the existence of the WO on the left and is that the two sides are equal. This assertion is true only under specific assumptions. Nevertheless, we shall see in \S 3.5 that the following weakened formulation of the IP is always valid. If the WO on both sides of (12)_- exist, then they are related by equality (12)_-.

We emphasize that in these arguments the relations (12) and (12)_- are considered independently of one another. In turn, each of them is understood, as always, as a set of two independent equalities.

From (11) we obtain the IP for the scattering operators (4.1):

\[
\mathcal{S}(\varphi(H), \varphi(H_{0}); \mathfrak{I}) = \mathcal{S}(H, H_{0}; \mathfrak{I}) \mathfrak{E}_{n}(\Omega_{n}) + \mathcal{S}'(H, H_{0}; \mathfrak{I}) \mathfrak{E}_{n}(\Omega_{n}). \quad \text{(13)}
\]

To formulate the IP in terms of scattering matrices it is, first of all, necessary to coordinate the decompositions of the space \( \mathfrak{H}_{b_{0}} = \mathfrak{H}_{b_{0}}^{(a)} \) into direct integrals diagonalizing the operators \( \varphi_{0}(h) \) and \( \varphi_{0}^{(a)} \). We shall suppose that the decomposition of the operator \( \varphi_{0}(h) \) is constructed from some fixed decomposition (4.2) by means of the change of variables \( \mu = \varphi(\lambda) \). According to (13), the scattering matrix \( s(\mu) \) corresponding to the operator \( \mathcal{S}(h, h_{0}; \mathfrak{I}) \) then acts in the space

\[
b_{0}(\mu) = \sum_{\varphi(\lambda) = \mu} b_{0}(\lambda)
\]

for a.e. \( \mu \in \mathfrak{H}_{b_{0}} \) according to the formula

\[
s(\mu) = \left( \sum_{\varphi(\lambda) = \mu} \mathcal{S}(\lambda) \right) \oplus \left( \sum_{\varphi(\lambda) = \mu} \mathcal{S}'(\lambda) \right). \quad \text{(14)}
\]

4. Finally, we present auxiliary technical arguments, used in the proof of the IP under concrete assumptions.

If the WO \( W_{\pm} \) exists for a pair \( H_{0}, H \), then by the intertwining property the relation (1.23) holds. The analogous relation for the group \( \theta_{0}(t) = \exp(-it\mathfrak{H}) \) is useful for verification of the IP.

**Lemma 3.** Suppose for a pair \( H_{0}, H \) and a set \( \Omega_{n} \), where \( \nu = \pm \), the local WO

\[
W_{\pm}(H, H_{0}; \mathfrak{I}, \Omega_{n}) = : W_{\pm} (\Omega_{n})
\]

exists. Then the existence of the WO \( W_{\pm} (\mathfrak{H}_{0}, h, H_{0}; \mathfrak{I} \mathfrak{E}_{n}(\Omega_{n})) \) and the validity of equality (12)_- for it are equivalent to the relation

\[
s \lim_{t \rightarrow \pm \infty} (W_{\pm} (\Omega_{n}) - \mathcal{I}) \theta_{0}(t) \mathcal{E}_{n}(\Omega_{n}) = 0. \quad \text{(15)}
\]

**Proof.** By the intertwining property

\[
u(t) W_{\pm} (\Omega_{n}) = W_{\pm} (\Omega_{n}) \theta_{0}(t),
\]

relation (15) is equivalent to

\[
s \lim_{t \rightarrow \pm \infty} (W_{\pm} (\Omega_{n}) - \mathcal{I}) \theta_{0}(t) = 0.
\]

The last equality means that for \( f \in \mathfrak{E}_{n}(\Omega_{n}) \mathfrak{K} \) and \( t \rightarrow \pm \infty \), the limit \( u(-t) \mathcal{I} \theta_{0}(t) f \) exists and is equal to \( W_{\pm} (\Omega_{n}) f \).

**Lemma 3** shows that the proof of the IP reduces to the verification of relation (15). We note that if the WO \( W_{\pm} = W_{\pm}(H, H_{0}; \mathfrak{I}) \) exists, the operator \( W_{\pm} (\Omega_{n}) \) in (15) can be replaced by \( W_{\pm} (\mathfrak{I} \mathfrak{E}_{n}(\Omega_{n})) \). In view of the equality \( W_{\pm} (\mathfrak{I} \mathfrak{E}_{n}(\Omega_{n})) = W_{\pm} \mathfrak{E}_{n}(\Omega_{n}) \), the relation obtained coincides with (15). According to (7) to prove (15) it suffices to consider each of the component intervals \( \Omega_{n} \), \( n = 1, 2, \ldots \), separately. The following technical assertion is often applied in this circumstance.

**Lemma 4.** Suppose the function \( \varphi(\lambda) \) is admissible on the interval \( \Lambda \) and \( \varphi \in L_{2}(\Lambda) \). Then

\[
l_{- \infty}^{\infty} ds \left| \int_{\Lambda} \exp(-is\lambda - it\varphi(\lambda)) \psi(\lambda) d\lambda \right|^{2} = 0, \quad \varphi'(\lambda) > 0. \quad \text{(16)}
\]

**Proof.** It suffices to verify (16) for \( \varphi \in C_{0}^{\infty}(\Lambda) \). The simplest way to see this is to write (16) in operator notation. Suppose \( B \) is multiplication by \( \varphi(\lambda) \) in the space \( L_{2}(\Lambda) \), while the operator \( T_{f} : L_{2}(\Lambda) \rightarrow L_{2}(\mathbb{R}) \) acts according to the formula

\[
(T_{f})(\lambda) = \int_{\Lambda} e^{-i\lambda\lambda} f(\lambda) d\lambda.
\]

Then (16) is equivalent to

\[
l_{- \infty}^{\infty} \int_{\Lambda} \exp(-itB) \psi(\lambda) d\lambda = 0, \quad \varphi'(\lambda) > 0.
\]
In view of the boundedness of $T$, it suffices to verify this equality on any dense set in $L^2_1(\Lambda)$.

For $\varphi \in C_0^\infty(\Lambda)$ it is possible to integrate by parts in the integral over $\Lambda$. Then

$$
(T \exp(-itB))\varphi(s) = -i \int_\Lambda \exp(-is\lambda - it\varphi(\lambda)) \varphi(\lambda)(s + i \varphi'(\lambda))^{-1} d\lambda. \quad (17)
$$

Carrying out the differentiation on the right, we see that the integrand in modulus does not exceed

$$
|\varphi'(\lambda)(s + i |\varphi'(\lambda)|)|^{-1} |\varphi(\lambda)\varphi''(\lambda)(s + i |\varphi'(\lambda)|)|^{-2}.
$$

For an admissible function $\varphi$ there is the estimate $|\varphi'(\lambda)| \geq \alpha > 0$ on the compact set $\text{supp} \varphi$ and $\varphi'' \in L^1_\text{(loc)}(\Lambda)$. Therefore, the modulus of the function (17) can be estimated by $c(s + a|t|)^{-1}$, and hence the entire integral (16) can be estimated by $|t|^{-1}$. □

5. In the formulation of the IP for global WO $W_a(\varphi(H), \varphi(H_0); \lambda)$ it was convenient for us to introduce local WO for the pair $H_0, H$. A systematic local formulation of the IP consists in the following. Let the function $\varphi$ be admissible on an interval $\Lambda$. We extend $\varphi$ to $\mathbb{R} \setminus \Lambda$ so that $\varphi \in L^1_\text{(loc)}$, while it is otherwise arbitrary. It is possible, for example, to take $\varphi(\lambda) = 0$ for $\lambda \notin \Lambda$. Then for any selfadjoint operator $H$ the function $\varphi(H)$ is well defined. The local version of the IP consists in the hypothesis that the existence of the WO $W_a(\varpi(H), \varpi(H_0); \lambda, \Lambda)$ implies the existence of the limit on the left-hand side of the equality

$$
s\lim_{t \to \pm \infty} n^{-1/2} U_0(t) E_{H_0}(\lambda) = W_a(H, H_0; \lambda, \Lambda), \quad \varphi'(\lambda) > 0, \quad (18)
$$

and the equality itself. If $\varphi'(\lambda) < 0$ for $\lambda \in \Lambda$, then the limit (18) must naturally be taken as $t \to \mp \infty$. As for the validity of the local formulation of the IP, what was said in Part 3 regarding the "global" WO remains true. It is not hard to demonstrate that the validity of (18) for all component intervals of the open set $\Omega$ is equivalent to the formulation (11).

The IP also has substance for the weak WO (2.1), including the local operators (2.6). We note that the IP for weak WO is not equivalent to the IP for strong WO, and none of these two formulations follows from the other. It is important to note, that the strong formulation in concrete applications (see Part 4 of §5.3) may be derived from the weak formulation by means of Theorem 2.1.

The IP is also meaningful in application to WO of the form (2.12) for a pair of unitary operators, where we consider functions $\varphi : T \to T$. The same is true for mappings $\varphi : T \to T$ and $\varphi : T \to \mathbb{R}$ relating selfadjoint and unitary operators. In particular, for $\varphi(\lambda) = (\lambda - i)(\lambda + i)^{-1}$ the validity of the IP implies that the WO for the pair of selfadjoint operators $H_0, H$ and their Cayley transforms $U_0 = \mathcal{H}_H$, $U = \mathcal{H}_H$ (see (2.13)) exist simultaneously and coincide.

If the operators $H_0, H$ are positive, then application of the IP is most often connected with use of the pairs $(H_0 + I)^{-\alpha}, (H + I)^{-\alpha}$, $\alpha > 0$, and $\exp(-H_0), \exp(-H)$.

§7. The stationary approach. Formulas for the WO

In the stationary approach the WO are defined in terms of the resolvents of the operators $H_0$ and $H$. This definition has two advantages. First of all, in applications, investigation of the resolvent turns out, as a rule, to be a simpler problem than the direct study of the unitary group. Secondly, in stationary terms, formula representations for the WO and the scattering operator and matrix can be given which are often used in physics and are convenient for verification of various properties of these objects. In the present section we give only a heuristic description of the stationary approach and present representations for the WO. Precise formulations and their proofs may be found in Chapter 5.

1. Below we shall need a generalization of the Parseval equality to vector-functions with values in a Hilbert space $\mathcal{H}$. Suppose $f_j \in L^2_1(\mathbb{R} ; \mathcal{H}) \cap L^2_1(\mathbb{R}; \mathcal{H})$, $j = 0, 1$, and

$$
f_j(\lambda) = (2\pi)^{-1/2} \int_{-\infty}^{\infty} \exp(\pm i\lambda t)f_j(t) dt.
$$

Then

$$
\int_{-\infty}^{\infty} (\tilde{f}_1(\lambda), \tilde{f}_2(\lambda)) d\lambda = \int_{-\infty}^{\infty} f_1(t), f_2(t) dt. \quad (1)
$$

The proof of (1) can be obtained by decomposing both sides with respect to some orthonormal basis in $\mathcal{H}$ and applying the scalar Parseval equality.

The starting point for stationary scattering theory is the transformation of the time-dependent definition (1.1) to an expression for the WO in terms of the resolvents $R_0(z)$ and $R(z)$ of the operators $H_0$ and $H$. Under such a transformation it suffices to assume that there exists only the weak WO (2.1), and even the weak limit can be understood in the Abel sense. Namely, we suppose that for $\omega(t) = 2\pi \exp(-2\pi t)$ (2.11) holds, i.e., for a pair of elements $f_0 \in \mathcal{H}$, $f \in \mathcal{H}$, and $f_0 = f_0 f_0^\dagger$, $f = f f^\dagger$,

$$
\lim_{t \to 0} 2\pi \int_0^\infty e^{-2\pi t} (\mathcal{H} U_0(\pm t)f_0^\dagger, U(\pm t)f_0^\dagger) dt = (\hat{\mathcal{H}} f_0^\dagger, f)
$$

exists.

Suppose $\theta(t) = 1$ for $t \geq 0$ and $\theta(t) = 0$ for $t < 0$. We apply equality (1) to the functions

$$
\theta(t) e^{-it} U_0(\pm t)f_0^\dagger, \quad \theta(t) e^{-it} U(\pm t)f_0^\dagger.
$$

Considering the expressions (1.4.4) for the resolvents of the operators $H_0$
and \( H \), we then find that
\[
2\pi \int_0^\infty e^{-2\varepsilon t}(3U_0(t)f_0^{(a)} + U(t)f^{(a)}) \, dt
= \pi^{-1} \int_{-\infty}^\infty (3R_0(\lambda \pm i\varepsilon)f_0^{(a)} + R(\lambda \pm i\varepsilon)f^{(a)}) \, d\lambda.
\]

We have thus established

**Lemma 1.** The existence of the weak Abelian WO \( \tilde{z}_\pm(H, H_0; \mathcal{I}) \) is equivalent to the existence, for any \( f_0 \in \mathcal{H}_0, \ f \in \mathcal{H} \), of the limit of the expression (2) as \( \varepsilon \to 0 \), and
\[
(\tilde{z}_\pm(H, H_0; \mathcal{I})f_0^{(a)}, f^{(a)}) = \lim_{\varepsilon \to 0} \pi^{-1} \int_{-\infty}^\infty (3R_0(\lambda \pm i\varepsilon)f_0^{(a)} + R(\lambda \pm i\varepsilon)f^{(a)}) \, d\lambda.
\]

Thus, the definition of the WO \( \tilde{z}_\pm(H, H_0; \mathcal{I}) \) can equivalently be reformulated in terms of the resolvents of the operators \( H_0 \) and \( H \). If the WO \( W_\pm(H, H_0; \mathcal{I}) \) exists, relation (3) can be considered as a stationary representation of its sesquilinear form. The definition of the stationary WO differs from (3) only in that the limit as \( \varepsilon \to 0 \) is taken under the integral sign. Namely, we adopt

**Definition 2.** Let \( \mathcal{L}_0, \mathcal{L} \) be dense linear manifolds in \( \mathcal{H}_0, \mathcal{H} \) and let for any \( f_0 \in \mathcal{L}_0, \ f \in \mathcal{L} \)
\[
\lim_{\varepsilon \to 0} \pi^{-1} \varepsilon (3R_0(\lambda \pm i\varepsilon)f_0, R(\lambda \pm i\varepsilon)f) =: a_\pm(f_0, f; \lambda)
\]

exists for a.e. \( \lambda \in \mathbb{R} \) (the corresponding set of full measure in \( \mathbb{R} \) may depend on \( f_0 \) and \( f \)). The stationary WO \( \mathcal{L}_\pm = \mathcal{L}_\pm(H, H_0; \mathcal{I}) \) for the operators \( H_0, H \) and the identification \( \mathcal{I} \) is the operator defined on \( \mathcal{L}_0 \times \mathcal{L} \) by the sesquilinear form
\[
(\mathcal{L}_\pm f_0, f) = \int_{-\infty}^\infty \lim_{\varepsilon \to 0} \pi^{-1} \varepsilon (3R_0(\lambda \pm i\varepsilon)f_0, R(\lambda \pm i\varepsilon)f) \, d\lambda.
\]

We postpone a discussion of the correctness of this definition to \$\text{§}5.2\$. There we shall see that when the limit (4) exists the operator \( \mathcal{L}_\pm \) possesses the usual properties of WO. It is bounded, satisfies the relations
\[
\mathcal{L}_0^{(a)} \subset N(\mathcal{L}_0), \quad R(\mathcal{L}_0) \subset \mathcal{L}_0^{(a)}
\]
and has the intertwining property
\[
E(\Lambda)\mathcal{L}_\pm = \mathcal{L}_\pm E(\Lambda).
\]
Moreover, we have the representation
\[
(\mathcal{L}_\pm E_0(\Lambda_\varepsilon) f_0, E(\Lambda)f) = \int_{\Lambda_\varepsilon \cap \Lambda} \lim_{\varepsilon \to 0} \pi^{-1} \varepsilon (3R_0(\lambda \pm i\varepsilon)f_0, R(\lambda + i\varepsilon)f) \, d\lambda.
\]

The limit on \( \varepsilon \) and integration over \( \lambda \) can be interchanged in (5) (and in (8)), provided \( f_0, f \) are replaced by \( f_0^{(a)}, f^{(a)} \). Thus, under condition (4) the weak Abelian WO \( \tilde{z}_\pm = \mathcal{L}_\pm \) exists. Under the conditions of definition (2), the stationary WO \( \mathcal{L}_\pm(H_0, H; \mathcal{I}) = \mathcal{L}_\pm(H, H_0; \mathcal{I}) \)
also exists. We further note that in view of the resolvent identity \( \text{(1.9.5)} \) the expression under the limit sign in (4) can be rewritten in any of the two forms
\[
\pi^{-1} \varepsilon (3R_0(\lambda \pm i\varepsilon)f_0, R(\lambda \pm i\varepsilon)f) = (\delta(\lambda, \varepsilon)f_0, f + V R_0(\lambda \pm i\varepsilon)f_0, \delta(\lambda, \varepsilon)f) \]
\[
(\delta(\lambda, \varepsilon)f_0, f + V R_0(\lambda \pm i\varepsilon)f_0, \delta(\lambda, \varepsilon)f)
\]
\[
(\delta(\lambda, \varepsilon)f_0, f + V R_0(\lambda \pm i\varepsilon)f_0, \delta(\lambda, \varepsilon)f) = (\delta(\lambda, \varepsilon)f_0, f) + V R_0(\lambda \pm i\varepsilon)f_0, \delta(\lambda, \varepsilon)f).
\]

where we have used the notation of \( \text{(1.4.5)} \).

**2.** Further study of properties of stationary WO is based on the representation
\[
(\mathcal{L}_\pm f_0, \mathcal{L}_\pm g_0) = \int_{-\infty}^\infty \lim_{\varepsilon \to 0} \pi^{-1} \varepsilon (3R_0(\lambda \pm i\varepsilon)f_0, R(\lambda \pm i\varepsilon)g_0) \, d\lambda,
\]
where \( f_0 \) and \( g_0 \) belong to a suitable dense set in \( \mathcal{H}_0 \). Here we present only a formal derivation of (11). By relation \( \text{(1.4.11)} \) and the inclusion \( \mathcal{L}_\pm f_0 \in \mathcal{H}^{(a)} \), for any Borel set \( \Lambda \)
\[
(E(\Lambda)\mathcal{L}_\pm f_0, f) = \int_{\Lambda} \lim_{\varepsilon \to 0} (\delta(\lambda, \varepsilon)f_0, f) \, d\lambda.
\]
Comparing this equality with (8) for \( \Lambda_\varepsilon = \mathbb{R} \), we find that for \( f_0 \in \mathcal{L}_0, \ f \in \mathcal{L} \)
\[
\lim_{\varepsilon \to 0} (\delta(\lambda, \varepsilon)f_0, f) = \lim_{\varepsilon \to 0} \pi^{-1} \varepsilon (3R_0(\lambda \pm i\varepsilon)f_0, R(\lambda \pm i\varepsilon)f).
\]
The relation (12) is altogether well defined. It formally implies that as \( \lambda \to 0 \)
\[
(\delta(\lambda, \varepsilon)f_0, f) \sim \pi^{-1} \varepsilon R(\lambda \pm i\varepsilon)R_0(\lambda \pm i\varepsilon)f_0.
\]
We set \( f = \mathcal{L}_\pm g_0 \) in (10) and consider the resolvent identity. In view of (13) this gives the relation
\[
\pi^{-1} \varepsilon (3R_0(\lambda \pm i\varepsilon)f_0, R(\lambda \pm i\varepsilon)g_0)
\]
\[
\sim \pi^{-1} \varepsilon (3 + V R_0(\lambda \pm i\varepsilon)f_0, R(\lambda \pm i\varepsilon)g_0) \]
\[
\sim \pi^{-1} \varepsilon (3 + V R_0(\lambda \pm i\varepsilon)f_0, R(\lambda \pm i\varepsilon)g_0).
\]
In contrast to (13) here the symbol \( \alpha(\lambda, \varepsilon) \sim \beta(\lambda, \varepsilon) \) has a precise meaning and indicates that both sides have coincident finite limits as \( \varepsilon \to 0 \) for a.e. \( \lambda \in \mathbb{R} \).
2. BASIC CONCEPTS OF SCATTERING THEORY

We now use the definition (5) for $f = \mathcal{Z}_\pm g_0$. Substituting (14) into the right-hand side of it, we complete the verification of (11)$_\pm$. We emphasize that we have omitted a precise formulation and justification of relation (13). Justification of it is possible (see §5.2) but it requires additional assumptions.

Along with $\mathcal{Z}_\pm(H_0; H_0; \mathfrak{J})$ we consider also the stationary WO $\mathcal{Z}^{(0)}_\pm = \mathcal{Z}^*_\pm(H_0, H_0; \mathfrak{J}^\star \mathfrak{J})$ for the pair $H_0$, $H_0$ and the identification $\mathfrak{J}^\star \mathfrak{J}$. According to definition (5),

$$
(\mathcal{Z}^{(0)}_\pm f_0, g_0) = \int_{-\infty}^{\infty} \lim_{\epsilon \to 0} \pi^{-1} e^{i(\lambda + \epsilon i)f_0} \mathfrak{J} \mathfrak{R}_0(\lambda + \epsilon i)g_0) d\lambda.
$$

Combining this equality with (11)$_\pm$, we find that

$$
\mathcal{Z}^*_\pm(H_0, H_0; \mathfrak{J}) \mathcal{Z}^*_\pm(H_0, H_0; \mathfrak{J}) = \mathcal{Z}^*_\pm(H_0, H_0; \mathfrak{J}^\star \mathfrak{J}).
$$

In particular, for an identification $\mathfrak{J}^\star \mathfrak{J}$ equivalent to $I$ (for example, for $\mathfrak{J}_0 = \mathfrak{J}$, $\mathfrak{J} = I$) from this it follows that the operator $\mathcal{Z}^*_\pm$ is an isometry on $\mathfrak{H}_0^{(a)}$.

3. Equality (16) is the key relation of the stationary approach. Indeed, assume that we have succeeded in establishing the existence of the weak time-dependent WO $\mathcal{W}^\star_\pm(H_0, H_0; \mathfrak{J})$ and $\mathcal{W}^\star_\pm(H_0, H_0; \mathfrak{J}^\star \mathfrak{J})$. These operators automatically coincide with the corresponding stationary WO. Therefore, the equality (16) for them implies that the relation (2.3) holds for the weak time-dependent WO. Hence, by Theorem 2.1 the strong WO $\mathcal{W}^\star_\pm(H, H_0; \mathfrak{J})$ exists.

Thus, verification of the existence of the strong time-dependent WO $\mathcal{W}^\star_\pm(H, H_0; \mathfrak{J})$ breaks into two stages. The first is the proof of the existence of the weak time-dependent WO, a problem considerably simpler than that for the strong WO. The second, which is more substantial, is the stationary proof of the identity (16). For that reason the entire scheme is called stationary.

§8. Stationary representations for the scattering operator and matrix

Continuing the considerations of the preceding section, we here obtain expressions of the scattering operator and matrix in terms of the limit values of the resolvents. For precise proofs of them, see §5.5.

1. We first establish a representation for the sesquilinear form of the stationary scattering operator $\mathcal{Z}^*_\pm \mathcal{Z}^\star_\pm$, or, more generally, the operator $\mathcal{Z}^*_\pm E(X) \mathcal{Z}^\star_\pm$, where $X$ is an arbitrary Borel set. This is done similarly to the derivation of the representation (7.11)$_\pm$. We again start from relation (7.13), which for now has only a formal meaning. It implies the asymptotics

$$
\pi^{-1} e^{i(\mathfrak{J} \mathfrak{R}_0(\lambda - i\epsilon)f_0, R(\lambda - i\epsilon)g_0)} = \left( (\mathcal{J} + V \mathcal{R}_0(\lambda - i\epsilon)f_0, \delta(\lambda, \epsilon) \mathcal{Z}^\star_\pm g_0) \right)
$$

$$
\sim \pi^{-1} e^{i(\mathfrak{J} + V \mathfrak{R}_0(\lambda - i\epsilon)f_0, R(\lambda - i\epsilon)\mathfrak{J} \mathfrak{R}_0(\lambda + i\epsilon)g_0)}
$$

as $\epsilon \to 0$.

As $\epsilon \to 0$, we now substitute the element $\mathcal{Z}^*_\pm g_0$ into the representation (7.8) for the form $(E(X) \mathcal{Z}^\star_\pm f_0, f_0)$. Then according to (1)

$$
(E(X) \mathcal{Z}^\star_\pm f_0, \mathcal{Z}^\star_\pm g_0)
$$

$$
= \int_{x} \lim_{\epsilon \to 0} \pi^{-1} e^{i(\mathfrak{J} + V \mathfrak{R}_0(\lambda - i\epsilon)f_0, R(\lambda - i\epsilon)\mathfrak{J} \mathfrak{R}_0(\lambda + i\epsilon)g_0)} d\lambda.
$$

The scalar product under the integral sign can be transformed identically by means of the equalities

$$
\pi^{-1} e^{i(\mathfrak{J} + V \mathfrak{R}_0^* \mathfrak{J} R(\lambda + i\epsilon)g_0)} = (\mathfrak{J} + V \mathfrak{R}_0^* \mathfrak{J} R(\lambda + i\epsilon)g_0) = \pi^{-1} e^{i(\mathfrak{J} + V \mathfrak{R}_0^* \mathfrak{J} R(\lambda + i\epsilon)g_0)}
$$

We here assume that $z = \lambda + i\epsilon$, and the dependence of $z$ is omitted in the notation, i.e., $R_0 = R_0(\lambda + i\epsilon)$, $\mathfrak{R}_0 = \mathfrak{R}_0(\lambda + i\epsilon)$, and similarly for $R$.

If the WO $\mathcal{W}^\star_\pm(H_0, H_0; \mathfrak{J})$ exists, various versions of equality (2) with $X = \mathbb{R}$ may be considered as stationary representations for the sesquilinear form of the scattering operator $S = \mathcal{W}^\star_\pm \mathcal{W}^\star_\pm$.

Another important transformation of the operator (3) is used in computing the scattering matrix. We set

$$
T_+ (z) = \mathfrak{J} V - V R(\lambda + i\epsilon) V,
$$

$$
T_- (z) = \mathfrak{J} V - V R(\lambda - i\epsilon) V
$$

and again use definition (1.4.5) and the resolvent identity (1.9.5). Then

$$
\pi^{-1} e^{i(\mathfrak{J} \mathfrak{R}_0 f_0, R(\lambda + i\epsilon)g_0)} = \pi^{-1} e^{i(\mathfrak{J} \mathfrak{R}_0 f_0, R(\lambda + i\epsilon)g_0)} - 2\pi i (\mathfrak{J} \mathfrak{R}_0 f_0, V \delta(\lambda, \epsilon) g_0) = \pi^{-1} e^{i(\mathfrak{J} \mathfrak{R}_0 f_0, R(\lambda + i\epsilon)g_0)}
$$

It is possible to carry out an analogous transformation by separating out $\pi^{-1} e^{i(\mathfrak{J} \mathfrak{R}_0 f_0, R(\lambda + i\epsilon)g_0)}$ as the first term on the right-hand side. In the second term $T_-$ is then replaced by $T_+$. By equalities (2), (3), and (7.13) from this we obtain the two representations

$$
(E(X) \mathcal{Z}^\star_\pm f_0, \mathcal{Z}^\star_\pm g_0) = (E_0(X) \mathcal{Z}^{(0)}_\pm f_0, g_0)
$$

$$
- 2\pi i \int_{x} \lim_{\epsilon \to 0} (T_+ (\lambda + i\epsilon) \delta(\lambda, \epsilon) f_0, \delta(\lambda, \epsilon) g_0) d\lambda.
$$

(6)
2. Equalities (6) make it possible to easily obtain representations for the scattering matrix $S(\lambda)$. Suppose $\mathcal{H}_0^{(a)}$ is decomposed into the direct integral (4.2), so that $f_0 \leftrightarrow \tilde{f}_0(\lambda)$. Since the operator $\mathcal{H}_+^{(0)}$ commutes with $H_0$ and annihilates on $\mathcal{H}_0^{(0)}$, a family of bounded operators $u_\pm(\lambda) : H_0(\lambda) \rightarrow H_0(\lambda)$ corresponds to it in this decomposition. Therefore,

$$
\left( E(X)\mathcal{H}_- f_0, \mathcal{H}_+ g_0 \right) = \left( E_0(X)\mathcal{H}_\pm^{(0)} f_0, g_0 \right) + \int_X \left( (S(\lambda) - u_\pm(\lambda))\tilde{f}_0(\lambda), \tilde{g}_0(\lambda) \right) d\lambda.
$$

Comparing this equality with (6) and considering the fact that $X$ is arbitrary, we find that for a.e. $\lambda \in \sigma_0$

$$
(\lambda - u_\pm(\lambda))\tilde{f}_0(\lambda), \tilde{g}_0(\lambda)) = -2\pi i \lim_{\varepsilon \to 0} (T_\pm(\lambda + i\varepsilon)\delta_0(\lambda, \varepsilon) f_0, \delta_0(\lambda, \varepsilon) g_0). \quad (7)
$$

Suppose now that the limit of $T_\pm(\lambda + i\varepsilon)$ as $\varepsilon \to 0$ exists and that a well-defined kernel $T_\pm(\mu, \nu; \lambda + i0)$ can be ascribed to the operator $P_0 T_\pm(\lambda + i0) P_0$ (see §1.5 and §5.4). It is additionally required that the points $\lambda, \mu$, and $\nu$ independently run through some set $\sigma_0$ of full measure in $\sigma_0$. The value of the kernel of the operator $P_0 T_\pm(\lambda + i0) P_0$ is then also well defined on the "triple" diagonal $\lambda = \mu = \nu$, and

$$
\lim_{\varepsilon \to 0} (T_\pm(\lambda + i\varepsilon)\delta_0(\lambda, \varepsilon) f_0, \delta_0(\lambda, \varepsilon) g_0) = \left( \mathcal{I}_0(\lambda, \lambda; \lambda + i0)\tilde{f}_0(\lambda), \tilde{g}_0(\lambda) \right). \quad (8)
$$

From (7) it now follows that

$$
S(\lambda) = u_\pm(\lambda) - 2\pi it_\pm(\lambda, \lambda; \lambda + i0) \quad (9)_\pm
$$

for a.e. $\lambda \in \sigma_0$. For WO $\mathcal{H}_\pm$ isometric on $\mathcal{H}_0^{(a)}$ we have $u_\pm(\lambda) = I(\lambda)$, so that (9) reduces to the equality

$$
S(\lambda) = I(\lambda) - 2\pi it_\pm(\lambda, \lambda; \lambda + i0). \quad (10)
$$

In this case the values of the kernels of the (distinct) operators $T_+(\lambda + i0)$ and $T_-(\lambda - i0)$ at the point $\mu = \nu = \lambda$ coincide with one another. For $\mathcal{I} = I$, of course $T_+ = T_-$, and hence

$$
S(\lambda) = I(\lambda) - 2\pi it(\lambda, \lambda; \lambda + i0). \quad (11)
$$

Additional facts regarding formula representations of the scattering matrix can be found in §7.2.

CHAPTER 3

Further Properties of the WO

In this chapter we collect facts from scattering theory going beyond the framework of the "compulsory minimum" of Chapter 2. The individual subsections have little relation to one another. To a considerable extent they can be read independently. At first reading this chapter may be omitted and consulted only when referred to in subsequent chapters.

§1. Perturbation by the boundary condition

We now consider an example in which the perturbation consists in a change of the boundary condition for the differential operator $-d^2/dx^2$. In this example the WO and the scattering operator and matrix can be computed explicitly. Even in this simple example, we see that the construction of scattering theory is bound up with profound questions of function theory.

1. Let $\mathcal{H} = L^2(\mathbb{R}_+)$, and $H_0 = -d^2/dx^2$ with the boundary condition $u(0) = 0$. As a "perturbed" operator $H = H^{(0)}$ we take the operator $-\frac{d^2}{dx^2}$ with the boundary condition $u'(0) = \alpha u(0), \alpha = \overline{\alpha}$. We first consider the case $\alpha = 0$. We introduce the Fourier sine and cosine transformations

$$
(\Phi_{s} f)(p) = (2/\pi)^{1/2} \int_{0}^{\infty} \sin(px)f(x) \, dx,
$$

$$
(\Phi_{c} f)(p) = (2/\pi)^{1/2} \int_{0}^{\infty} \cos(px)f(x) \, dx,
$$

mapping $L^2(\mathbb{R}_+)$ unitarily onto $L^2(\Xi_+), \Xi_+ = (0, \infty)$. Let $A$ be the operator of multiplication by $p^2$ in $L^2(\Xi_+)$. It can be verified in an elementary way that

$$
H_0 = \Phi_{c}^* A \Phi_{s}, \quad H^{(0)} = \Phi_{s}^* A \Phi_{c}. \quad (1)
$$

Thus, the operators $H_0$ and $H = H^{(0)}$ are unitarily equivalent to one another, and their spectra are simple, absolutely continuous, and coincide with the positive semiaxis. The WO $W_+(H, H_0)$ can easily be computed in terms of the transformations $\Phi_{s}$ and $\Phi_{c}$. We set

$$
(\Pi f)(p) = (2/\pi)^{1/2} \int_{0}^{\infty} \exp(\pm ipx)f(x) \, dx. \quad (2)
$$

97
3. Further Properties of the WO

Then $2i\Phi_\epsilon = \Pi_+ - \Pi_-, 2\Phi_\epsilon = \Pi_+ + \Pi_-$. According to (1) there are the equalities

$$U_0(t)f = (2\pi)^{-1} \left( \Pi_- - \Pi_+^* \right) U_A(t)\hat{f},$$

$$U(t)g = 2^{-1} \left( \Pi_+ + \Pi_-^* \right) U_A(t)\hat{g},$$

where $\hat{f} = \Phi_\epsilon f$, $\hat{g} = \Phi_\epsilon g$.

It follows from Lemma 2.6.4 that

$$s\lim_{t \to \pm\infty} \Pi_\pm U_A(t) = 0.$$  \hfill (4)

Indeed, for the upper index this relation is a corollary of (2.6.16) for the case $\phi(\lambda) = \lambda^2$, $\Lambda = \mathbb{R}_+$ (here $\Pi_\pm = (2\pi)^{-1/2} T$ in the notation of Lemma 2.6.4). For the lower index it is necessary, in addition, to pass to the compact conjugate in (2.6.16).

According to (3), (4) as $t \to \pm\infty$ we have the asymptotics

$$U_0(t)f \sim \pm i(2\pi)^{-1} \Pi_\pm U_A(t)\hat{f},$$

$$U(t)g \sim 2^{-1} \Pi_\pm U_A(t)\hat{g},$$

where the sign "~" means that the difference of the left- and right-hand sides tends to zero in $\mathcal{H}$. Thus, $U(t)g \sim U_0(t)f$ as $t \to \pm\infty$ if $\hat{f} = i\hat{g}$. This implies that all the WO $W_\pm(H, H_0)$ exist and $W_\pm(H, H_0) = \mp i\Phi_\epsilon^* \Phi_\epsilon$. The WO are unitary in $\mathcal{H}$, and the corresponding scattering operator is $S = -I$.

2. For arbitrary $\alpha$ we first convince ourselves of the existence of the WO $W_\pm(H, H_0)$. Since the domains $\mathcal{D}(H_0)$ and $\mathcal{D}(H)$ of these operators are essentially different, Theorem 2.5.1 cannot be applied directly. We first consider the WO $W_\pm(H, H_0; 3)$, where the identification $\mathcal{I}$ is the operator of multiplication by a function $\eta \in C_c^\infty(R_+)$ such that $\eta(x) = 0$ for $x \leq 1$ and $\eta(x) = 1$ for $x > 2$. Then for $u \in \mathcal{D}(H_0)$

$$(H\mathcal{I} - 3H_0)u = (H\mathcal{I} - 3H_0)u = -2\eta''u.$$  \hfill (5)

Suppose the set $D_0$ consists of functions $f \in C_c^\infty(\Xi_\alpha)$. We again use the expression (3) for $U_0(t)f$. Integrating the relation

$$(U_0(t)f)(x) = (2\pi)^{1/2}\int_0^\infty \sin px \exp(-ip^2) \hat{f}(p) dp$$

by parts twice (with the help of equality $e^{-ip^2} dp = i(2\pi)^{-1} dx e^{-ip^2}$), we obtain the estimate

$$|(U_0(t)f)(x)| + |(U_0(t)f)'(x)| \leq C t^{-2}, \quad 0 < x \leq 2.$$  \hfill (6)

Therefore, the integrand in (2.5.1) decays like $t^{-2}$ as $|t| \to \infty$, and hence, by Theorem 2.5.1, the WO $W_\pm(H, H_0; 3)$ exist.

51. Perturbation by the Boundary Condition

It remains to show that the identification $\mathcal{I}$ is equivalent to the identity (see Definition 2.1.8). By Proposition 2.1.9 for this it suffices to demonstrate the compactness of the operator

$$(\mathcal{I} - I)E_\alpha(\Lambda) = (\mathcal{I} - I)\phi(\epsilon) E_\alpha(\Lambda)\phi_\epsilon,$$  \hfill (7)

where $\Lambda = (0, a^2)$ is some bounded interval. The operator $\Phi_\epsilon$ is unitary, while the operator $(\mathcal{I} - I)\phi(\epsilon) E_\alpha(\Lambda)$ is an integral operator with kernel

$$(2\pi)^{-1/2}[\eta(x) - 1] \sin px x^\alpha(x), \quad \Lambda_1 = (0, a).$$  \hfill (8)

This kernel obviously belongs to $L_2(\mathbb{R} \times \Xi_\alpha)$, and hence the operator itself belongs to the Hilbert-Schmidt class $\Theta_\delta$. This implies the compactness of the operator (5) which completes the verification of the existence of the WO $W_\pm(H, H_0)$.

3. We now establish the completeness of the WO $W_\pm(H, H_0)$. We first note that for $\alpha < 0$ the operator $H = H^{(a)}$ has a (unique) simple negative eigenvalue $\lambda = \lambda_\alpha = -a^2$; the corresponding normalized eigenfunction is $w_\alpha(x) = (2|\alpha|)^{1/2} e^{ax}$. We denote by $P = P_\alpha$ the projection onto the orthogonal complement of $\nu = \nu_\alpha$, i.e., $P = I - (\cdot, \nu_\alpha)\nu_\alpha$ for $\alpha < 0$ and $P_\alpha = I$ for $\alpha \geq 0$. We shall establish the equality

$$R(W_\pm(H^{(a)}, H_0)) = P_\alpha \mathcal{H}$$  \hfill (9)

which ensures, in particular, the completeness of the WO $W_\pm(H^{(a)}, H_0)$. From this it follows that the absolutely continuous spectrum of $H$ is simple and coincides with $[0, \infty)$. Moreover, it follows from (6) that for $\alpha \geq 0$ there is no singular component in the spectrum in $H^{(a)}$, while for $\alpha < 0$ it consists of the single eigenvalue $\lambda_\alpha$. Thus, for all $\alpha$ the operators $H^{(a)}$ have no singular continuous spectrum, and the $P_\alpha$ project onto their absolutely continuous subspaces.

To prove relation (6) we start from the following explicit expression for the evolution operator:

$$(U(t)f)(x) = 2^{-1}(\pi|t|)^{-1/2}\int_0^\infty [\exp(4^{-1}\gamma^{-1}(x - x')^2) f(x')] dx', \quad \gamma = \gamma_\alpha = (\pi|t|)^{1/4},$$

where $\gamma = \gamma_\alpha = (\pi|t|)^{1/4}$

$$F(x) = f(x) - 2ae^{-ax} \int_0^x e^{ax} f(y) dy,$$  \hfill (10)

and $f \in \mathcal{P} \mathcal{H}$. For $\alpha < 0$ and $f = c\psi_\alpha$ in place of (7) we obviously have $U(t)f = e^{it^2} f$. For $f, \psi_\alpha = 0$ relation (8) is equivalent to the equality

$$F(x) = f(x) + 2ae^{-ax} \int_0^\infty e^{ax} f(y) dy, \quad \alpha < 0.$$  \hfill (11)
3. Further Properties of the WO

Formula (7) can be obtained from the explicit expression presented in Volume 2 of the course [21] for a solution of the heat equation. For this it is only necessary to change $t$ to $i t$.

Direct verification of (7) consists of the following. If the functions $f$ and $F$ decay sufficiently fast as $x \to \infty$, then in (7) it is possible to differentiate with respect to $t$ and $x$ under the integral sign. This shows that the function (7) satisfies the equation $i u / d t = -\partial^2 u / \partial x^2$. To verify the initial condition we extend the functions $f$ and $F$ by zero to the entire axis. The first term on the right of (7) can then be considered as $\exp(-iHt)f(x)$, where $H = -\partial^2 / \partial x^2$ in $L_2(R)$. Therefore, as $t \to 0$ it converges in $L_2(R)$ to $f(x)$. Similarly, the second term in (7) converges as $t \to 0$ in $L_2(R)$ to $F(-x)$. Since $F(x) = 0$ for $x < 0$, this implies that it tends to zero in $L_2(R)$. The boundary condition $u(0, t) = 0$ gives the connection (8) between $F$ and $f$. Indeed, differentiating (7) with respect to $x$, setting $x = 0$, and integrating once by parts, we find that the boundary condition is satisfied if $f(0) = F(0)$ and $f'(x) = F'(x) = 0$. Solving this differential equation for $F$, we obtain relation (8). The arguments presented require that the functions $f$ and $F$ be sufficiently smooth and decay rapidly as $|x| \to \infty$. With regard to the initial datum $f$ this may be assumed. Then for $\alpha \geq 0$ the function $F$ defined by equality (8) is also smooth and rapidly decreasing. For $\alpha < 0$ the rapid decay of $F$ is ensured by relation (9), valid for $(f, \psi_0) = 0$.

We now rewrite the expression for $(K f)(x)$ in operator notation. We set

$$
(K f)(x) = \alpha e^{-\alpha x} \int_{-\infty}^{\infty} e^{-i y} f(y) \, dy, \quad \alpha \geq 0, \quad (K f)(x) = -\alpha e^{-\alpha x} \int_{-\infty}^{\infty} e^{i y} f(y) \, dy, \quad \alpha \leq 0.
$$

For any $\alpha$ the operator $K = K_\alpha$ is bounded in $H$. Relation (8) for $\alpha > 0$ and (9) for $\alpha \leq 0$ can be written in the form $F = f - 2Kf$. We further introduce the unitary dilation operator $\mathcal{T}(x)$, $(\mathcal{T}(s)f)(x) = s^{-1/2} f(s^{-2} x), s > 0$, and the operator $\Gamma(t)$ of multiplication by the function $\exp(i t^2 x^2 / 4)$. Further, using the notation (2), we write the expression (7) in the form

$$
U(t)P f = \gamma_p \Gamma(t) \mathcal{T}(2|t|) \Phi \gamma \Gamma(t) - \Pi \gamma \Gamma(t) K P f, \quad t > 0.
$$

Since all the operators here are bounded, relation (11) extends by continuity from the dense set of smooth and compactly supported functions $f$ to the entire space $H$. A representation analogous to (11) also holds for the unperturbed group:

$$
U_0(t) = \mp i \gamma \Gamma(t) \mathcal{T}(2|t|) \Phi \gamma \Gamma(t).
$$

4. To find the asymptotics of the functions $U(t)g$ and $U_0(t)f$ we use the fact that $\Gamma(t) \to I$ as $|t| \to \infty$. It therefore follows from (11), (12) that for

$$
\Phi_\pm = \Phi_\pm^{(n)} = \Phi_\pm - \Pi \pm K_\alpha
$$

for any $g \in P H$, $f \in H$

$$
U(t)g \sim \gamma \Gamma(t) \mathcal{T}(2|t|) \Phi g, \quad U_0(t)f \sim \mp i \gamma \Gamma(t) \mathcal{T}(2|t|) \Phi_\pm f.
$$

§1. Perturbation by the Boundary Condition

Comparison of these relations shows that $U(t)g \sim U_0(t)f$ as $t \to \pm \infty$ if $\Phi_\pm g = \mp i \Phi_\pm f$. From this we find that $f = f_\pm = \pm i \Phi_\pm f$. Thus, for any $g \in P H$ the function $U(t)g$ has free asymptotics as $t \to \pm \infty$, i.e., $P H \subset R(W_\pm(H, H_0))$. To conclude the proof of (6) it remains to notice that $R(W_\pm(H, H_0)) \subset \mathcal{C}(\alpha) \subset P H$ whenever the WO $W_\pm(H, H_0)$ exists.

We have simultaneously obtained the expressions for the WO

$$
W_\pm(H, H_0) = \mp i \pi \Phi_\pm \Phi_\pm^* f,
$$

and also an expression for the scattering operator

$$
S(H, H_0) = -\Phi_\pm \Phi_\pm^* P \Phi_\pm^* \Phi_\pm.
$$

5. Finally, we compute the scattering matrix. According to (16), for this it is necessary to find the connection between the operators $\Phi_\pm$. We note first of all the identity

$$
\Phi_\pm P \Phi_\pm^* = I,
$$

which follows from the isometricity of the operator (15).

Let $M = M_\alpha$ be the operator of multiplication by the function $m_\alpha(p) = \alpha(\alpha - ip)^{-1}$. Let us show that

$$
\Pi \gamma K P = M \Pi \gamma P.
$$

It suffices to verify (18) for compactly supported functions $f$. For $\alpha \geq 0$, combining definitions (2) and (10) and interchanging the order of integration, we find that

$$
(\pi/2)^{1/2} \left( \Pi \gamma K f \right)(p) = \alpha \int_{-\infty}^{\infty} dy f(y) e^{ip y} \int_{-\infty}^{\infty} dx e^{ip x - ip y}.
$$

Computing the integral on $x$, we find that the right-hand side is here equal to $m(p)(\Pi \gamma f)(p)$. This proves relation (18) for the case $\alpha \geq 0$. Similarly, for $\alpha \leq 0$

$$
(\pi/2)^{1/2} \left( \Pi \gamma K f \right)(p) = -\alpha \int_{-\infty}^{\infty} dy f(y) e^{ip y} \int_{-\infty}^{\infty} dx e^{ip x - ip y} = m_\alpha(p) \int_{-\infty}^{\infty} dh f(h)(e^{ip y} - e^{ip y}).
$$

Moreover, for $P f = f$ the term with $e^{ip y}$ on the right drops out. Thus, the identity (18) is also valid for the case $\alpha < 0$. Passing to the complex conjugate in (18), we find also that

$$
\Pi \gamma K P = M^* \Pi \gamma P.
$$

We denote by $S = S_\alpha$ the operator of multiplication by the function

$$
S_\alpha(p) = (\alpha + ip)(\alpha - ip)^{-1}
$$

(19)
vanish. To complete the proof of (23) it remains to show that the sum of the terms linear in $\rho$ is also equal to zero, i.e.,

$$\rho \Pi \Pi^* \sigma - \Pi \Pi^* \sigma - \rho \Pi \Pi^* \sigma = 0.$$  

One of the methods of verifying (24) consists in the following. According to (2), $\Pi \Pi^*$ is a singular integral operator with kernel

$$2\pi^{-1} \int_{-\infty}^{\infty} e^{i\rho p + i0} (x) = 2\pi^{-1} i(\rho - p + i0)^{-1}.$$  

Similarly, the kernel of the operator $\Pi \Pi^*$ is equal to $2\pi^{-1} i(\rho + p')^{-1}$. From this it follows that the kernel of the operator on the left in (24) is identically zero.

\section{3-Completeness}

1. Suppose that WO $W_\pm = W_\pm (H, H_0 ; \mathfrak{J})$ exists. For an arbitrary $\mathfrak{J}$ it is, of course, hard to expect it to be complete in the sense of Definition 2.3.1. Moreover, relations (2.1.15) and (2.3.2) pertain, in essence, to different properties of the WO $W_\pm$. The concept of completeness is naturally connected only with a description of the range \( R(W_\pm) \), since for $f \in R(W_\pm)$ the asymptotics of $U(t)f$ can be expressed in terms of the "free" group \( U(t) \) by relation (2.3.1). In the general case a description of \( R(W_\pm) \), or, more precisely its closure \( \overline{R(W_\pm)} \), must be given in terms depending on $\mathfrak{J}$.

We consider the set $\mathfrak{M}_\pm = \mathfrak{M}_\pm (H, \mathfrak{J})$ of elements $g \in \mathfrak{M}$ for which

$$\lim_{t \to \pm \infty} \| U(t)g \| = 0.$$  

It is clear that $\mathfrak{M}_\pm$ is a subspace and $\mathfrak{M}(\mathfrak{J}) \subset \mathfrak{M}_\pm$. Moreover, since $\mathfrak{M}_\pm$ contains with any $g$ all elements of the form $U(s)g$, $s \in \mathbb{R}$, this subspace reduces the operator $H$. We further note that (1) is equivalent to the equality

$$\lim_{t \to \pm \infty} \| U(t)g \| = 0$$  

with a nonnegative operator $\mathfrak{F}_\mathfrak{J}$.

\textbf{Lemma 1.} Let the WO $W_\pm (H, H_0 ; \mathfrak{J})$ exist. Then the subspaces $\mathfrak{M}(W_\pm)$ and $\mathfrak{M}_\pm$ are mutually orthogonal.

\textbf{Proof.} Let $f \in \mathfrak{M}(W_\pm) \subset \mathfrak{M}(\mathfrak{J})$. Then for some $f_0 \in \mathfrak{M}_\pm$ (2.3.1) is satisfied. For $g \in \mathfrak{M}_\pm$ from this it follows that

$$\langle f, g \rangle = \langle Pf, g \rangle = \lim_{t \to \pm \infty} \langle U(t)Pf, U(t)g \rangle$$

$$= \lim_{t \to \pm \infty} \langle U(t)Pf_0, U(t)g \rangle$$

$$= \lim_{t \to \pm \infty} \langle U(t)Pf_0, \mathfrak{F} U(t)g \rangle.$$  

By (1) this expression is equal to zero. $\Box$
3. FURTHER PROPERTIES OF THE WO

**Corollary 2.** If the WO \( W_{\pm}(H, H_0; \mathcal{J}) \) exists, then

\[ R(W_{\pm}) \subseteq \mathfrak{H}_{\pm} := \mathcal{N}_{\pm} \cap \mathcal{N}_{\pm}. \]  

(3)

Thus, the range of the WO is always imbedded in the subspace \( \mathfrak{H}_{\pm} = \mathfrak{N}_{\pm}(H, 3) \) which reduces \( H \) and is contained in \( \mathfrak{N}(\mathcal{N}) \), but, generally speaking, is narrower. This justifies

**Definition 3.** The operator \( W_{\pm}(H, H_0; \mathcal{J}) \) is called \( \mathcal{J} \)-complete if in (3) the equality holds, i.e., \( R(W_{\pm}) = \mathfrak{H}_{\pm} \).

We emphasize that under the condition of \( \mathcal{J} \)-completeness of the WO \( W_{\pm}(H, H_0; \mathcal{J}) \) the closure of its range can be described in terms independent of \( H_0 \).

2. The \( \mathcal{J} \)-completeness of the “direct” WO \( W_{\pm}(H, H_0; \mathcal{J}) \) can be related to the existence of the “adjoint” WO \( W_{\pm}(H_0, H; \mathcal{J}^*) \) without additional a priori assumptions of the type (2.1.8). This relation can again be realized (cf. Theorem 2.2.1) in terms of the WO for the auxiliary “triple” \( H_0, H, \mathcal{J}^* \).

**Theorem 4.** Let the WO \( W_{\pm}(H, H_0; \mathcal{J}) \) exist. Then the existence of the WO \( W_{\pm}(H_0, H; \mathcal{J}^*) \) is equivalent to the following two conditions: (1) the operator \( W_{\pm}(H, H_0; \mathcal{J}) \) is \( \mathcal{J} \)-complete, (2) the strong WO \( W_{\pm}(H_0, H; \mathcal{J}^*) \) exists.

**Proof.** Suppose the WO \( W_{\pm}(H, H_0; \mathcal{J}) \) exists. It is then adjoint to the WO \( W_{\pm}(H_0, H; \mathcal{J}) = W_{\pm}^*, \) and its kernel is equal to \( \mathfrak{N}_{\pm} \). By (2.1.13) from this it follows that \( \mathfrak{N}_{\pm} = \mathfrak{N}(W_{\pm}) \). This coincides with the definition of \( \mathcal{J} \)-completeness of \( W_{\pm} \). The existence of the operator

\[ \Omega_{\pm} := \text{W}_{\pm}(H_0, H; \mathcal{J}) = \text{W}_{\pm}(H_0, H; \mathcal{J}^*) \]

follows from Theorem 2.1.7 on multiplication of WO.

Conversely, suppose that \( W_{\pm} \) is \( \mathcal{J} \)-complete and \( \Omega_{\pm} \) exists. The limit of \( \mathcal{N}(t = 0) \mathcal{J} \mathcal{U}(t) \mathcal{P} f = \mathcal{J} \mathcal{U}(t) \mathcal{P} f \) Clearly exists and is equal to zero for \( f \in \mathfrak{N}_{\pm} \). Further, \( \mathfrak{N}(\Omega_{\pm}) = R(W_{\pm}) \) by the \( \mathcal{J} \)-completeness of \( W_{\pm} \). For \( f \in R(W_{\pm}) \) some \( f_0 \in \mathfrak{N}(\mathcal{J}) \) relation (2.3.1) is satisfied, and hence also

\[ \lim_{t \to \pm \infty} \| \mathcal{N}(t) \mathcal{J} \mathcal{U}(t) f - \mathcal{U}(t) \mathcal{J} \mathcal{U}(t) P_0 f_0 \| = 0. \]

To prove the existence of the WO \( W_{\pm}(H_0, H; \mathcal{J}) \) it remains to consider the existence of \( \Omega_{\pm} \).

If (2.18) is satisfied it is not necessary to consider the auxiliary triple \( H_0, H, \mathcal{J}^* \).

**Corollary 5.** Suppose the operator \( W_{\pm}(H, H_0; \mathcal{J}) \) exists and condition (2.1.8) holds. Then the existence of the WO \( W_{\pm}(H_0, H; \mathcal{J}^*) \) is equivalent to \( \mathcal{J} \)-completeness of the WO \( W_{\pm}(H, H_0; \mathcal{J}) \).

\[ \mathfrak{N}(\mathcal{J}) \]

§2. \( \mathcal{J} \)-COMPLETENESS

**Proof.** Under condition (2.1.8) the WO \( W_{\pm}(H_0, H; \mathcal{J}) \) exists and (1) is equal to \( P_0 \).

\[ \mathfrak{N}(\mathcal{J}) \]

It is natural to compare this assertion with Corollary 2.3.10, bearing in mind that under conditions (2.1.8) and (2.3.9) the definitions of \( \mathcal{J} \)-completeness and completeness coincide.

The concept of \( \mathcal{J} \)-completeness and assertions connected with it are preserved for the Abelian WO \( \mathfrak{N}(H, H_0; \mathcal{J}) \) if the limit of (1) is understood in the Abel sense.

3. We now present sufficient conditions on the operator \( \mathcal{J} \) under which for any selfadjoint \( H \)

\[ \mathfrak{N}(H, \mathcal{J}) = \mathfrak{S}(\mathcal{J}) \]

In this case the definition of \( \mathcal{J} \)-completeness reduces to equality (2.3.3).

**Lemma 6.** Suppose zero is not a point of the essential spectrum of the non-negative operator \( \mathfrak{S}(\mathcal{J}) \). Then equality (4) is satisfied for any selfadjoint operator \( H \).

**Proof.** The assumption regarding the spectrum of \( \mathfrak{S}(\mathcal{J}) \) is equivalent to \( \mathfrak{S}(\mathcal{J}) = A + K \), where the operator \( A = A^* \) is boundedly invertible \((A \equiv A^*, a > 0)\), while \( K \) is compact (finite-dimensional). Let \( g \in \mathfrak{N}(\mathcal{J}) \). By Lemma 1.4.1 it follows from (2) that \( \| A U(t) P g \| \to 0 \) as \( t \to \pm \infty \), and hence \( P g = 0 \). Thus \( \mathfrak{N}(\mathcal{J}) \subset \mathfrak{S}(\mathcal{J}) \) which proves (4).

Combining Theorem 4 with Lemma 6, we obtain

**Corollary 7.** Suppose \( \mathcal{J} \) satisfies the hypothesis of Lemma 6. Then the existence of the WO \( W_{\pm}(H, H_0; \mathcal{J}) \) and \( W_{\pm}(H_0, H; \mathcal{J}^*) \) implies equality (2.3.3).

We note that for a Fredholm operator \( \mathcal{J} \) (see Part 3 of §1.6) the condition of Lemma 6 is clearly satisfied.

If the point zero belongs to the essential spectrum of the operator \( \mathfrak{S}(\mathcal{J}) \), then equality (4) generally speaking, is violated. In this connection we consider two examples.

**Example 8.** Suppose \( H \) is the operator of multiplication by \( x \) in \( \mathfrak{S}(\mathcal{J}) = L_2(\mathbb{R}) \). \( \Phi : L_2(\mathbb{R}) \to L_2(\mathbb{R}) \) the Fourier transform and \( \Gamma_{\pm} \) the multiplication by the indicator \( x_{\pm} \) of the semiaxis \( \pm x > 0 \). Consider the two identifications \( \mathfrak{N}(\mathcal{J}) = \Phi \Gamma_{\pm} : L_2(\mathbb{R}) \to L_2(\mathbb{R}) \). Then

\[ \lim_{t \to \pm \infty} \| \mathcal{N}(t) \mathcal{J} \mathcal{U}(t) g \| = 0, \quad \lim_{t \to \pm \infty} \| \mathcal{N}(t) \mathcal{U}(t) g \| = \| g \|. \]

From this it follows that \( \mathfrak{N}_{\pm} = \mathfrak{S}(\mathcal{J}) \) for \( \mathcal{J} = \mathcal{J}_a \) and \( \mathfrak{N}_{\pm} = \{ 0 \} \) for \( \mathcal{J} = \mathcal{J}_a \). On the contrary, if the sign of time is changed, then \( \mathfrak{N}_{\pm} = \{ 0 \} \) for \( \mathcal{J} = \mathcal{J}_a \) and \( \mathfrak{N}_{\pm} = \mathfrak{S}(\mathcal{J}) \) for \( \mathcal{J} = \mathcal{J}_a \). In this example the operators \( \mathcal{J}_a, \mathcal{J}_a^*, \) and \( \mathcal{J}_a, \mathcal{J}_a^* \) are projections in \( L_2(\mathbb{R}) \) onto the Hardy classes (see §2.1) of functions analytic in the upper and lower half-planes. For both operators the point zero is an eigenvalue of infinite multiplicity. For each of the identifications \( \mathfrak{N}(\mathcal{J}) \) the relation (4) is satisfied for one and is violated for the other sign.

**Example 9.** Suppose \( H \) is the operator of multiplication by \( x \) in \( \mathfrak{S}(\mathcal{J}) = L_2(\mathbb{R}) \), and \( Z : L_2(\mathbb{R}) \to L_2(\mathbb{R}) \) is the operator of restriction to the positive semiaxis. In the notation of
3. Scattering for multiplicative perturbations

Problems with multiplicative perturbations arise, for example, in the study of scattering of waves by inhomogeneities of a medium. Here we consider such problems in an abstract formulation. For them it is necessary to construct scattering theory in a pair of spaces. In this circumstance it is natural to consider at once entire collections of identifications that are equivalent to one another.

Let \( H_0 \) be an arbitrary selfadjoint operator in a Hilbert space \( \mathcal{H}_0 \) with scalar product \((\cdot, \cdot)_0\). We consider some bounded positive definition operator \( M \),

\[
0 < m_0 I \leq M \leq m_1 I < \infty.
\]  

(1)

We denote by \( \mathcal{K} \) the Hilbert space with scalar product

\[
(f, g) = (Mf, g)_0, \quad f, g \in \mathcal{K}_0. 
\]  

(2)

By (1) the norms in \( \mathcal{K}_0 \) and \( \mathcal{K} \) are equivalent, while different, with respect to scalar product and norm, the spaces \( \mathcal{K}_0 \) and \( \mathcal{K} \) coincide elementwise. We define the Hamiltonian \( H \) in the space \( \mathcal{K} \) on the set \( \mathcal{D}(H) = \mathcal{D}(H_0) \) by the relation \( H = M^{-1}H_0 \). Starting from the definition of the adjoint operator and using the fact that \( H_0 = H_0^* \), it is easy to see that \( H \) is also selfadjoint.

Since \( H_0 \) and \( H \) act in different Hilbert spaces, in the construction of scattering theory for the pair \( H_0, H \) it is necessary to introduce an identification operator \( J: \mathcal{K}_0 \to \mathcal{K} \). For \( J \) it is possible, for example, to take the operator \( J = M^{-1/2} \), which maps \( \mathcal{K}_0 \) unitarily onto \( \mathcal{K} \). However, it is more convenient to choose an operator \( J \) such that the "perturbation" \( HJ - H_0 \) has the simplest possible structure. In applications the identity mapping \( I_0: \mathcal{K}_0 \to \mathcal{K}_0 \) is usually suitable. Of course the inverse operator to \( I_0 \) exists (and is also the identity), but, according to (2), \( I_0 = M \), and hence this identification is not isometric.

Thus, for multiplicative perturbations an identification \( J \) has a number of special properties. For such \( J \) the properties of the WO \( W_\pm(H, H_0; 3) \) are essentially the same as in the case of a single space (for the identity identification). They are described in the next assertion, which is valid for arbitrary selfadjoint operators \( H_0 \) and \( H \). This assertion follows immediately from

the results of \$2.3 \) (see Theorem 6, Proposition 11, and Corollary 12 of that section).

**Lemma 1.** Suppose the operator \( J: \mathcal{K}_0 \to \mathcal{K} \) is boundedly invertible, condition (2.1.8) is satisfied, and the WO \( W_\pm(H_0, H; 3) \) exist. Then the WO \( W_\pm(H_0, H; 3) \) and \( W_\pm(H_0, H; 3) \) are isometries on \( \mathcal{K}_0 \) and \( \mathcal{K} \), respectively, and both are complete. Moreover, for any operator \( J_0: \mathcal{K} \to \mathcal{K}_0 \) satisfying (2.3.5), the WO \( W_\pm(H_0, H; J_0) = W_\pm(H_0, H; J_0) \) exists.

In applications, the operator \( M \) satisfies the relation

\[
\lim_{\varepsilon \to 0^+} (M^n - I)U_\varepsilon(t)P_0 = 0, \quad \alpha \in \mathbb{R}. 
\]  

(3)

We note that by (1) the operator \((M^n - I)(M^n - I)^{-1}\) is bounded for any \( \alpha, \beta \in \mathbb{R} \). Therefore, assuming (3) for some \( \alpha \), we find that this relation is satisfied for all \( \alpha \).

For \( \alpha = 1 \) condition (3) means that the identification \( J = I_0 \) satisfies condition (2.1.8). For \( \alpha = -1/2 \), from (3) it follows that this identification is \( H_0 \)-equivalent (in the sense of Definition 2.1.8) to the unitary operator \( M^{-1/2}: \mathcal{K}_0 \to \mathcal{K} \).

We now apply Lemma 1 to the pair of operators \( H_0, H \) in question and the identification \( J = I_0: \mathcal{K}_0 \to \mathcal{K} \). The next assertion shows that for multiplicative perturbations all the properties of the WO (the most important of them is completeness) are derived from the existence of the "direct" and "adjoint" WO.

**Theorem 2.** Suppose that conditions (1), (3) are satisfied, \( H = M^{-1}H_0 \), and the WO \( W_\pm(H_0, H; I_0) \) exist. Then they are isometric and complete. Moreover, for any operator \( N \in \mathcal{D}(\mathcal{K}_0) \), \( H_0 \)-equivalent to the identity, the WO

\[
W_\pm(H_0, H; NI_0^{-1}) = W_\pm(H_0, H; I_0)
\]  

exist.

4. Equations of second order in time

We shall establish the connection between the scattering theory for the Schrödinger equation and the scattering theory for an equation of second order in time. For this we use the invariance principle (IP).

Let us consider the Cauchy problem (cf. Part 1 of §1.4)

\[
\frac{d^2\psi}{dt^2} + H\psi = 0, \quad \psi(0) = f, \quad \frac{d\psi(0)}{dt} = g,
\]  

(1)

with a positive operator \( H \) (this means that \( H \geq 0 \) and \( 0 \notin \sigma^0(H) \)) in the Hilbert space \( \mathcal{K} \). In (1) it is assumed that the function \( \psi(t) \) is twice
continuously differentiable and that \( \psi(t) \in \mathcal{D}(H) \) for all \( t \in \mathbb{R} \). By equation (1) the “energy”
\[
\|\psi\|^2 + \|H^{1/2} \psi\|^2, \quad \psi_t = \frac{d\psi}{dt},
\]
does not depend on \( t \), and therefore the solution of problem (1) is unique. It is not hard to see that for \( f \in \mathcal{D}(H) \), \( g \in \mathcal{D}(H^{1/2}) \) a solution exists and is given by the formula
\[
\psi(t) = \cos(H^{1/2}t)f + H^{-1/2} \sin(H^{1/2}t)g.
\]  
A convenient method of investigating problem (1) consists in the reduction to a system of first order in \( t \). Namely, by the substitution
\[
\psi_t = x_1, \quad H^{1/2} \psi = x_2
\]  
(1) reduces to the problem
\[
i \frac{dx}{dt} = hx, \quad x(0) = a,
\]  
where
\[
h = \begin{pmatrix} 0 & -iH^{1/2} \\ iH^{1/2} & 0 \end{pmatrix}, \quad x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \quad a = \begin{pmatrix} g \\ H^{1/2}f \end{pmatrix}.
\]  
The operator \( h \) is selfadjoint in the space \( \mathcal{H} = \mathcal{H} \otimes \mathcal{H} \) on the domain \( \mathcal{D}(h) = \mathcal{D}(H^{1/2}) \otimes \mathcal{D}(H^{1/2}) \). It can be “diagonalized” by the equalities
\[
h = \mathcal{F}\left( \begin{pmatrix} H^{1/2} \\ 0 \\ 0 \end{pmatrix} \right) \mathcal{F}^*, \quad \mathcal{F} = 2^{-1/2} \begin{pmatrix} -i & i \\ 1 & 1 \end{pmatrix}, \quad \mathcal{F}^* = \mathcal{F}^{-1}.
\]  
(7)

For any \( a \in \mathcal{D}(h) \) the solution of problem (5) is given by the formula
\[
x(t) = \exp(-iht) a.
\]  
(8)

From this, with the help of equalities (4), (6) and (7), it is possible to extract the representation (3). The conservation of the quantity (2) here is a consequence of the unitarity of the operator \( \exp(-iht) \) in \( \mathcal{H} \). Assuming the connection (4), it is possible to replace the study of the asymptotics as \( t \to \pm \infty \) of the solution \( \psi(t) \) of problem (1) by the analogous question for the function (8).

Along with (1), we consider the free problem
\[
\frac{d^2 \psi_0}{dt^2} + H_0 \psi_0 = 0, \quad \psi_0(0) = f_0, \quad \frac{d\psi_0(0)}{dt} = g_0,
\]
of the same type with a positive operator \( H_0 \) acting in the same space \( \mathcal{H} \). This equation we also reduce to a system of first order \( i dx_0/dt = h_0 x_0 \), \( x_0(0) = a_0 \), of the form (5), (6).

As explained in the Introduction, comparison of the asymptotics in the space \( \mathcal{H} \) of the functions (8) and
\[
x_0(t) = \exp(-i h_0 t) a_0
\]  
(9)
is conveniently carried out in terms of the WO for the pair \( h_0, h \). It follows from (7) that the existence of each of the WO \( W_+(h, h_0) \), and \( W_-(h, h_0) \) is equivalent to the existence of both WO \( W_+(H^{1/2}, H_0^{1/2}) \), and
\[
W_+(h, h_0) = \mathcal{T} \left( \begin{pmatrix} W_+(H^{1/2}, H_0^{1/2}) & 0 \\ 0 & W_-(H^{1/2}, H_0^{1/2}) \end{pmatrix} \right) \mathcal{T}^*.
\]

This equality shows that the completeness of each of the WO \( W_+(h, h_0) \) and \( W_-(h, h_0) \) is equivalent to the completeness of both of the WO \( W_+(H^{1/2}, H_0^{1/2}) \).

Suppose now that for the pair of operators \( H_0, H \) and the function \( \varphi(\lambda) = \lambda^{1/2} \) the IP holds. Then from the existence of the WO \( W_+(H, H_0) \) we obtain the existence of the WO \( W_{\pm}(H^{1/2}, H_0^{1/2}) \) and their coincidence with \( W_\pm(H, H_0) \). We thus have

**Theorem 1.** Suppose for the pair \( H_0, H \) both WO \( W_\pm = W_\pm(H, H_0) \) exist and are complete, and for \( \varphi(\lambda) = \lambda^{1/2} \) this pair the IP holds. Then the WO for the pair \( h_0, h \) also exist, are complete, and can be constructed by the formula
\[
W_\pm(h, h_0) = 2^{-1} \begin{pmatrix} W_\pm + W_\mp & i(W_\pm - W_\mp) \\ i(W_\pm + W_\mp) & W_\pm + W_\mp \end{pmatrix}.
\]  
(10)

Under the conditions of this assertion for the functions (8), where \( a \in \mathcal{H}_0(a) \), and (9) there is the relation
\[
\lim_{t \to \pm \infty} \|x(t) - x_0(t)\| = 0, \quad a_0 = W_\pm^{-1}(h, h_0)a.
\]  
(11)

Here the operator \( W_\pm(h, h_0) \) is constructed according to formula (10) in terms of the WO for the auxiliary Schrödinger equations with Hamiltonians \( H_0 \) and \( H \). In virtue of (4), in the original variables relation (11) means that as \( t \to \pm \infty \) the following asymptotics hold in \( \mathcal{H} \)
\[
\frac{\partial \psi}{\partial t} \sim \frac{\partial \psi_0}{\partial t}, \quad H^{1/2} \psi(t) \sim H_0^{1/2} \psi_0(t).
\]  
(12)

In applications, the asymptotics of \( \psi(t) \) in other metrics are also of interest. For example, if the metrics of \( \|H^{1/2}u\| \) and \( \|H_0^{1/2}u\| \) are equivalent, then it is natural to replace the second of relations (12) by the equality
\[
\lim_{t \to \pm \infty} \|H^{1/2}_0(\psi(t) - \psi_0(t))\| = 0.
\]

In terms of equations of first order the proof of this asymptotics requires consideration of the WO for the pair \( h_0, h \) with a nontrivial identification. In this regard see the paper of T. Kato [110] and Volume 3 of the course [18].

§5. The IP for Abelian WO

For the WO (2.1.1) the IP formulated in Parts 1 and 3 of §2.6 is, generally speaking, not true. At the same time the absolute formulation of it presented
there turns out to be valid for the Abelian WO defined by equality (2.2.9).

Suppose the function \( \varphi \) is admissible (in the sense of Definition 2.6.2) relative to the selfadjoint operators \( H_0 \) and \( H \); then \( h_0 = \varphi(H_0) \), \( h = \varphi(H) \), and \( \Omega_x = \{ \lambda \in \Omega : \pm \varphi'(\lambda) > 0 \} \).

**Theorem 1.** If the local Abelian WO \( \mathfrak{A}_x(H, H_0; \mathcal{I}_x, \Omega_x) \) where \( \nu = "+" \) or \( \nu = "-" \), exists, the Abelian WO \( \mathfrak{A}_x(h, h_0; \mathcal{I}_x, \Omega_x) \) also exists and

\[
\mathfrak{A}_x(h, h_0; \mathcal{I}_x, \Omega_x) = \mathfrak{A}_x(H, H_0; \mathcal{I}_x, \Omega_x).
\]

For brevity we set

\[
\mathfrak{A}_x(H, H_0; \mathcal{I}_x, \Omega_x) = \mathfrak{A}_x(\Omega_x).
\]

If both WO \( \mathfrak{A}_x(\Omega_x) \) and \( \mathfrak{A}_x(\Omega_x) \) exist, then by Theorem 1 the "global" WO

\[
\mathfrak{A}_x(h, h_0; \mathcal{I}_x, \Omega_x) = \mathfrak{A}_x(\Omega_x) + \mathfrak{A}_x(\Omega_x)
\]

also exists.

We recall that definition (2.2.9) does not depend on the choice of the averaging kernel \( \omega(t) = e\omega(\varepsilon t) \) provided that \( \omega \) satisfies the conditions listed in Part 3 of §2.2. It is convenient for us to prove Theorem 1 for the Cesàro definition (2.2.10) of the WO. For the WO (2.2.9), and in particular (2.2.10), a relation of the form (2.1.23) and also the assertion of Lemma 2.6.3 are preserved. Therefore, to prove Theorem 1 it suffices to show that the validity of the equality

\[
\lim_{t \to 0} T^{-1} \int_0^T \| (\mathfrak{A}_x(\Omega_x) - \mathcal{I}) U_0(\pm t)f \|^p dt = 0
\]

for all \( f \in E_{\Omega}(\mathfrak{A}_x) \) implies the analogous equality for \( u_0(t) = \exp(-ih_0t) \):

\[
\lim_{t \to 0} T^{-1} \int_0^T \| (\mathfrak{A}_x(\Omega_x) - \mathcal{I}) U_0(\pm t)f \|^p dt = 0.
\]

We establish a more general assertion.

**Theorem 2.** Let \( H \) be an arbitrary selfadjoint operator, let the function \( \varphi \) be admissible on the set \( \Omega_x \) with respect to \( H \), and let \( K \) be a bounded operator. Suppose that for any \( f \in E_{\Omega}(\mathfrak{A}_x) \)

\[
\lim_{t \to 0} T^{-1} \int_0^T \| Ku(\pm t)f \|^p dt = 0, \quad 0 < p < \infty.
\]

Then for \( u(t) = \exp(-i\varphi(H)t) \) and any \( g \in E_{\Omega}(\mathfrak{A}_x) \)

\[
\lim_{t \to 0} T^{-1} \int_0^T \| Ku(\pm t)g \|^p dt = 0.
\]

**Proof.** Because of the uniform boundedness of the integrands relations (1) and (2) for different \( p \) are equivalent to one another. We use (1) for

\[
p = 2 \quad \text{and establish (2) for } p = 4.
\]

To be specific we consider (1), (2) for \( \nu = "+" \) and the sign " + " at the variable \( t \). We shall show that if (1) holds for some element \( f \in \mathfrak{A}_x \) with a finite quantity (2.5.2), then (2) is satisfied for elements \( g = \psi(H)f \), where \( \psi \in C_{\mathcal{I}}(\Omega_x) \) and \( \Omega_x \) is any of the component intervals of \( \Omega_x \). This is sufficient, since linear combinations of such elements \( g \) form a dense set in \( E_{\Omega}(\mathfrak{A}_x) \).

We first obtain a convenient representation for \( \| Ku(t)g \|^p \). For an arbitrary \( x \in \mathfrak{A}_x \) from the spectral theorem it follows that

\[
(\mathfrak{A}_x(\Omega_x) - \mathcal{I}) U_0(\pm t)f \|^p dt = 0.
\]

By the estimate (2.5.3) for \( f \in \mathfrak{A}_x \) the function \( d(E(\lambda)f, x)/d\lambda \) belongs to \( L_2(\mathbb{R}) \), and its Fourier transform is equal to \( (2\pi)^{-1} L_2(\mathbb{R}) \). Since, of course, also \( \exp(-it\varphi(t)) \psi(t) \in L_2(\mathbb{R}) \), it is possible to apply the Parseval equality to the integral (3). This leads to the representation

\[
(\mathfrak{A}_x(\Omega_x) - \mathcal{I}) U_0(\pm t)f \|^p dt = 0.
\]

We establish a more general assertion.

**Theorem 2.** Let \( H \) be an arbitrary selfadjoint operator, let the function \( \varphi \) be admissible on the set \( \Omega_x \) with respect to \( H \), and let \( K \) be a bounded operator. Suppose that for any \( f \in E_{\Omega}(\mathfrak{A}_x) \)

\[
\lim_{t \to 0} T^{-1} \int_0^T \| Ku(\pm t)f \|^p dt = 0, \quad 0 < p < \infty.
\]

Then for \( u(t) = \exp(-i\varphi(H)t) \) and any \( g \in E_{\Omega}(\mathfrak{A}_x) \)

\[
\lim_{t \to 0} T^{-1} \int_0^T \| Ku(\pm t)g \|^p dt = 0.
\]

**Proof.** Because of the uniform boundedness of the integrands relations (1) and (2) for different \( p \) are equivalent to one another. We use (1) for

\[
p = 2 \quad \text{and establish (2) for } p = 4.
\]
the first of these terms tends to zero as $t \to \infty$ in the usual sense (without averaging). Applying the Schwarz inequality, we find that the square of the integral over $\Lambda_t$ does not exceed

$$\int_{\Lambda_t} |G(t,s)|^2 \, ds \int_{\mathbb{R}} |(U(s)f, K^* K u(t)g)|^2 \, ds.$$ 

The first factor here tends to zero by (7). The second factor, according to (2.5.3), is estimated by $2\pi r(f)\|K^* K\|g\|^2$ and is hence uniformly bounded with respect to $t$.

We further show that the term in (6) corresponding to the interval $(0, at)$ tends to zero in the Cesàro sense, i.e.,

$$\lim_{T \to \infty} T^{-1} \int_0^T dt \int_0^{at} G(t, s)(U(s)f, K^* K u(t)g) \, ds = 0. \quad (9)$$

We apply the Schwarz inequality to the integral on $s$ in (9), extend the interval to $(0, aT)$ and take into account equality (8). By interchanging the order of integration, we reduce the verification of (9) to the proof that

$$\lim_{T \to \infty} T^{-1} \int_0^T dt \int_0^{aT} |(K^* K U(s)f, u(t)g)|^2 \, ds = 0. \quad (10)$$

We now use the estimate (2.5.3) for the integral on $t$ and the operator $h = \varphi(H)$. We note that the element $g = \varphi(H) f \in \mathcal{F}_h$ and hence

$$\int_0^T |(K^* K U(s)f, u(t)g)|^2 \, dt \leq 2\pi r(h)\|K^* K U(s)f\|^2.$$ 

From this it is evident that (10) is a consequence of relation (1) for $p = 2$. Thus, the entire expression (6) tends to zero in the Cesàro sense. This completes the verification of equality (2) for $p = 4$. □

As already noted, Theorem 1 follows directly from Theorem 2. According to Theorem 1, the IP in a somewhat weakened form is always valid for the ordinary WO.

**Corollary 3.** Suppose the WO in both sides of (2.6.12)$_+$ or (2.6.12)$_-$ exist. They are then related by equality (2.6.12)$_+$ or (2.6.12)$_-$. 

For functions $\varphi$ piecewise linear on $\Omega$ (where $\varphi(\lambda) = a\lambda + b$, for $\lambda \in \Omega_\lambda$) the IP, of course, is true for any pairs $H_\lambda$, $H$. If, however, $\varphi'(\lambda)$ is strictly monotone on at least one of the intervals $\Omega_\lambda$, then for a given $H_\lambda$ it is possible to choose the operator $H$ in such a way that (for $\lambda = I$) the WO $W_{\lambda}(H, H_\lambda)$ exists, while the WO $W_{\lambda}(\varphi(H), \varphi(H_\lambda))$ does not. The pertinent construction can be found in the original paper of M. Wollenberg [140] or in the book [30].

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**CHAPTER 4**

**Scattering for Relatively Smooth Perturbations**

This chapter essentially decomposes into two independent parts comprised of §§1, 2 and §§3–7.

In the first part we study the Friedrichs-Faddeev model in which an operator of multiplication is perturbed by an integral operator with a smooth matrix kernel. Application of the stationary method requires investigation of the resolvent of the full Hamiltonian. Such an investigation is carried out in §1 by means of a suitable integral equation. It is important that in an auxiliary Banach space of Hölder vector-valued functions this equation is Fredholm. In §2 within the framework of the Friedrichs-Faddeev model the stationary scheme of §§2.7, 2.8 is realized.

In §§3–7 we expound the abstract theory of smooth perturbations. In §3 there is introduced convenient, unitarily invariant concept of smoothness (Kato smoothness) of any operator $G$ relative to a selfadjoint Hamiltonian $H$. In §4 there are presented two sufficient conditions on a pair $H$, $G$ which ensure $H$-smoothness of the operator $G$. In the construction of scattering theory in §5 it is assumed that the perturbation $V = H - 2H_0$ admits a factorization $V = G^* G_0$, where the factors $G_0$ and $G$ are smooth relative to the operators $H_0$ and $H$ respectively. Since the concept of $H$-smoothness can be formulated in an equivalent way in terms of the resolvent of $H$ as well as in terms of its unitary group $U(t)$, for the perturbations considered here the stationary and time-dependent approaches practically coalesce. In §5 there is given a direct proof of the existence of time-dependent WOs. The results of §5 can also be obtained by stationary means, but this is put off to §§7.3, 7.4. It is there explained that in the case of smooth perturbations for the WO and the scattering operator and matrix, the formula representations of §§2.7, 2.8 hold.

In applications smoothness with respect to an unperturbed operator $H_0$ is usually verified by direct computations. On the other hand, the study of smoothness relative to the full Hamiltonian $H$ is a more substantial problem. In §6 we present a method making it possible for relatively compact perturbations to reduce the question of $H$-smoothness of $G$ to an investigation of $H_0$-smoothness of the operators $G_0$ and $G$. However, $H_0$-smoothness
must here be understood in a certain stronger sense. In §7 the method of §6 is used to study the singular spectrum of the operator $H$. To a large extent §§6, 7 are based on considerations developed in connection with the Friedrichs-Faddeev model. However, in contrast to §§1, 2 the assumptions are made not regarding the perturbation $V$ itself, but regarding the factors $G_0$ and $G$ in the relation $V = G^* G_0$. Because of this it is possible to avoid introducing an auxiliary Banach space.

§1. The Friedrichs-Faddeev model

Any selfadjoint operator $H_0$ with absolutely continuous spectrum of constant (possibly infinite) multiplicity can be realized (see §1.5) as the operator of multiplication by the independent variable $(\lambda, \mu)$ in the Hilbert space $L_2(\sigma; \mathbb{H})$. Here $\sigma$ is the core of the spectrum of the operator $H_0$, and $\mathbb{H}$ is an "auxiliary" Hilbert space whose dimension is equal to the multiplicity of the spectrum. In the Friedrichs-Faddeev model the case is considered where $\sigma = \sigma$ is a closed interval, while the perturbation $V$ of the operator $H_0$ is an integral operator with smooth kernel $v(\lambda, \mu)$. Within the framework of this model it is possible not only to construct the theory, i.e., prove the existence and completeness of WO $W_\infty(H, H_0)$ (corresponding to $\mathcal{J} = I$) but also to verify the absence of singular continuous spectrum for the operator $H = H_0 + V$.

1. We shall give a precise description of the Friedrichs-Faddeev model. For simplicity we assume that the interval $\sigma = [a, b]$ is finite. Let $\mathcal{H} = L_2(\sigma; \mathbb{H})$, $(H_0 f)(\lambda) = \lambda f(\lambda)$,

$$(V f)(\lambda) = \int_a^b v(\lambda, \mu) f(\mu) d\mu. \quad (1)$$

The "kernel" $v(\lambda, \mu)$ is assumed to be a compact operator in $\mathbb{H}$. We denote by $| \cdot |$ and $(\cdot, \cdot)$ the norm (including the operator norm) and the scalar product in $\mathbb{H}$. We assume that $v(\lambda, \mu)$ is Hölder-continuous, i.e.,

$$\sup_{\lambda, \mu, \lambda', \mu' \in \sigma} |v(\lambda, \mu) + v(\lambda', \mu') - v(\lambda, \mu)| (|\lambda' - \lambda| + |\mu' - \mu|)^{-\alpha_0} < \infty, \quad (2)$$

and that it vanishes at the end points of the interval:

$$v(\lambda, a) = v(\lambda, b) = v(a, \mu) = v(b, \mu) = 0. \quad (3)$$

The kernel $v(\lambda, \mu) = v^*(\mu, \lambda)$ is Hermitian, so that the perturbation $V$ is symmetric.

The operator $V$ is compact in $\mathcal{H}$ if the kernel $v(\lambda, \mu)$ is compact in $\mathbb{H}$ and is uniformly bounded for $\lambda, \mu \in \sigma$. Therefore, by the theorem of H. Weyl the essential spectrum of the operator $H = H_0 + V$ coincides with the spectrum $\sigma(H_0) = \sigma$ of the operator $H_0$. In particular, outside the segment $\sigma$ the spectrum of $H$ is exhausted by eigenvalues of finite multiplicities accumulating only at the end points $a$ and $b$. Scattering theory makes it possible to study the spectrum of the operator $H$ in considerably more detail.

We note that according to the classical Fredholm result (see the book [7]) for $\alpha_0 > 1/2$ and $\dim \mathbb{H} < \infty$ condition (2) ensures the membership of the operator $V = V^*$ in the trace class. However, for $\dim \mathbb{H} = \infty$ trace class scattering theory is obviously not applicable to the model in question.

For the operator $H = H_0 + V$ in place of the resolvent $R(z) = (H - z)^{-1}$ itself it is more convenient to consider the operator

$$T(z) = V - V R(z) V, \quad (4)$$

in terms of which the resolvent can be expressed by the equality

$$R(z) = R_0(z) - R_0(z) T(z) R_0(z), \quad R_0(z) = (H_0 - z)^{-1}. \quad (5)$$

According to (4), the operator-valued function $T(z)$ is meromorphic in $\rho(H_0)$ and has poles only at points of the discrete spectrum of the operator $H$. We denote by $\Pi = \rho(H_0)$ the (closed) complex plane with a cut along the spectrum $\sigma = \sigma(H_0)$ of the operator $H_0$. Results on the continuity of $T(z)$ with respect to $z \in \Pi$ needed for the stationary construction of scattering theory are formulated in the next assertion.

**Theorem 1.** Suppose condition (2), where $\alpha_0 > 1/2$, and condition (3) are satisfied. Then the point spectrum $\sigma^{(0)}(\sigma)$ of the operator $H$ is finite, and its eigenvalues have finite multiplicities. The operator $T(z)$ is an integral operator, and its kernel $t(\lambda, \mu; z)$ is Hölder continuous with respect to the arguments $\lambda, \mu \in \sigma$ and $z \in \Pi$, $z \neq \sigma^{(0)}$, with any exponent $\alpha < \alpha_0$, i.e.,

$$|t(\lambda', \mu', z') - t(\lambda, \mu; z)| \leq C(|\lambda' - \lambda| + |\mu' - \mu| + |z' - z|)^{\alpha}. \quad (6)$$

Here the constant $C$ does not depend on $z, z' \in \Pi$ outside arbitrary neighborhoods of the points of $\sigma^{(0)}$. On the boundary of the square $\sigma \times \sigma$ the kernel $t(\cdot, \cdot; z)$ vanishes. The values of the kernel $t$ are compact operators in $\mathbb{H}$.

By (4), $T(\overline{z}) = T^*(z)$. In terms of the kernel $t(\lambda, \mu; z)$ this means that

$$t(\lambda, \mu; \overline{z}) = t^*(\mu, \lambda; z). \quad (7)$$

From Theorem 1 it is not hard to derive that, in addition to eigenvalues, the spectrum of the operator $H$ may contain only an absolutely continuous component. This, however, we postpone to §2.

2. To study the operator $T(z)$ we apply the equation

$$T(z) = V - V R_0(z) T(z), \quad \text{Im} \, z \neq 0, \quad (8)$$

which is equivalent to the resolvent identity (1.9.5) for $\mathcal{J} = I$. Assuming that the operator $T(z)$ is an integral operator, we write equation (8) in terms of its kernel

$$t(\lambda, \mu; z) = v(\lambda, \mu) - \int_a^b v(\lambda, \nu) (\nu - z)^{-1} t(\nu, \mu; z) d\nu. \quad (9)$$
From the solvability of this equation for \( \text{Im} \, z \neq 0 \) established below it follows that the integral operator with kernel \( t(\lambda, \mu; z) \) satisfies equation (8). Therefore, this operator coincides with \( T(z) \), i.e., \( T(z) \) is, in fact, itself an integral operator with kernel \( t(\lambda, \mu; z) \).

For fixed \( \mu \) (9) is considered as an equation for the function \( t(\lambda, \mu; z) \) of the variable \( \lambda \). In order to avoid dealing with operator-valued functions, we apply both sides of (9) to an element \( \varphi \in \mathfrak{h} \) and write it in the form

\[
t(z) = v + A(z)t(z).
\]

Here \( v \) is the vector-valued function \( v(\cdot) = v(\cdot, \mu)\varphi \), \( t(z) \) is the vector-valued function \( t(\cdot; z) = t(\cdot, \mu; z)\varphi \), and \( A(z) = -VR_0(z) \) is the integral operator with kernel \( -v(\lambda, \nu)(v(\nu - z))^{-1} \). We shall consider equation (10) in the auxiliary Banach space of Hölder continuous functions. It is important that on the basis of Theorem 1.2.6 equation (10) in this space extends by continuity to all \( z \in \Pi \).

Namely, let \( C^\alpha = C^\alpha(\sigma; h) \) be the Banach space of vector-valued functions taking values in \( h \) with norm

\[
\| f \|_\alpha = \sup_{\lambda, \lambda'} (|f(\lambda)| + |f(\lambda') - f(\lambda)| |\lambda' - \lambda|^{\alpha}), \quad 0 < \alpha \leq 1.
\]

(11)

This space is not separable. We therefore further introduce the closure \( \hat{C}^\alpha = \hat{C}^\alpha(\sigma; h) \) of the class \( C^\alpha(\sigma; h) \) of infinitely differentiable functions with respect to the norm (11). We denote by \( C^\alpha_0 \) and \( \hat{C}^\alpha_0 \) the subspaces of functions \( f \) in \( C^\alpha \) and \( \hat{C}^\alpha \) vanishing at the end points of \( \sigma \). We note that \( C^\alpha_0 \subset C^\alpha_0 \), and \( \hat{C}^\alpha_0 \subset \hat{C}^\alpha_0 \), for any \( \alpha < \alpha_1 \). We shall need a criterion for compactness in the space \( \hat{C}^\alpha_0 \). Suppose a set of vector-valued functions \( f_\alpha(\lambda) \) is compact in \( h \) for any fixed \( \sigma \) and are uniformly (with respect to \( \lambda \)) Hölder continuous with some exponent \( \alpha_1 \). Then for any \( \alpha < \alpha_1 \) the set \( \{ f_\alpha \} \) belongs to \( \hat{C}^\alpha_0 \) and is compact in this space. In the scalar case \( \dim h = 1 \) this assertion is altogether analogous (see, for example [10]) to the familiar criterion for compactness in the space of continuous functions. Its generalization to the vector-valued case is achieved at the expense of the compactness of \( \{ f_\alpha(\lambda) \} \) in \( h \).

We shall consider the operator \( A(z) \) in the space of scales \( C^\alpha_0 \) and \( \hat{C}^\alpha_0 \). It follows from (2) that for any \( s \in [0, 1] \)

\[
|v(\lambda', \mu') - v(\lambda', \mu) + v(\lambda, \mu)| \leq C|\lambda' - \lambda|^{\alpha_1} \mu' - \mu^{\alpha_1 - s}.
\]

For \( f \in C^\alpha_0 \), \( \alpha < \alpha_0 \), from this we easily obtain the estimate

\[
\| [v(\lambda', s) - v(\lambda', s)] f(s) \|_{\alpha_1 - s} \leq C|\lambda' - \lambda|^{\alpha_1} \| f \|_s.
\]

(12)

We now apply Theorem 1.2.6 to the integral on the right-hand side of the equality

\[
(A(z)f)(\lambda') - (A(z)f)(\lambda) = \oint_{\sigma} \frac{v(\lambda, \mu' - v(\lambda', \mu)}{\mu - z} f(\mu) d\mu.
\]

From (1.2.4) and (12) it follows that

\[
[|A(z)f(\lambda') - (A(z)f)(\lambda)| + |z' - z|^{\alpha_1 - s}] 
\times \| [A(z)f(\lambda') - (A(z)f)(\lambda)] - (A(z')f)(\lambda') + (A(z')f)(\lambda) \| 
\leq C|\lambda' - \lambda|^{\alpha_1} \| f \|_s,
\]

(13)

Setting here \( \lambda' = a \) (or \( \lambda' = b \)) and considering condition (3), we find also an estimate for the function \( (A(z)f)(\lambda) \) itself. The estimate for the first term on the left-hand side of (13) shows that the operator \( A(z) \), for all \( z \in \Pi \), takes the space \( C^\alpha_0 \) into \( C^\alpha_0 \), where \( \alpha_1 \) is any number less than \( \alpha_0 \). Further, we note that the operators \( v(\lambda, \mu) \) are compact in \( h \) for all \( \lambda, \mu \in \sigma \), while \( s \) is arbitrary. Therefore, for any \( \alpha < \alpha_0 \) and \( \alpha < \alpha_0 \), the operator \( A(z) : C^\alpha_0 \rightarrow C^\alpha_0 \) is compact. Moreover, for \( s = \alpha_1 \alpha_0^{-1} \) from the estimate (13) for the second term on the left-hand side we obtain Hölder dependence (with exponent \( \alpha(1 - s) \)) of this operator on the parameter \( z \in \Pi \). We thus have

**Lemma 2.** For \( z \in \Pi \) and any \( \alpha < \alpha_0 \), \( \alpha_1 < \alpha_0 \), the operator \( A(z) : C^\alpha_0 \rightarrow C^\alpha_0 \) is compact. The operator-valued function \( A(z) \) is homomorphic in \( z \) on the open set \( \rho(H_0) \) and is Hölder continuous with exponent \( \gamma = (\alpha_0 - \alpha_1)\alpha_0^{-1} \) for \( z \in \Pi \).

We further note that for large \( z \) the norm of the operator \( A(z) \) does not exceed \( C|z|^{\alpha_1} \). Below, the operator \( A(z) \) is considered in the space \( C^\alpha_0 \) for \( \alpha \in (0, \alpha_0) \). To construct the operator \( (I - A(z))^{-1} \), we first study the set \( \mathfrak{N} \subset \Pi \) of “singular” points \( z \) at which the equation \( f(A(z)) \) has nontrivial solutions. This set can be decomposed into three components \( \mathfrak{N} = \mathfrak{N} + A^{\alpha_1} \cup A^{\alpha_0} \cup A^{\alpha_1} \) where \( \mathfrak{N} \subset \rho(H_0) \), while \( A^{\alpha_1} \) belongs to the upper (lower) bank of the cut along \( \sigma \). By Lemma 2 all these sets do not depend on the choice of \( \alpha \in (0, \alpha_0) \). We shall need

**Lemma 3.** Suppose \( \lambda \in \sigma \) and \( f = A(\lambda_0 + i\varepsilon) f \). Then \( f(\lambda_0) = 0 \).

**Proof.** We set \( g(e) = A(\lambda_0 \pm i\varepsilon) f \). Since \( A(z) = -VR_0(z) \) for \( z \in \rho(H_0) \) and, by hypothesis, \( g(0) = f \), it follows that

\[
(VR_0(\lambda_0 \pm i\varepsilon) f, \lambda_0 \pm i\varepsilon) f = (g(e) - g(0), \lambda_0 \pm i\varepsilon) f - (g(e), R_0(\lambda_0 \pm i\varepsilon) f),
\]

(14)

where the scalar product is taken, of course, in \( L_2(\sigma; h) \). According to Lemma 2 there is the estimate \( \| g(e) - g(0) \| \leq C\varepsilon, \gamma > 0, \) and hence

\[
(g(e) - g(0), R_0(\lambda_0 \pm i\varepsilon) f) \leq C\varepsilon \int_{|\lambda - \lambda_0| < \varepsilon} d\lambda = O(1)
\]

as \( \varepsilon \to 0 \). We take the imaginary part of both sides of (14). Here by the symmetry of \( V \) the left-hand side becomes zero, and hence

\[
\text{Im}(f, R_0(\lambda_0 \pm i\varepsilon) f) = o(1)
\]
as \( \varepsilon \to 0 \). At the same time, by equalities (1.2.6), (1.2.7) for \( a < \lambda_0 < b \), this expression tends to \( 2\pi i f(\lambda_0) \). For \( \lambda_0 = a \) or \( \lambda_0 = b \) the desired equality \( f(\lambda_0) = 0 \) follows from condition (3). \( \square \)

According to Lemma 3, both equations \( f = A(\lambda_0 \pm i0)f \) have solutions simultaneously, and these solutions coincide. We denote by \( \mathcal{N} \) the set \( \mathcal{A}^{(a)} = \mathcal{A}^{(b)} \), considered as a subset of \( \sigma \subset \mathbb{R} \). Actually, we have

**Lemma 4.** Suppose \( \alpha_0 > 1/2 \). Then the set \( \mathcal{N} \cup \mathfrak{N} \) coincides with the point spectrum \( \sigma^{(p)} \) of the operator \( H \). Moreover, the multiplicity of an eigenvalue \( z \) of the operator \( H \) is equal to the multiplicity of the eigenvalue \( 1 \) of the operator \( A(z) \).

**Proof.** For the operator \( A(z) = -\mathcal{V} \mathcal{R}_0(z) \) the relations

\[
(H_0 - z)\psi + V\psi = 0 \quad \text{and} \quad f = A(z)f
\]

are formally equivalent, and the corresponding eigenvalues \( \psi \) and \( f \) are connected by the equalities

\[
f = -V\psi \quad \text{and} \quad \psi = \mathcal{R}_0(z)f.
\]

By conditions (2) \( V\psi \in C^0_0 \) for \( \psi \in \mathcal{N} \) and any \( \alpha < \alpha_0 \), and hence \( \sigma^{(p)} \subset \mathcal{N} \cup \mathfrak{N} \). Conversely, for \( f \in C^0_0 \) the inclusion \( \psi = \mathcal{R}_0(z)f \in \mathcal{N} \) is obvious for \( z \in \rho(H_0) \), whence \( \mathfrak{N} \subset \sigma^{(p)} \). For \( z = \lambda_0 \pm i0 \), \( \lambda_0 \in \sigma \), taking account of Lemma 3 and the finiteness of the norm (11), we have

\[
|\psi(\lambda)| = |\lambda - \lambda_0|^{-1}|f(\lambda)| \leq C|\lambda - \lambda_0|^{-n}
\]

and \( \psi \) is square-integrable if \( 2\alpha > 1 \). From this we obtain the inclusion \( \mathcal{N} \subset \sigma^{(p)} \). The same argument also establishes that the multiplicities of the eigenvalues \( z \) and 1 of the operators \( H \) and \( A(z) \) coincide. \( \square \)

From Lemma 4 it follows, of course, that the set \( \mathfrak{N} \) may lie only on the real axis.

3. We can now prove Theorem 1.

**Proof of Theorem 1.** That the eigenvalues (including those in the continuous spectrum) of the operator \( H \) have finite multiplicity is a direct consequence of Lemma 4 and the compactness of the operator \( A(z) \).

We ascertain that the set \( \sigma^{(p)} \) is finite. In view of the boundedness of \( H \), it suffices to verify that \( \mathcal{N} \cup \mathfrak{N} \) has no finite accumulation points. Assume otherwise. Let, for example, \( \lambda_n \in \mathcal{N} \) and \( \lambda_n \to \lambda_0 \) as \( n \to \infty \). According to Lemma 2, the set of normalized solutions \( f_n \) in \( C^0_0 \) of the equation \( A(z_n)f_n = f_n \), \( z_n = \lambda_n \pm i0 \), is compact in \( C^0_0 \). It may be assumed that \( f_n \to f_0 \) in \( C^0_0 \) as \( n \to \infty \). Then \( A(z_0)f_0 = f_0 \). By the orthogonality of the eigenfunctions \( \psi_n = \mathcal{R}_0(z_n)f_n \) and \( \psi_0 = \mathcal{R}_0(z_0)f_0 \) of the selfadjoint operator \( H \)

\[
\int (\lambda - \lambda_0)^{-1}f_0(\lambda)(\lambda - \lambda_0)^{-1}f_0(\lambda)\,d\lambda = 0.
\]

According to Lemma 3, we have the equalities \( f_n(\lambda_0) = 0 \) and \( f_0(\lambda_0) = 0 \), and hence

\[
|f_0(\lambda)| \leq C|\lambda - \lambda_0|^n, \quad |f_0(\lambda)| \leq C|\lambda - \lambda_0|^n
\]

with a constant \( C \) independent of \( n \). For \( 2\alpha > 1 \) this makes it possible to estimate the integral (15) in a neighborhood of the point \( \lambda_0 \) uniformly with respect to \( n \). Now the relation \( \|f_0 - f_0\|_0 \to 0 \), \( n \to \infty \), shows that in the integral (15) it is possible to pass to the limit as \( n \to \infty \). Thus, from (15) we obtain the equality \( (\lambda - \lambda_0)^{-1}f_0(\lambda) = 0 \) which contradicts the normalization \( \|f_0\|_0 = 1 \). As concerns the finiteness of the set \( \mathfrak{N} \), only the verification that there is no accumulation at the end points \( a \) and \( b \) merits attention. In justifying the passage to limit in (15) it is necessary to consider that by (3) \( f_0(\lambda_0) = 0 \), \( f_0(\lambda_0) = 0 \). Inequalities (16) are now satisfied if on the right-hand side \( \lambda_0 \) is replaced by \( \lambda_0 \). Similarly to the previous case, in the case \( 2\alpha > 1 \) this makes it possible to pass to the limit as \( n \to \infty \) in the integral (15).

To prove the results of Theorem 1 regarding the kernel \( t(\lambda, \mu; z) \) of the operator (4) we start from the integral equation (9) written in the form (10).

By Lemma 2 and the Fredholm alternative (see Part 1 of §1.7) the operator \( I - A(z) \) is invertible in the space \( C^0_0 \) for all \( z \in \Pi \setminus \mathfrak{N} \). Moreover, the constant \( C \) in the estimate

\[
\|t(\lambda, \mu; z) - t(\lambda, \mu; z)\|_0 \leq C\|t(\lambda, \mu; z)\|_0, \quad \alpha < \alpha_0.
\]

From this it follows that equation (9) is solvable, and the function \( t(\lambda, \mu; z) \) is Hölder continuous in the first variable with any exponent \( \alpha < \alpha_0 \). In view of the symmetry (7), the same is true with regard to smoothness in the second variable.

It remains to study the smoothness of \( t(\lambda, \mu; z) \) with respect to the spectral variable \( z \). By equation (10)

\[
t(z') - t(z) = (I - A(z'))^{-1}(A(z') - A(z))t(z).
\]

From Lemma 2 and the estimate (17) it follows that for any \( \alpha_1 \in (0, \alpha_0) \) and \( \gamma < \alpha_0 - \alpha_1 \)

\[
\|t(z') - t(z)\|_0 \leq C|z' - z|^\gamma.
\]

Since the operator \( (I - A(z'))^{-1} \) is bounded in the space \( C^0_0 \) for any sufficiently small \( \alpha_1 \), comparing (18) and (19) we find that the kernel \( t(\lambda, \mu; z) \) is Hölder continuous in \( z \) with any exponent \( \gamma < \alpha_0 \).
4. We make some remarks regarding the necessity of the conditions of Theorem 1. Without condition (3) the operator $A(z)$ must be considered in the space $\mathcal{C}^0(\sigma; \mathfrak{h})$ (rather than $\mathcal{C}^0_0(\sigma; \mathfrak{h})$). Here by Theorem 1.2.6 and the remark to it Lemmas 2 and 4 are preserved if the end points of the interval $\sigma$ are excluded from consideration. The eigenvalues of the operator $H$ have no accumulation points distinct from $a$ and $b$. The estimate (6) in Theorem 1 for the kernel of the operator (4) is valid for all $\lambda, \mu \in \sigma$ and $z$ lying outside arbitrary neighborhoods of the points of $\sigma^{(0)}$ and of the end points $a$ and $b$. On the boundary of the square $\sigma \times \sigma$ the kernel $i(\cdot, \cdot; z)$, of course, now does not vanish.

If in place of (3) we have the equality

$$v(\lambda, a) = v(a, \mu) = 0,$$

then only the point $b$ must be excluded from consideration. The eigenvalues of $H$ may also accumulate only at this point.

We shall show that even in the absence of condition (3) the discrete spectrum of the operator $H$ remains finite. We shall establish, for example, the absence of accumulation to the left of the point $a$. To this end we consider the auxiliary operator $H_1 = H_0 + V_1$ where $V_1$ is an integral operator of the form (1) with kernel

$$v_1(\lambda, \mu) = v(\lambda, \mu) - \xi(\lambda) - \xi(\mu), \quad \xi(\lambda) = v(\lambda, a) - 2^{-1} v(a, a).$$

For this kernel the condition $v_1(\lambda, a) = \overline{v_1}(a, \mu) = 0$ is satisfied, and hence the eigenvalues of $H_1$ cannot accumulate at the point $a$ (from the left or the right). At the same time the operators $H$ and $\overline{H}$ differ by a finite-dimensional (two-dimensional) operator. Therefore (see, for example, the book [4]), the total multiplicity of the discrete spectra of them in the interval $(-\infty, a)$ can differ only by a finite number (two). Finiteness of the discrete spectrum of the operator $H$ follows from this.

We shall now consider generalizations of Theorem 1 to the case where condition (2) is satisfied only for some $\alpha_0 > 0$. This requires application of the results of §1.8. The assumption (3) is now dropped.

In the proof of Theorem 1 the condition $\alpha_0 > 1/2$ was used only to verify the inclusion $\mathcal{M} \subset \sigma^{(0)}$ and also to prove the finiteness of the set $\sigma^{(0)}$. Therefore, the estimate (6) for the kernel $i(\lambda, \mu; z)$ is retained if the spectral parameter $z$ is separated from points of the set $\mathfrak{R} \cup \mathcal{M}$ and also the end points $a$ and $b$. To describe the structure of the set $\mathcal{M}$, in which we agree to include the end points $a$ and $b$, we now need Theorem 1.8.3. It is important here that the operator $A(z)$ can be approximated by finite-dimensional operators. This follows from the next elementary assertion of general character.

**Lemma 5.** In the space $\mathcal{C}^0(\sigma; \mathfrak{h})$, where $\alpha \in (0, 1)$, any compact operator can be approximated in norm by finite-dimensional operators.

**Proof.** It suffices to construct a sequence of finite-dimensional operators $T_n$ such that

$$\lim_{n \to \infty} T_n = I.$$  \hspace{1cm} (20)

Then for any $A \in \mathcal{E}_\infty$, we have $\|A - A_n\| \to 0$, $n \to \infty$, where the operators $A_n = T_n A$ are finite-dimensional. To construct $T_n$ it suffices to consider the space $\mathcal{C}^0(\sigma)$ which corresponds to $\dim \mathfrak{h} = 1$. Passage to the general case is realized by the relation $T_n = T_n \otimes I_n$, where $T_n$ are operators in $\mathcal{C}^0(\sigma)$ satisfying (20) and $P_n$ is the orthogonal projection in $\mathfrak{h}$ onto the first $n$ vectors of some fixed orthonormal basis. Further, in place of $\mathcal{C}^0(\sigma)$ it is possible to consider the subspace $\mathcal{C}^0(\sigma)$ of it consisting of periodic functions and differing from $\mathcal{C}^0(\sigma)$ by a one-dimensional subspace. Moreover, by a change of variables the matter reduces to the case $\sigma = [-\pi, \pi]$. Finally, in the space $\mathcal{C}^0([-\pi, \pi])$ as $T_n$ it is possible to take an integral operator with kernel $T_n(\lambda - \mu)$ where

$$T_n(\lambda) = (2\pi n)^{-1} (1 - \cos \lambda)(1 - \cos \lambda)^{-1},$$

i.e., $T_n f$ is the $n$th Cesàro mean of the Fourier series of $f$. The rank of $T_n$ is equal to $2n + 1$, while relation (20) for it can be verified in complete analogy to the proof of the same fact for continuous functions (see, for example, [6]).

By this Lemma and Remark 1.8.4, Theorem 1.8.3 is applicable to the operator-valued function $A(z)$ in the space $\mathcal{C}^0 = \mathcal{C}^0(\sigma; \mathfrak{h})$. Therefore, the set $\mathcal{M}$ is closed and has Lebesgue measure zero.

We shall now consider the discrete spectrum of the operator $H$. We note first of all that in the proof of the inclusion $d^{(0)} \subset \mathcal{M} \cup \mathfrak{R}$ in Lemma 4 only the condition $\alpha_0 > 0$ was used. Indeed, for $H \psi = \lambda \psi$ the function $f = -V \psi \in \mathcal{C}^0$ and satisfies the equation $A(\lambda \pm i0) f = f$. Such an element $f$ is not $0$, since otherwise we would have $H_0 \psi = \lambda \psi$. In addition, it becomes clear that the multiplicity of the eigenvalue $\lambda$ of the operator $H$ does not exceed the multiplicity of the eigenvalue $1$ of the operator $A(\lambda \pm i0)$. Since for $\lambda \neq a$ and $\lambda \neq b$ the operator $A(\lambda \pm i0)$ is compact in the space $\mathcal{C}^0$, from this we obtain the finite multiplicity of the eigenvalues of the operator $H$. Moreover, in complete analogy to the proof of Theorem 1 it can be verified that $\mathfrak{R} \subset \sigma^{(0)}$ and that the eigenvalues of the operator $H$ have no accumulation points in $(-\infty, a)$ and $(b, \infty)$. We thus have...
Theorem 6. Suppose for some $\alpha_0 > 0$ condition (2) is satisfied. Then the set $\mathcal{N}$ is closed and has Lebesgue measure zero. The eigenvalues of the operator $H$ belong to the set $\mathfrak{M} = \mathcal{N} \cup \mathfrak{M}$, they have no accumulation points off the segment $\sigma$, and, with possible exception of the points $a$ and $b$, they have finite multiplicity. The kernel $t(\lambda, \mu; z)$ of the operator $T(z)$ assumes compact values and is Hölder continuous in the arguments $\lambda, \mu \in \sigma$ and $z \in \Pi \mathfrak{M}$ with any exponent $\alpha < \alpha_0$. The constant $C$ in (6) is uniform with respect to $z, z' \in \Pi$ arbitrary neighborhoods of the points of the set $\mathfrak{M}$.

We note that in the investigation of eigenvalues of the operator $H$ lying outside $\sigma$ it is possible to appeal to Theorem 1.6. All the information regarding the discrete spectrum follows also from the theorem of H. Weyl.

Theorems 1 or 6, generally speaking, fail without the assumption of compactness of the operators $v(\lambda, \mu)$ in $\mathfrak{h}$. Nevertheless, under conditions (2) and (3) the operator $A(z)$ is, as before, bounded in the spaces $C^0_0(\sigma; \mathfrak{h})$ for $\alpha \in (0, \alpha_0)$ and is Hölder continuous with respect to $z \in \Pi$. Therefore, for all $z \in \Pi$, equation (10) is properly set, and, for small perturbations, it is uniquely solvable by the contraction mapping principle. More precisely, we have

Theorem 7. Suppose conditions (2), with $\alpha_0 > 0$, and (3) are satisfied, and the operator $A(z) : C^0_0(\sigma; \mathfrak{h}) \to C^0_0(\sigma; \mathfrak{h})$ satisfies the estimate

$$\sup_{z \in \Pi} \| A(z) \| < 1$$

for some $\alpha \in (0, \alpha_0)$ (the assumption $v(\lambda, \mu) \in C^0_0(\sigma; \mathfrak{h})$ is not imposed). Then the operator $H$ is absolutely continuous and, in particular, has no point spectrum. The kernel $t(\lambda, \mu; z)$ depends on the arguments $\lambda, \mu \in \sigma$ and $z \in \Pi$ in a Hölder continuous fashion with any exponent $\alpha < \alpha_0$. Moreover, on the boundary of the square $\sigma \times \sigma$ the kernel $t(\cdot, \cdot; z)$ vanishes.

Under conditions (2) and (3) the assertion of Theorem 7 is clearly valid for the operator $H_\varepsilon = H_0 + \varepsilon V$ if $\varepsilon$ is sufficiently small.

§2. Scattering in the Friedrichs-Faddeev model

Here the results of §1 are applied to the stationary construction of scattering theory in the Friedrichs-Faddeev model. In this model the WO are represented by singular integral operators. These representations differ in form from the representations of 2.7 but are equivalent to them. In connection with this we note that in the model considered the assumptions regarding the perturbation do not have explicit unitary-invariant character. The formula representations obtained are thus also not unitary-invariant. At the same time investigation of the Friedrichs-Faddeev model makes it possible to display specific features and simplifications in cases where a spectral analysis of the unperturbed operator can be carried out explicitly.

1. As in §1, we assume that $\mathcal{H} = L_2(\sigma; \mathfrak{h})$, $(H_0 f) (\lambda) = \lambda f(\lambda), H = H_0 + V$, where $V$ is the integral operator (1.1). As before, it is assumed that the kernel $v$ is Hermitian, i.e., $v(\lambda, \mu) = v^*(\mu, \lambda)$ and (except in Theorem 3) it takes compact values in $\mathfrak{h}$.

We carry out the considerations directly under the conditions of Theorem 1.6 which are more general as compared with Theorem 1.1. By $\mathcal{N}$ we denote the set of points $\lambda \in (a, b)$ for which the equations $A(\lambda \pm i0) f = f$ have nontrivial solutions. The points $a$ and $b$ are also included in this set. Under the conditions of Theorem 1.6 the set $\mathcal{N}$ is closed and has Lebesgue measure zero.

We formulate the main results of this section.

Theorem 1. Suppose condition (1.2) holds for some $\alpha_0 > 0$. Then the WO $W_0(\mathcal{H}, H_0)$ exist and are complete. On the set $\sigma \mathcal{N}$ the spectrum of the operator $H$ is absolutely continuous.

According to Lemma 1.4, under the conditions of Theorem 1.1 the set $\mathcal{N}$ is exhausted by the eigenvalues of the operator $H$. Without condition (1.3) this is true relative to all interior points of the interval $\sigma$. Combining these assertions with Theorem 1, we obtain

Corollary 2. Suppose condition (1.2) is satisfied for some $\alpha_0 > 1/2$. Then the operator $H$ has no singular continuous spectrum.

For small perturbations the set $\mathcal{N}$ is empty, and the assumption regarding compactness of $v$ is not necessary. In this case we have

Theorem 3. Suppose the conditions of Theorem 1.7 are satisfied. Then the spectrum of the operator $H$ is absolutely continuous, and the WO $W_0(\mathcal{H}, H_0)$ exist and are unitary.

The assumption regarding compactness of the operators $v(\lambda, \mu)$ is not explicitly used in the proof of Theorem 1. It is needed only for the application of the results of §1. Theorem 3 is therefore essentially a corollary of Theorem 1.

By Corollary 2.3.9 under the conditions of Theorems 1 and 3 the WO $W_0(\mathcal{H}, H)$ also exist.

2. To prove Theorem 1 we consider the kernel $t(\lambda, \mu; z)$ of the operator (1.4). It is possible to define in its terms the two singular integral operators $\mathcal{U}_+ (\text{the stationary WO})$

$$\mathcal{U}_+(f) (\lambda) = f(\lambda) - \int_{\sigma} t(\lambda, \mu, \mu \pm i0)(\lambda - \mu \pm i0)^{-1} f(\mu) d\mu$$

In a precise sense the operators $\mathcal{U}_+$ are defined in $\mathcal{H}$ by relation (1) on the set $\mathcal{D} = C^0_0(\sigma \mathcal{N}; \mathfrak{h})$ of smooth, vector-valued functions vanishing in neighborhoods of points of $\mathcal{N}$. Since $\sigma \mathcal{N}$ is an open set of full measure, $\mathcal{D}$ is dense in $\mathcal{H}$.
We shall verify that the operators $\mathcal{U}_\pm$ are isometric on $\mathcal{D}$ and hence extend by continuity to isometric operators on all of $\mathcal{H}$. To this end for $\varepsilon > 0, \tau > 0$ we consider the auxiliary integral operators

$$
(\mathcal{F}_\varepsilon(e, \tau)f)(\lambda) = \int_\sigma t(\lambda, \mu; \mu \mp i\varepsilon)(\lambda - \mu \pm i\tau)^{-1} f(\mu) d\mu.
$$

(2)

According to Theorems 1.2.6 and 1.6 for $f \in \mathcal{D}$ and $\varepsilon \to 0, \tau \to 0$ the functions $\mathcal{F}_\varepsilon(e, \tau)f$ converge to $\mathcal{F}_\pm f$ uniformly with respect to $\lambda \in \sigma$, and hence

$$
\lim_{\varepsilon \to 0} \mathcal{F}_\pm(e, \tau)f = \mathcal{F}_\pm f, \quad f \in \mathcal{D},
$$

(3)

in the sense of strong convergence in $\mathcal{H}$. We shall also need the identity

$$
T(z_2) - T(z_1) = (z_1 - z_2)T(z_2) - T(z_1)R_0(z_1)T(z_1), \quad z_j \in \rho(H),
$$

following from definition (1.4), the equality $T(z)R_0(z) = V R(z)$, and the Hilbert identity for $R(z)$. In terms of kernels this means that

$$
T(\lambda, \mu; z_2; z_1) = (z_1 - z_2) \int_\sigma T(\lambda, \nu; z_2)(\nu - z_2)^{-1}(\nu - z_1)^{-1} t(\nu, \mu; z_1) d\nu,
$$

(4)

$z_j \in \rho(H)$.

**Lemma 3.** The operators $\mathcal{U}_\pm$ are isometric.

**Proof.** In (4) we set $z_1 = \lambda + i\varepsilon, z_2 = \lambda \mp i\varepsilon$ and consider the symmetry equality (1.7). We then find that

$$
\mathcal{F}_\pm(e, 2\varepsilon) + \mathcal{F}_\pm(e, 2\varepsilon) = \mathcal{F}_\pm(e, e) \mathcal{F}_\pm(e, e).
$$

We compute the quadratic form of this operator on an element $f \in \mathcal{D}$ and, in accordance with (3), pass to the limit as $\varepsilon \to 0$. This leads to the equality $\|\mathcal{U}_\pm f\| = \| f \|$, $f \in \mathcal{D}$, for the operator $\mathcal{U}_\pm = I - \mathcal{F}_\pm$. □

The intertwining property of the operators $\mathcal{U}_\pm$ can be established in the same simple manner.

**Lemma 4.** For any $z \in \rho(H)$

$$
R(z)\mathcal{U}_\pm = \mathcal{U}_\pm R_0(z).
$$

(5)

**Proof.** We again start from identity (4), where we set $z_1 = \mu + i\varepsilon, z_2 = z$. Multiplying it by the function

$$
(\lambda - z)^{-1}(\mu - z \mp i\varepsilon)^{-1} = (\lambda - \mu \pm i\varepsilon)^{-1}[(\mu - z \mp i\varepsilon)^{-1} - (\lambda - z)^{-1}],
$$

we find that

$$
R_0(z)T(z)R_0(z \pm i\varepsilon) + R_0(z)\mathcal{F}_\pm(e, e) - \mathcal{F}_\pm(e, e)R_0(z \pm i\varepsilon) = R_0(z)T(z)R_0(z \pm i\varepsilon).
$$

We apply this operator to an element $f \in \mathcal{D}$ and pass to the limit as $\varepsilon \to 0$ in accordance with (3). In view of (1.5) the relation thus obtained is equivalent to (5) for $\mathcal{U}_\pm = I - \mathcal{F}_\pm$. □

The equality $\mathcal{U}_\pm^* = \mathcal{U}_\mp^* \mathcal{F}_\mp$ for an arbitrary bounded function follows from (5) in the standard way (cf. §2.1). The equality $H \mathcal{U}_\pm = \mathcal{U}_\pm H_0$ also holds.

3. We now establish absolute continuity of the spectrum of the operator $H$ on the set $\sigma \setminus \mathcal{N}$. To this end we compute by Stone's formula (1.4.7) the spectral measure of $H$. For $f \in C^0 = C^0(\sigma; h)$, $\alpha > 0$, we consider the function

$$
u(\lambda, z; f) = f(\lambda) \int_\sigma t(\lambda, \mu; z)(\mu - z)^{-1} f(\mu) d\mu.
$$

(6)

By Theorems 1.2.6 and 1.6 this function is Hölder continuous in the variables $\lambda \in \sigma$ and $z \in \Pi \mathcal{M}$. Therefore, for $\lambda \in \sigma \setminus \mathcal{N}$ the "diagonal" values $u(\lambda, \lambda \pm i0; f)$ are well defined.

**Lemma 5.** Suppose $f, g \in C^0$ and the Borel set $\Lambda \subset \sigma \setminus \mathcal{N}$. Then for any sign $\leq$ \leq$

$$
(E(\Lambda)f, g) = \int_\Lambda (u(\lambda, \lambda \pm i0; f), u(\lambda, \lambda \pm i0; g)) d\lambda.
$$

(7)

**Proof.** According to (6) and (1.5)

$$
(R(z)f)(\mu) = (\mu - z)^{-1} u(\mu, z; f).
$$

Therefore, by definition (1.4.5)

$$
(\delta(\lambda, e)f, g) = \pi^{-1} e(R(\lambda \pm i0)f, R(\lambda \pm ie)g)
$$

$$
= \pi^{-1} e \int_\sigma \frac{(u(\mu, \lambda \pm i0; f), u(\mu, \lambda \pm i0; g))}{(\mu - \lambda)^2 + \varepsilon^2} d\mu.
$$

For $\lambda \in \sigma \setminus \mathcal{N}$ we here let $\varepsilon$ tend to zero. On the basis of equalities (1.2.6), (1.2.7) we then find that

$$
\lim_{\varepsilon \to 0} (\delta(\lambda, e)f, g) = (u(\lambda, \lambda \pm i0; f), u(\lambda, \lambda \pm i0; g)),
$$

where the limit is uniform with respect to $\lambda \in [\beta_1, \beta_2]$ if $[\beta_1, \beta_2]\cap\mathcal{N} = \emptyset$. Hence, for $\Lambda = [\beta_1, \beta_2]$ and $\varepsilon \to 0$ it is possible to pass to the limit under the integral sign in (1.4.7). This proves (7) for the case of intervals and hence also for any Borel sets. □

On the set $\sigma \setminus \mathcal{N}$ the measure (7) is absolutely continuous. Since, moreover, the sets $\mathcal{N}$ and $\mathcal{M}$ have Lebesgue measure zero, from this we obtain the equality

$$
P^{(\alpha)} = E(\sigma \setminus \mathcal{N}).
$$

(8)

To prove the completeness of the WO $\mathcal{U}_\pm$ we write the representation (7) in terms of the operators (1). We here need
LEMMA 6. For \( f \in C^\alpha \) and \( \lambda \in \sigma \setminus N \)
\[
(\mathcal{U}_\pm^* f)(\lambda) = u(\lambda, \lambda \pm i0; f).
\] (9)

PROOF. We start from the relation
\[
(f, g) - (f, \mathcal{T}_\pm^*(e, \varepsilon)g) = \int_\sigma (u(\lambda, \lambda \pm i\varepsilon; f), g(\lambda)) d\lambda.
\] (10)

To prove it, it is necessary to substitute the definition (6) into the right-hand side of (10), interchange the integration on \( \lambda \) and \( \mu \), and use equality (1.7). According to (2), the expression obtained is equal to the left-hand side of (10).

Further, assuming that \( g \in \mathcal{D} \), in (10) we pass to the limit as \( \varepsilon \to 0 \). By (3) and the continuity of \( u(\lambda, z; f) \) with respect to \( \lambda \in \sigma \) and \( z \in \Pi \setminus \Pi_0 \), this leads to the equality
\[
(f, \mathcal{U}_\pm g) = \int_\sigma (u(\lambda, \lambda \pm i0; f), g(\lambda)) d\lambda.
\]

Since \( g \in \mathcal{D} \) is arbitrary, the desired relation (9) follows immediately from it.

Combining equalities (7) and (9), we find that
\[
(E(\lambda)f, g) = \int_\lambda (\mathcal{U}_\pm^* f)(\lambda), (\mathcal{U}_\pm^* g)(\lambda)) d\lambda.
\]

Here we set \( \Lambda = \sigma \setminus N \) and consider (8). We then obtain the relation
\[
\mathcal{U}_\pm^* E = \mathcal{F}^{(a)},
\] (11)

meaning the completeness of the WO \( \mathcal{U}_\pm \). Thus, \( R(\mathcal{U}_\pm) = \mathcal{F}^{(a)} \), and on this subspace the operator \( H \) is unitarily equivalent to \( \tilde{H}_0 \).

To complete the proof of Theorem 1 it remains to consider the time-dependent WO (2.1.1) (for \( J = I \)). We shall here need the elementary

LEMMA 7. Suppose \( f \) is a Hölder continuous function on the segment \([a, b]\). Then
\[
\lim_{t \to \pm \infty} \lim_{\varepsilon \to 0} \int_a^b f(x)e^{itx}(x \pm i\varepsilon)^{-1} dx = 0.
\] (12)

PROOF. We add and subtract the value of \( f \) at zero and note that the function \(|f(x) - f(0)|x^{-1}\) is absolutely integrable. An integral of the form (12), where the role of \( f(x) \) is played by \( f(x) - f(0) \), tends to zero by the Riemann-Lebesgue lemma. It remains to note that for \( \varepsilon > 0 \) and \( \pm t > 0 \) the integral of \( \exp(itx)(x \pm i\varepsilon)^{-1} \) over \( \mathbb{R} \) is equal to zero, and hence
\[
\int_{-\infty}^\infty e^{itx}(x \pm i\varepsilon)^{-1} dx = -\int_{|x| \geq a} e^{itx} x^{-1} dx = -\int_{|y| \geq |t|} e^{\pm iy} y^{-1} dy.
\]
The last integral, of course, tends to zero as \( |t| \to \infty \). □

It is now not hard to see that the time-dependent WO \( W_\pm(H, \mathcal{H}) \) exist and coincide with \( \mathcal{U}_\pm \). In view of the intertwining property and the isometricity of \( \mathcal{U}_\pm \)
\[
\|U(-t)U_0(t)f - \mathcal{U}_\pm f\|^2 = 2 \text{Re} \langle (I - \mathcal{U}_\pm)U_0(t)f, U_0(t)f \rangle.
\] (13)

For \( f \in \mathcal{D} \) the use of definition (1) for the operator \( \mathcal{U}_\pm \) gives a representation in the form of a double integral for the right-hand side of (13)
\[
2 \text{Re} \int_\sigma \int_\sigma (\lambda - \mu \pm i0)^{-1} e^{i(\lambda - \mu)\varepsilon} w_\pm(\lambda, \mu) d\lambda d\mu
\] (14)

where
\[
w_\pm(\lambda, \mu) = (t(\lambda, \mu; \mu \mp i0) f(\mu), f(\lambda))
\]
is a Hölder continuous function of the variables \( \lambda, \mu \in \sigma \). Applying now Lemma 7 in the variable \( \lambda - \mu \) to the integral (14), we see that it tends to zero as \( t \to \pm \infty \). Thus, according to (13), the limit \( U(-t)U_0(t)f \) as \( t \to \pm \infty \) exists and is equal to \( \mathcal{U}_\pm f \), i.e., \( W_\pm(H, \mathcal{H}) = \mathcal{U}_\pm \). Equality (11) shows that the WO \( W_\pm(H, \mathcal{H}) \) are complete. This completes the proof of Theorem 1.

4. We further obtain expressions for the scattering operator and matrix. Since \( \mathcal{W}_\pm = I - \mathcal{F}_\pm \), by (13) for \( f, g \in \mathcal{D} \)
\[
(Sf, g) = (f, g) + \lim_{t \to 0} \left( -\mathcal{F}_+^*(2\varepsilon, \varepsilon) - \mathcal{F}_-(e, \varepsilon) + \mathcal{F}_+^*(2\varepsilon, 2\varepsilon) \mathcal{F}_-(e, \varepsilon) \right) f, g).
\] (15)

According to (2) and the symmetry equality (1.7), the kernel of the operator in square brackets is equal to
\[
(\lambda - \mu + i\varepsilon)^{-1} t(\lambda, \mu; \lambda + 2i\varepsilon) - (\lambda - \mu - i\varepsilon)^{-1} t(\lambda, \mu; \mu + i\varepsilon)
\]
\[
+ \int_\sigma t(\lambda, \nu; \lambda + 2i\varepsilon)(\nu, \mu; \mu + i\varepsilon) d\nu.
\] (16)

Applying the identity (4) for \( z_1 = \mu + i\varepsilon \), \( z_2 = \lambda + 2i\varepsilon \) to the last integral, we write the second term on the right-hand side of (15) in the form
\[
-2i \lim_{\varepsilon \to 0} \int_\sigma \int_0^\infty (\lambda - \mu)^2 + e^{2i\varepsilon} - 1)^{-1} t(\lambda, \mu; \mu + i\varepsilon) f(\mu), g(\lambda) d\lambda d\mu.
\]

On the basis of relations (1.2.6), (1.2.7), in the Poisson integral with respect to the variable \( \lambda - \mu \) obtained it is possible to pass to the limit as \( \varepsilon \to 0 \). This leads to a representation for the sesquilinear form of the scattering operator
\[
(Sf, g) = (f, g) - 2\pi i \int_\sigma \langle (t(\lambda, \lambda; \lambda + i0) f(\lambda), g(\lambda)) d\lambda.
\]

Thus, \( S \) acts as multiplication by the operator-valued function (the scattering matrix)
\[
S(\lambda) = I - 2\pi it(\lambda, \lambda; \lambda + i0) : \zeta \to \zeta.
\] (17)
By Theorem 1.6 the scattering matrix is well defined for $\lambda \in \sigma \setminus \mathcal{N}$. From the unitarity of $S$ in $\mathcal{H}$ it follows that the operator (17) is also unitary in $\mathcal{H}$. Besides, the unitarity of the operator (17) follows directly from the equality (1.7) and the identity (4), where it is necessary to set $\lambda = \mu$, $z_2 = z_1 = \lambda + ie$ and let $e$ tend to zero. The properties of $S(\lambda)$ formulated below are corollaries of Theorem 1.6.

**Theorem 9.** Under the conditions of Theorem 1 the scattering matrix for the pair $H_\alpha, \ H$ is given, for $\lambda \in \sigma \setminus \mathcal{N}$, by relation (17). The operator $S(\lambda)$ is unitary in $\mathcal{H}$, differs from the identity operator by a compact operator, and $S(\lambda)$ depends in a Hölder continuous way with exponent $\alpha < \alpha_0$ (in the operator norm of $\mathcal{H}$), on $\lambda \in \sigma \setminus \mathcal{N}$.

5. The representation (17) for the scattering matrix is, of course, a realization of equality (2.8.1) within the framework of the Friedrichs-Faddeev model. As concerns the WO, it can be seen in the following manner that the operators (1) coincide with the stationary WO of §2.7. By virtue of the second equality of (2.7.10) the relation (2.7.5) can be written in the form

$$
\langle \mathcal{A}_{\mu} f, f \rangle = \langle \mathcal{A}_{\mu} f, f \rangle - \int_{-\infty}^{\infty} \lim_{\epsilon \to 0} u(\lambda, \epsilon) f(\lambda + i\epsilon) R_0(\lambda + i\epsilon) f(\lambda + i\epsilon) d\lambda.
$$

(18)

This expression is equal to the sesquilinear form of the operator (1) in the circumstance in question.

For an infinite interval $\sigma$ (for example, for $\sigma = \mathbb{R}_+$ or $\sigma = \mathbb{R}$) the scheme of investigation described and the conclusions of Theorems 1 and 9 continue to hold (see the original paper of Faddeev [79]) under particular assumptions regarding the kernel $u(\lambda, \mu)$ at infinity. Namely, it is necessary to require that

$$
|u(\lambda, \mu)| \leq C(1 + |\lambda| + |\mu|)^{-\theta_0}, \quad \theta_0 > 1/2,
$$

and also assume an analogous estimate for the Hölder differences of $u(\lambda, \mu)$. Of course, this scheme extends also directly to the case where $\sigma$ is the finite union of nonintersecting intervals.

In conclusion we note that in the proof of the absence of singular continuous spectrum it is not possible to relax the condition $\alpha_0 > 1/2$ in (1.2). Actually, as shown in [71], even for a one-dimensional perturbation $V = (v, v)v$ with $v \in C_0^\infty$ and any $\alpha < 1/2$ the operator $H$, generally speaking, has nontrivial singular continuous spectrum. Some details in this regard can be found in §6.7.

§3. Kato smoothness

1. Suppose $H$ is a selfadjoint operator in the Hilbert space $\mathcal{H}$, $\mathcal{H}$ is an auxiliary Hilbert space, $G: \mathcal{H} \to \mathcal{H}$ is an $H$-bounded operator (see Part 1 of §1.9), and $\Lambda$ is an arbitrary finite interval. In contrast to the original paper of Kato [109] (see also Volume 4 of the course [18]) the operator $G$ is not assumed to be closed (or to admit closure). Therefore, the operator $R(\lambda)^* G^*$ must be defined by equality (1.9.1). We also use the notation $G_\lambda = GE(\lambda)$. We further recall that the operator $\delta(\lambda, \epsilon) = \delta_H(\lambda, \epsilon)$ is defined by relation (1.4.5).

The concept of Kato smoothness is introduced in unitarily invariant terms and admits several equivalent formulations. We precede the definition of this concept with the following assertion.

**Theorem 1.** The following conditions are equivalent:

$$
\gamma_1^2 = (2\pi)^{-1} \sup_{f \in \mathcal{D}(H), \|f\| = 1} \int_{-\infty}^{\infty} \|G(\lambda i) f\|^2 dt < \infty,
$$

(1)

$$
\gamma_2^2 = (2\pi)^{-2} \sup_{\|f\| = 1, \epsilon > 0} \int_{-\infty}^{\infty} \left(\|G(\lambda + i\epsilon) f\|^2 + \|G(\lambda - i\epsilon) f\|^2\right) d\lambda < \infty,
$$

(2)

$$
\gamma_3^2 = \sup_{\|f\| = 1, \epsilon > 0} \int_{-\infty}^{\infty} \|G\delta(\lambda, \epsilon) f\|^2 d\lambda < \infty,
$$

(3)

$$
\gamma_4^2 = \sup_{\lambda \in \mathbb{R}, \epsilon > 0} \|G\delta(\lambda, \epsilon) G^*\| < \infty,
$$

(4)

$$
\gamma_5^2 = \sup_{\lambda \in \mathbb{R}} |\Lambda|^{-1} \|G E(\Lambda)\| < \infty.
$$

(5)

All the constants $\gamma_j = \gamma_j(G), \ j = 1, \ldots, 5$, are equal to one another.

**Proof.** We first note that the suprema in (2) and (3) may be computed on elements of $\mathcal{D}(H)$ or on compactly supported elements of the form $f = E(X) f$, where the set $X = X(f)$ is bounded. Indeed, suppose, for example, that for any $n$ and $X_n = (-n, n)$

$$
\int_{-\infty}^{\infty} \|G\delta(\lambda, \epsilon) E(X_n) f\|^2 d\lambda \leq \gamma_3^2 \|f\|^2.
$$

Since $E(X_n) f \to f$ as $n \to \infty$, for all $\lambda \in \mathbb{R}$ the integrand converges to $\|G\delta(\lambda, \epsilon) f\|^2$. Therefore, on the basis of (Fatou's) Theorem 1.1.4 it is possible to pass to the limit as $n \to \infty$ under the integral sign in the last equality. This gives (3).

We shall first establish the equality $\gamma_1 = \gamma_2 = \gamma_3$. From the expression (1.4.4) for the resolvent in terms of the unitary group and the vector Parseval equality (2.7.1) it follows that for $f \in \mathcal{D}(H)$

$$
\int_{-\infty}^{\infty} \|G(\lambda + i\epsilon) f\|^2 d\lambda = 2\pi \int_{0}^{\infty} e^{-2\epsilon t} \|G(\pm t) f\|^2 dt.
$$

(6)

We add equalities (6) and (6) and take the supremum on $\epsilon$. Here by Theorem 1.1.2 (of Lebesgue) the factor $\exp(-2\epsilon t)$ drops out on the right-hand side. Passing further to the supremum on $f \in \mathcal{D}(H)$, $\|f\| = 1$, we obtain the equality $\gamma_1 = \gamma_2$. From (1.4.4) and (2.7.1) it also follows that

$$
\int_{-\infty}^{\infty} (GR(\lambda + i\epsilon) f, GR(\lambda - i\epsilon) f) d\lambda = 0.
$$
Since by definition (1.4.5)
\[ (2\pi)^2 \| G\delta(\lambda,\varepsilon)f \|^2 = \| GR(\lambda + i\varepsilon)f \|^2 + \| GR(\lambda - i\varepsilon)f \|^2 - 2 \text{Re}(GR(\lambda + i\varepsilon)f, GR(\lambda - i\varepsilon)f), \]
this shows that \( \gamma_2 = \gamma_3 \).

We shall now verify that \( \gamma_3 \leq \gamma_4 \leq \gamma_5 \leq \gamma_3 \). To prove the first of these estimates we use the inequality
\[ \int_{-\infty}^{\infty} \| G\delta(\lambda,\varepsilon)f \|^2 d\lambda \leq \sup_{\lambda} \| G\delta(\lambda,\varepsilon) \|^{1/2} \int_{-\infty}^{\infty} \| \delta(\lambda,\varepsilon) \|^{1/2} d\lambda. \]
On the right-hand side the first factor does not exceed \( \gamma_3^2 \) while the integral is equal to \( \| f \|^2 \) according to (1.4.6). From this it follows that \( \gamma_3 \leq \gamma_4 \).

Further, under condition (5) for any bounded set \( \mathcal{X} \)
\[ (E(\lambda)G_x^*f, G_x^*f) \leq \gamma_3^2 \| \lambda \| \| f \|^2. \]
Thus, the element \( G_x^*f \) is absolutely continuous and
\[ \frac{d(E(\lambda)G_x^*f, G_x^*f)}{d\lambda} \leq \gamma_3^2 \| f \|^2. \]
Applying now the spectral theorem we obtain the inequality
\[ (\delta(\lambda,\varepsilon)G_x^*f, G_x^*f) = \varepsilon \pi^{-1} \int_{-\infty}^{\infty} \frac{d(E(\mu)G_x^*f, G_x^*f)}{d\mu} d\mu \]
\[ \leq \sup_{\lambda} \frac{d(E(\lambda)G_x^*f, G_x^*f)}{d\lambda} \pi^{-1} \int_{-\infty}^{\infty} (|\mu - \lambda|^2 + \varepsilon^2)^{-1} d\mu \]
\[ \leq \gamma_3^2 \| f \|^2. \]
This implies that
\[ \| G\delta(\lambda,\varepsilon)^{1/2} E(\lambda(X)) \| \leq \gamma_3. \]
Since \( X \) is arbitrary the same sort of inequality holds also without the operator \( E(\lambda(X)) \) on the left-hand side, i.e., \( \gamma_4 \leq \gamma_3 \). Finally, according to Stone's formula (1.4.7),
\[ |2^{-1} (G(E(\lambda) + E(\lambda)), f, g)|^2 = \lim_{\varepsilon \to 0} \int_{A} (G \delta(\lambda,\varepsilon)f, g) d\lambda. \]
We estimate the right-hand side by Schwarz' inequality in terms of
\[ \lim_{\varepsilon \to 0} \int_{A} \| G\delta(\lambda,\varepsilon)f \|^2 d\lambda \| g \|^2 |A| \leq (\gamma_3 \| f \| \| g \|)^2 |A|. \]
Comparing the last two relations, we find that \( \gamma_3 \leq \gamma_3 \).  

**Definition 2.** If \( G \) is \( H \)-bounded and one of the inequalities (1)–(5) holds (and then all of them), then the operator \( G \) is called Kato smooth relative to the operator \( H \) (\( H \)-smooth). The common value of the quantities \( \gamma_1, \ldots, \gamma_3 \) we denote by \( \gamma_H(G) \). As a rule, we call Kato smoothness simply smoothness.

### §3. Kato Smoothness

We make two further remarks regarding the definition of \( H \)-smoothness.

**Remark 3.** The quantities \( \gamma_2 - \gamma_4 \) do not change if the suprema on \( \varepsilon \) in (2)–(4) are taken only over the interval \( (0, \varepsilon_0) \) for some \( \varepsilon_0 > 0 \). Indeed, denote for the time being the corresponding quantities by \( \gamma_2' - \gamma_4' \). The proof of Theorem 1 is unchanged if the numbers \( \gamma_j \) are replaced by \( \gamma_j' \), \( j = 2, 3, 4 \). Therefore, \( \gamma_1 = \gamma_2' = \gamma_3' = \gamma_4 = \gamma_2 \), and hence \( \gamma_j = \gamma_j' \), \( j = 2, 3, 4 \), by Theorem 1.

**Remark 4.** It is possible to give other expressions for the number \( \gamma_H(G) \).

For example, for each of the signs "\( + " \)
\[ \gamma_H^2(G) = (2\pi)^{-1} \sup_{t \in \mathcal{F}(H), \| t \| = 1} \int_{0}^{\infty} \| GU(\pm t)f \|^2 dt, \]  
\[ \gamma_H^2(G) = (2\pi)^{-1} \sup_{t \in \mathcal{F}(H), \| t \| = 1} \int_{0}^{\infty} \| GU(\pm t)f \|^2 dt, \]
and also
\[ \gamma_H^2(G) = (2\pi)^{-1} \sup_{t \in \mathcal{F}(H), \| t \| = 1} \| GU(\pm t)f \|^2 \]
\[ \| GR(\lambda \pm i\varepsilon)f \|^2. \]

**Proof.** To prove (8) we set \( f = U(\mp s)f_0 \). Then
\[ \int_{0}^{\infty} \| GU(\pm t)f_0 \|^2 dt = \int_{0}^{\infty} \| GU(\pm t)f_0 \|^2 dt. \]
Letting \( s \) tend to infinity here, we see that the supremum in (8) and (1) coincide. Now (9) follows directly from the identity (6). Finally, we establish (10). We note first of all that equality (4) can be equivalently rewritten in the form
\[ \gamma_H^2(G) = (2\pi)^{-1} \sup_{t \in \mathcal{F}(H), \| t \| = 1} \| GU(\pm t)f_0 \|^2. \]

The quantity (11), of course, is bounded above by the expression (10). To prove the opposite inequality we note that by the representation (1.4.4)
\[ \| GR(\lambda \pm i\varepsilon)f \|^2 \leq \int_{0}^{\infty} \| GU(\pm t)f_0 \|^2 e^{-\varepsilon t} dt, \]
\[ f \in \mathcal{F}(H). \]
Therefore, by the Schwarz inequality
\[ \| GR(\lambda \pm i\varepsilon)f \|^2 \leq (2\pi)^{-1} \int_{0}^{\infty} \| GU(\pm t)f_0 \|^2 dt. \]
Adding these estimates for the signs "\( + " \) and "\( - " \), we find that the right-hand side of (10) does not exceed the quantity (1). This proves equality (10).  

**Remark.** We further note that the class of \( H \)-smooth operators is invariant under multiplication on the left by bounded operators. This follows from any of the conditions (1)–(5). Moreover, by (5) this class is also invariant relative to multiplication on the right by bounded operators which commute with \( E(\cdot) \).

It is important that the concept of \( H \)-smoothness can be formulated both in time-dependent (1) and stationary (2)–(5) terms. In §5 we shall see that...
the first of these definitions is very convenient for verification of the existence of the \( W_0 \) (2.1.1). On the other hand, the \( H \)-smoothness itself, as a rule, has to be verified in stationary terms.

2. From definition (5) of \( H \)-smoothness of an operator \( G \) it follows (see inequality (7)) that for any bounded set \( X \) the elements \( G_{A} f \) are absolutely continuous. We therefore have

\[
\bigcup_{A} R(G_{A}) \subset \mathcal{H}^{(A)}, \tag{12}
\]

where the union is taken over all possible finite intervals \( A \). In particular, for \( N(G) = 0 \) the operator \( H \) is absolutely continuous.

**Proof.** Only the last assertion needs clarification. Since \( R(G_{A}) \subset N(G_{A}) = \mathcal{H} \), it follows from (12) that

\[
\mathcal{H}^{(A)} \subset \bigcap_{A} N(G_{A}).
\]

If \( GE(\Lambda)f = 0 \) for any bounded \( \Lambda \), then for \( N(G) = 0 \) we necessarily have \( f = 0 \). The subspace \( \mathcal{H}^{(A)} \) is thus trivial. \( \square \)

An operator \( G \) smooth with respect to \( H \) is a priori assumed to be \( H \)-bounded. Actually, \( H \)-smoothness imposes further restrictions on \( G \).

**Proposition 6.** Suppose the operator \( G \) is \( H \)-smooth. Then its relative \( H \)-bound is equal to zero, i.e., for any \( \varepsilon > 0 \)

\[
\| Gg \|^2 \leq C \| Hg \|^2 + C(\varepsilon) \| g \|^2, \quad g \in \mathcal{D}(H). \tag{13}
\]

**Proof.** Relation (11) implies that

\[
\tau \| G(\lambda + it)f \|^2 \leq \pi \tau \| f \|^2.
\]

Here we set \( \lambda = 0 \) and \( g = R(\pm i t) f \). Then

\[
\tau \| Gg \|^2 \leq \pi \tau \| Hg \|^2 + \tau \| g \|^2.
\]

For \( \tau = \pi \tau \varepsilon^{-1} \) (and \( \tau \to \infty \)) from this we obtain (13). \( \square \)

**Remark 7.** The a priori assumption regarding \( H \)-boundedness of \( G \) can also be removed. It suffices to assume that the operator \( G \) is defined only on the set \( \mathcal{S} \) (see Part 4 of §1.5) of compactly supported elements. Here the definition (5) remains meaningful, while in definitions (1)–(3) it is necessary to take the supreme over \( f \in \mathcal{S} \). As is evident from the proof of the inequality \( \gamma_{4} \leq \gamma_{2} \) in Theorem 1, finiteness of the quantity (5) already guarantees \( H \)-boundedness of the operator \( G \).

According to definition (2), the vector-valued function \( GR(\lambda)f \) belongs to the Hardy classes \( H_{\pm}^{2}(\mathcal{S}) \) in the upper and lower half-planes. The assertion of Theorem 1.2.2 regarding the existence of radial (and angular) limit values for a.e. \( \lambda \) for functions of the class \( H_{\pm}^{2} \) generalizes to vector-valued functions (see the book [20]). In the vector case the limit values exist in the strong sense which can be established by duplication of the scalar elements. We set

\[
\lim_{\tau \to 0} G(\lambda \pm it)f =: F^{(\pm)}(\lambda), \quad F^{(\pm)} = F_{G}^{(\pm)}, \tag{14}
\]

and \( F(\lambda) = (2\pi)^{-1} (F^{(+)}(\lambda) - F^{(-)}(\lambda)) \). Passing to the limit under the integral sign in (3) on the basis of Fatou's lemma, we find that

\[
\int_{-\infty}^{\infty} \| F(\lambda) \|^2 d\lambda \leq \gamma_{2}^{2}(H) \| f \|^2, \tag{15}
\]

The existence of the strong limits (14) is used in §§7.3, 7.4 in justifying the formula representations of §§2.7, 2.8 for the \( W_0 \) and the scattering matrix. In this chapter in the derivation of the invariance principle in §5 it is sufficient for us that the weak limit exists for a.e. \( \lambda \):

\[
w\lim_{\tau \to 0} G(\lambda, \varepsilon)f = F(\lambda), \quad F = F_{G}.
\]

This fact is proved in §5.1 (see Lemma 6) without application of theorems on vector-valued functions. By (16)

\[
\| F(\lambda) \| \leq \lim_{\varepsilon \to 0} \| G(\lambda, \varepsilon)f \|.
\]

Therefore, inequality (15) for the weak limit is also justified.

We shall illustrate the concept of relative smoothness with the example of an operator \( H \) of multiplication. In this case the natural class of \( H \)-smooth operators is the class of integral operators.

**Example 8.** Suppose \( \Omega \) is any interval of the real axis, possibly infinite, let \( H \) be multiplication by \( \lambda \) in \( \mathcal{H} = L_{2}(\Omega) \), and let \( G \) be an integral operator with kernel \( G(\lambda, \mu) \) for which

\[
\sup_{\mu \in \Omega} \int_{\Omega} |G(\lambda, \mu)|^{2} d\mu =: b < \infty.
\]

Then the operator \( G \) is \( H \)-smooth, and \( \gamma_{2}(G) < b \).

**Proof.** Applying the Schwarz inequality, we find that

\[
\| G(\lambda) f \|^{2} = \int_{\Omega} d\lambda \left| \int_{\Omega} G(\lambda, \mu) f(\mu) d\mu \right|^{2}
\leq \int_{\Omega} d\lambda \int_{\Omega} \left| G(\lambda, \mu) \right|^{2} d\mu \| f(\nu) \|^{2} d\nu.
\]

By condition (17) this quantity does not exceed \( |\lambda| \| f \|^{2} \). It remains to use definition (5). \( \square \)

An \( H \)-smooth operator \( G \) may not admit closure. In the preceding example such a \( G \) is obtained if it is assumed that \( \mathcal{H} = L_{2}(\mathbb{R}) \) and \( G(\lambda, \mu) = g(\lambda) \). For \( g \in L_{2}(\mathbb{R}) \) condition (17) is satisfied. At the same time the operator \( G \) does not admit closure. This can be seen by considering the sequence \( f_{n}(\lambda) = n^{-1} f(\lambda n^{-1}) \) where \( f \in L_{2}(\mathbb{R}) \cap L_{1}(\mathbb{R}) \) and \( \int f(\lambda) d\lambda = a \neq 0 \). Clearly, \( \| f_{n} \| \to 0 \) as \( n \to \infty \), but \( Gf_{n} = ag \).
3. The assumption of $H$-smoothness of the operator $G$ imposes too stringent conditions on the operator $H$. Thus, for $N(G) = \{0\}$ the operator $H$ must be absolutely continuous. The concept of local $H$-smoothness is considerably more flexible.

**Definition 9.** An operator $G$ is called $H$-smooth on a Borel set $\Lambda$ if the operator $GE(\Lambda)$ is $H$-smooth.

Now a priori $H$-boundedness of the operator $G$ is not assumed. In order that the product $GE(\Lambda)$ be well defined (on $D(H)$) it suffices to assume that $E(\Lambda)E(\Lambda) \subset D(H) \subset D(G)$. It is clear that under the conditions of Definition 9 the operator $G$ is smooth on any Borel set $\Lambda \subset \Lambda$. Further, according to Proposition 5, for $N(G) = \{0\}$ the spectrum of the operator $H$ on the set $\Lambda$ is absolutely continuous. The next result is convenient for practical verification of local $H$-smoothness.

**Theorem 10.** Suppose $G$ is $H$-bounded and for a Borel set $\Lambda$

$$\sup_{\lambda \in \Lambda, 0 < \varepsilon < 1} \|G\delta(\lambda, \varepsilon)G\| < \infty. \tag{18}$$

Then the operator $G$ is $H$-smooth on the closure $\bar{\Lambda}$ of the set $\Lambda$.

**Proof.** By the continuity of $GR(\lambda + i\varepsilon)$ in $\lambda$ for $\varepsilon > 0$ inequality (18) extends to all of $\bar{\Lambda}$, i.e.,

$$\sup_{\lambda \in \bar{\Lambda}, 0 < \varepsilon < 1} \varepsilon^{1/2}\|GR(\lambda + i\varepsilon)\| < \infty. \tag{19}$$

We must show that

$$\sup_{\lambda \in \bar{\Lambda}, 0 < \varepsilon < 1} \varepsilon^{1/2}\|GE(\bar{\Lambda})R(\lambda + i\varepsilon)\| < \infty. \tag{20}$$

In view of (19) in the desired estimate (20) it may be assumed that $\lambda \in \mathbb{R}\setminus\bar{\Lambda}$. Since the set $\mathbb{R}\setminus\bar{\Lambda}$ is open, each point $\lambda$ of it lies in some component interval. We denote by $\lambda^*, \lambda' \in \bar{\Lambda}$, the end point of the latter closest to $\lambda$. Then

$$\inf_{\mu \in \bar{\Lambda}} |\mu - \lambda| = |\lambda^* - \lambda|. \tag{21}$$

According to the Hilbert identity, it is now possible to write the estimate

$$\varepsilon^{1/2}\|GR(\lambda + i\varepsilon)E(\bar{\Lambda})\| \leq (\varepsilon^{1/2}\|GR(\lambda' + i\varepsilon)\||I + (\lambda - \lambda')R(\lambda + i\varepsilon)E(\bar{\Lambda})||. \tag{22}$$

For $\lambda' \in \bar{\Lambda}$, the first factor on the right-hand side is uniformly bounded by (19). It remains to note that on the basis of (21)

$$\|R(\lambda + i\varepsilon)E(\bar{\Lambda})\| = \sup_{\mu \in \bar{\Lambda}} |\mu - \lambda - i\varepsilon|^{-1} \leq |\lambda' - \lambda|^{-1},$$

and therefore the second factor in (22) does not exceed 2. □

### §4. Sufficient conditions for smoothness

1. We shall first give a sufficient condition for smoothness of "commutator" type. This condition makes it possible to construct $H$-smooth operators as soon as an auxiliary antisymmetric operator $A$ is found such that the commutator $[H, A]$ is positive. We present immediately a more general assertion of this type in which it is not assumed that $A$ is antisymmetric.

**Theorem 1.** Suppose $H$ is selfadjoint, the operator $G$ is bounded relative to \(|H|^{1/2}\), $\delta \geq 0$, and $\Lambda$ is any Borel set. We suppose that for some $|H|^{1/2}$, bounded operator $A$ for all $\lambda \in \Lambda$ and $f \in D(|H|^{1/2})$ there is the inequality

$$\Re((H - \lambda)f, Af) \geq \|Gf\|^2.$$ \tag{1}

Then the operator $G(H - i\kappa)A_f \geq G_{\delta}$ is $H$-smooth on $\bar{\Lambda}$.

**Proof.** From (1) we obtain the estimate

$$\|G\|^{2} \leq \|\delta(H - \lambda)f\|\|A\|, \quad \lambda \in \Lambda, \quad f \in D(|H|^{1/2}).$$

We apply it to the element $f = |H - i\kappa R(\lambda + i\varepsilon)g$. Then

$$\Re(G_{\delta}R(\lambda + i\varepsilon)g, g) \geq \|G\|^{2} \leq \|\delta(H - \lambda)R(\lambda + i\varepsilon)\|\|A\|\|H - i\kappa R(\lambda + i\varepsilon)\|\|g\|^{2}.$$ \tag{2}

By the spectral theorem both factors depending on $\varepsilon$ on the right-hand side do not exceed 1. Thus, the right- and therefore the left-hand side of (2) for $\lambda \in \Lambda$ can be estimated in terms of $C\|g\|^{2}$. It remains to use Theorem 3.10. □

**Corollary 2.** On any bounded subset of $\Lambda$ the operator $G$ is $H$-smooth.

We note that it suffices to verify inequality (1) on any subset of the space $D(|H|^{1/2})$ that is dense in this space with respect to its metric.

**Remark 3.** If the operator $A$ is antisymmetric, i.e., $A = i\bar{A}$ where $\lambda \in A^{*}$, then on the right-hand side of (1) the term $\Re(f, Af)$ drops out, and (1) takes the form

$$i(\bar{A}f, Hf) - i(Hf, \bar{A}f) \geq \|Gf\|^2.$$ \tag{3}

This inequality for the commutator $i[H, \bar{A}]$ guarantees global smoothness of the operator $G_{\delta}$. Giving up symmetry of $A$ makes it possible to obtain local smoothness conditions.

In concrete cases it is usually difficult to find an antisymmetric operator $A$ for which the commutator $[H, A]$ is strictly positive. Nevertheless, the previous constructions are in a certain sense reversible. Namely, we have

**Proposition 4.** Suppose the operator $G$ is $H$-smooth. Then there exists a bounded operator $A = -A^{*}$ such that

$$(Af, Hg) - (AHf, g) = (Gf, Gg)$$ \tag{3}

for any $f, g \in D(H)$. 

4. SCATTERING FOR RELATIVELY SMOOTH DISTURBANCES

134
4. SCATTERING FOR RELATIVELY SMOOTH PERTURBATIONS

Proof. We shall construct two operators \( A = A_\pm \) satisfying (3). We define them in terms of the sesquilinear forms

\[
a_\pm(f, g) = \pm i \int_0^\infty \langle G(U(t)f, G(U(t)g) \rangle \, dt.
\]

By definition (3.1) of \( H \)-smoothness of the operator \( G \) for \( f, g \in \mathcal{D}(H) \) the integral (4) converges and according to the Schwarz inequality can be estimated by

\[
\left\{ \int_{-\infty}^\infty \| G(U(t)f) \|^2 \, dt \int_{-\infty}^\infty \| G(U(t)g) \|^2 \, dt \right\}^{1/2} \leq 2 \pi \gamma_2^2(G) \| f \| \| g \|.
\]

Therefore, to the form (4) there corresponds a bounded operator \( A_\pm \) such that \( (A_\pm f, g) = a_\pm(f, g) \) for \( f, g \in \mathcal{D}(H) \). Since \( a_\pm(f, g) = -a_\pm(g, f) \), the operators \( A_\pm \) are antisymmetric. By definition (4)

\[
(A_\pm U(\pm s)f, U(\pm s)g) = \pm i \int_0^{\infty} \langle G(U(\pm t)f, G(U(\pm t)g) \rangle \, dt.
\]

Differentiating this equality with respect to \( s \), we find that

\[
-(A_\pm U(\pm s)HF, U(\pm s)g) + (A_\pm U(\pm s)f, U(\pm s)HG) = \langle G(U(\pm s)f, G(U(\pm s)g) \rangle.
\]

Setting here \( s = 0 \) we obtain (3). \( \square \)

2. In applications it is sometimes convenient to formulate sufficient conditions for \( H \)-smoothness in terms of a diagonal for \( H \) decomposition of \( \mathcal{F} \) into a direct integral. We suppose that on a compact interval \( \Lambda = [a, b] \) the spectrum of the operator \( H \) is absolutely continuous and has constant (possibly infinite) multiplicity \( k \). We consider (see §1.5) a unitary mapping \( \mathcal{F} \) of the space \( E(\Lambda) \mathcal{F} \) onto \( L_2(\Lambda; h) \), \( \dim h = k \), under which \( H \) turns into a multiplication operator: if \( f \in E(\Lambda) \mathcal{F} \) and \( \mathcal{F} f = \hat{f} \), then \( (\mathcal{F} E(X)f)(\lambda) = \mathcal{F}(\lambda)\hat{f}(\lambda) \). Along with \( L_2(\Lambda; h) \) we consider the space \( C^\alpha(\Lambda; h) \), \( \alpha > 0 \), of Hölder continuous vector-valued functions on \( \Lambda \) with values in \( h \). The norm in this space is given by relation (1.11). Regarding the operator \( G \colon \mathcal{F} \rightarrow \mathcal{F} \) it is assumed that \( E(\Lambda) \mathcal{F} \subset C(G) \) and the operator \( G_\Lambda = G|_{E(\Lambda)} \) is bounded.

Definition 5. An operator \( G \) is called strongly \( H \)-smooth (with exponent \( \alpha \in (0, 1] \)) on \( \Lambda \) if the operator \( \mathcal{F} G_\Lambda \) maps \( \mathcal{F} \) continuously into \( C^\alpha(\Lambda; h) \), i.e., for \( g = G^*_f \)

\[
\left\| \hat{g}(\lambda) \right\|_h \leq C \| f \|_h, \quad \left\| \hat{g}(\lambda) - \hat{f}(\lambda) \right\|_h \leq C(1 - \mu)|\lambda|, \quad \mu = \mu_\Lambda
\]

where \( C \) depends neither on \( f \) nor on \( \lambda, \mu \in \Lambda \).

It is necessary to bear in mind that the definition presented is not unitarily invariant, since it depends on the choice of the mapping \( \mathcal{F} \). The next assertion clarifies the term introduced.

Lemma 6. An operator which is strongly \( H \)-smooth on any segment \( \Lambda \) is Kato smooth there.

Proof. According to (1.5.3) for \( g = G^*_f \) and any interval \( X \)

\[
\left\| E(X)G^*_f \right\|_2 \leq \left\| E(X \cap \Lambda)g \right\|_2
\]

\[
= \int_X \left\| \hat{g}(\lambda) \right\|^2 d\lambda \leq sup_{\lambda \in \Lambda} \left\| \hat{g}(\lambda) \right\|^2 |X|.
\]

Thus under the first of conditions (5) the operator \( G_\Lambda \) satisfies inequality (3.5). By Definition 3.9 this implies that \( G \) is \( H \)-smooth on \( \Lambda \). \( \square \)

Actually, a stronger assertion, also formulated in unitarily invariant terms, holds for strongly \( H \)-smooth operators. As in §1.9, we set \( E(\Lambda)G^* \subset G_\Lambda \).

Theorem 7. Suppose the operators \( G_1 \) and \( G_2 \) are strongly \( H \)-smooth on \( \Lambda \). Then the spectral measure \( G_1 E(\mathcal{S})G_2 \) is weakly differentiable inside \( \Lambda \), and its derivative is Hölder continuous on \( \Lambda \) in the operator norm in \( \mathcal{S} \). If, moreover, the operators \( G_2 \) are \( H \)-bounded, then the operator-valued function \( G_1 \mathcal{S}(\lambda)G_2 \) is Hölder continuous with respect to the parameters \( \lambda \in (a, b) \) and \( \varepsilon \geq 0 \). Under the additional assumption of \( |H| \) boundedness of the operators \( G_1, \theta_1 \in [0, 1] \) and \( \theta_1 + \theta_2 = 1 \), the operator-valued function \( G_1 R(z)G_2 \) of the variable \( z \) is Hölder continuous in norm for \( Re z \in \Lambda \) up to the cut along \( \Lambda \) (with the exception of the end points of \( \Lambda \)).

Proof. Suppose the operators \( G_j, j = 1, 2 \), satisfy Definition 5 with exponent \( \alpha > 0 \). Then for any \( f_j \in \mathcal{S} \) the elements \( g_j = G_j^*f_j \) the function

\[
\mathcal{S}(\lambda) = \mathcal{S}(\lambda; f_1, f_2) = (g_1(\lambda), g_2(\lambda))
\]

is Hölder continuous on \( \Lambda \) and

\[
|\mathcal{S}(\lambda) - \mathcal{S}(\mu)| \leq C|\lambda - \mu|^\alpha \| f_1 \| \| f_2 \|.
\]

Since

\[
E(X)g_1, g_2 = \int_X \mathcal{S}(\lambda) \, d\lambda, \quad X \subset \Lambda,
\]

it follows from (6) that the weak derivative of \( G_j E(X)G_j^* \) exists and is Hölder continuous in norm (with exponent \( \alpha > 0 \)). By Theorem 1.2.6 this implies in turn that the function

\[
(G_1 R(z)E(\Lambda)G_2^*f_1, f_2) = \int_\Lambda (\lambda - z)^{-1} \mathcal{S}(\lambda; f_1, f_2) \, d\lambda
\]

is Hölder continuous (uniformly with respect to \( f_j \) with \( \| f_j \| \leq 1 \) in \( z \) for \( a < Re z < b \) up to the cut along \( \Lambda \). To complete the consideration of the operator-valued function \( G_1 R(z)E(\Lambda)G_2^* \) it remains to observe that by the subordination condition for the operators \( G_j \) the remainder \( G_2 R(z)E(\Lambda)G_1^* \) is holomorphic in the entire strip \( a < Re z < b \). In a similar way, to
prove the assertion regarding $G_\lambda \delta(\alpha, \epsilon)G_\lambda^2$ it suffices to use the smoothness of $G_\lambda R(z)R(\xi)E(\eta(\lambda))G_\lambda^2$ for $\alpha < \Re z < \beta$. □

§5. The WO for smooth perturbations

With the help of the time-dependent definition (3.1) in the theory of (Kato) smooth perturbations the existence of the WO $W_\lambda(H, H_0; 3)$ can be verified in an entirely elementary manner. The existence of the adjoint WO $W_\lambda(H_0, H; 3^\ast)$ is simultaneously established in view of the symmetry of the conditions imposed. We also consider local WO here. In addition, the invariance principle is proved.

1. Suppose, as always, that $H_0$ and $H$ are selfadjoint operators in Hilbert spaces $\mathcal{H}_0$ and $\mathcal{H}$, while $3: \mathcal{H}_0 \to \mathcal{H}$ is a bounded identification. We suppose that the perturbation $V = H3 - 3H_0$ can be represented in the form (1.9.2), where the operators $G_0: \mathcal{H}_0 \to \mathcal{H}$ and $G: \mathcal{H} \to \mathcal{H}$ are bounded relative to $H_0$ and $H$, respectively. For $f_0 \in \mathcal{H}(H_0)$, $f \in \mathcal{H}(H)$ there exists the derivative

$$
\frac{\partial}{\partial t} (3U(t)f_0, U(t)f) = -i(3H_0U(t)f_0, U(t)f) + i(3U(t)f_0, HU(t)f) = i(G_0U(t)f_0, GU(t)f).
$$

From this it follows that for the operator (2.1.2)

$$
(W(t_2)f_0, f) - (W(t_1)f_0, f) = i \int_{t_1}^{t_2} (G_0U(s)f_0, GU(s)f) \, ds. \tag{1}
$$

**Theorem 1.** Suppose relation (1.9.3) holds, where the operators $G_0$ and $G$ are smooth relative to the operators $H_0$ and $H$, respectively. Then both WO $W_\lambda(H, H_0; 3)$ exist.

**Proof.** Applying the Schwarz inequality and considering definition (2.1) of $H$-smoothness of the operator $\mathfrak{G}$, we estimate the expression (1) by

$$(2\pi)^{1/2} \gamma_\mu(G) \| s \int_{t_1}^{t_2} \| G_0U(s)f_0 \|^2 \, ds \right)^{1/2}$$

This gives the estimate

$$
\| W(t_2)f_0 - W(t_1)f_0 \| \leq (2\pi)^{1/2} \gamma_\mu(G) \left( \int_{t_1}^{t_2} \| G_0U(s)f_0 \|^2 \, ds \right)^{1/2}. \tag{2}
$$

By the $H_0$-smoothness of the operator $G_0$, the last integral tends to zero as $t_j \to \pm \infty$, $j = 1, 2$. Therefore, for $f_0 \in \mathcal{H}(H_0)$, and hence also for any $f_0 \in \mathcal{H}_0$, the strong limit of the vector-valued function $W(t)f_0$ as $t \to \pm \infty$ exists. □

Corollary 2. The WO $W_\lambda(H_0, H; \lambda^*)$ exist under the conditions of Theorem 1. Moreover, the WO $W_\lambda(H, H_0; \lambda^*)$ are $3$-complete (see Definition 3.2.3), while the WO $W_\lambda(H_0, H; \lambda^*)$ are $3^\ast$-complete.

**Proof.** In view of the symmetry of the conditions of Theorem 1, it can be applied to the pair $H_0$, $H$ and the identification $3^\ast$. This gives the existence of $W_\lambda(H_0, H; \lambda^*)$. The assertion regarding $3^\ast$- (or $3^\ast$-) completeness of the WO follows from Theorem 3.2.4. □

In particular, for $\mathfrak{H}_0 = \mathfrak{H}$ and $3 = I$ under the conditions of Theorem 1 the WO $W_\lambda(H, H_0; \lambda^*)$ and $W_\lambda(H_0, H; \lambda^*)$ are complete in the sense of Definition 2.3.1, i.e., their ranges coincide with the absolutely continuous subspaces of the operators $H$ and $H_0$, respectively.

**Remark 3.** The assertion of Theorem 1 remains in force if in place of (1.9.2) the representation

$$
V = \sum_{k=1}^{\infty} G_\lambda k G_0 k
$$

holds, where all the operators $G_0 k: \mathfrak{H}_0 \to \mathfrak{H}$ and $G k: \mathfrak{H} \to \mathfrak{H}$ are smooth relative to $H_0$ and $H$, respectively. This case reduces to that already considered by passing to the new operators

$$
G_0 f = (G_0, 1, \ldots, G_0, f), \quad G f = (G, f, \ldots, G, f),
$$

acting from $\mathfrak{H}$ into the "vector" space $\mathfrak{H} \otimes \mathbb{C}^n$. For them (3) can be written in the standard form $V = G^* G_0$.

**Remark 4.** In the proof of Theorem 1 we have used to full extent only the smoothness of the operator $G$ relative to $H$. As concerns the pair $G_0, H_0$, for the existence of $W_\lambda(H, H_0; 3)$ it suffices that the integral

$$
\int_0^\infty \| G_0U(s) f_0 \|^2 \, ds
$$

converges for some set of elements $f_0$ dense in $\mathfrak{H}_0$ (or at least in $\mathfrak{H}_0^{(a)}$).

**Remark 5.** In the proof of Theorem 1 it was not assumed that $f_0 \in \mathfrak{H}_0^{(a)}$.

Therefore, under the conditions of that theorem there exists the strong limit as $t \to \pm \infty$ of the operator-valued function $W(t)$ (without the projection $P_0$ on the right). This agrees with Proposition 3.5, which asserts the absolute continuity of $H_0$ under the additional assumption $N(G_0) = \{0\}$. Similarly, the strong limit of $W^*(t)$ exists without the projection $P$ on the right. Thus, for $\mathfrak{H}_0 = \mathfrak{H}$ and $3 = I$ the WO $W_\lambda(H, H_0)$ is not only complete but also unitary.

In a real situation the WO $W_\lambda(H, H_0)$ are usually not unitary. Therefore, the assertion of Theorem 1 is "too strong," while its conditions are rarely satisfied. In connection with this we further remark that for $H_0\psi = \lambda \psi$ under the conditions of Theorem 1 we must have $H^2 \psi = \lambda^2 \psi$. Such invariance of the discrete spectrum is hardly likely. Therefore, the conditions of Theorem 1 practically require that both operators $H_0$ and $H$ be absolutely continuous.
2. Scattering theory for perturbations that are smooth only locally is free from the "shortcomings" mentioned. We will now consider equality (1.9.3), giving a precise meaning to (1.9.2), only on the sets $\mathcal{E}_0$ and $\mathcal{E}$ of compactly supported elements $f_0$ and $f$. Along with it, it is assumed that $\mathcal{E}_0 \subset \mathcal{D}(G_0)$ and $\mathcal{E} \subset \mathcal{D}(G)$. For locally smooth perturbations an analogue of Theorem 1 holds. In it we consider the local WO defined by relation (2.2.5).

**Theorem 6.** Suppose the perturbation $V = H_3 - 3H_0$ can be represented in the form $G^2G_0$, where the operators $G$ and $G_0$ are smooth on some interval $\Lambda$ relative to the operators $H_0$ and $H$, respectively. Then the WO $W_\pm(H, H_0 ; \mathcal{E}, \Lambda)$ exist.

**Proof.** Since the operators $G_0E_0(\Lambda)$ and $GE(\Lambda)$ are smooth relative to $H_0$ and $H$, the existence of the limit

$$\lim_{t \to \pm \infty} E(\Lambda)U(-t)3U_0(t)E_0(\Lambda)$$

(4) can be verified in full analogy to the proof of Theorem 1. The limit of an expression of the form (4) with the additional factor $P_0$ on the right then exists a fortiori. Therefore, to prove the theorem it is only necessary to show that

$$\lim_{t \to \pm \infty} E(\Lambda)U_0(t)E_0(\Lambda)P_0 = 0, \quad \Lambda' = \mathbb{R} \setminus \Lambda. \quad (5)$$

It suffices to verify relation (5) on any dense set in $E_0(\Lambda)^{E(a)}$. Hence, the operator $E_0(\Lambda)$ in (5) can be replaced by $E_0(\Lambda_0)$, where $\Lambda_0$ is any bounded and strictly interior subinterval of $\Lambda$. Suppose the simple closed contour $\Gamma$ encompasses $\Lambda_0$, does not intersect $\Lambda'$, and is traversed counterclockwise. By formula (1.4.8), applied to the resolvent of the operator $H_0E_0(\Lambda_0)$ in the space $E_0(\Delta)^{E(a)}$, we have

$$2\pi iE_0(\Lambda_0) = -\int_{\Gamma} R_0(z)E_0(\Lambda_0) \, dz.$$

Moreover, we apply (1.4.8) to the resolvent of the operator $HE(\Lambda')$ in $E(\Lambda')^{E(a)}$. Since $\Lambda'$ lies outside $\Gamma$,

$$\int_{\Gamma} E(\Lambda')R(z) \, dz = 0.$$

From this it follows that

$$2\pi iE(\Lambda')3E_0(\Lambda_0) = \int_{\Gamma} E(\Lambda')(R(z)3 - 3R_0(z))E_0(\Lambda_0) \, dz. \quad (6)$$

In view of the resolvent identity (1.9.4) for $z \in \Gamma$

$$E(\Lambda')(R(z)3 - 3R_0(z))E_0(\Lambda_0) = -(GR(\mathcal{Z})E(\Lambda'))^*G_0R_0(z)E_0(\Lambda_0),$$

where the operator $GR(\mathcal{Z})E(\Lambda')$ is bounded.

According to (6), to prove relation (5) it remains to verify that the vector-valued function

$$\varphi(t) = G_0R_0(z)U_0(t)E_0(\Lambda_0)f, \quad z \in \Gamma,$$

tends (strongly) to zero as $|t| \to \infty$. By definition (3.1) of the $H_0$-smoothness of the operator $G_0E_0(\Lambda_0)$ the function $\|\varphi(t)\|^2$ is integrable on $t$. Moreover, $\|\varphi(t)\| \leq C < \infty$ for a bounded $\Lambda_0$. From this we find the desired relation

$$\|\varphi(t)\| = o(1), \quad |t| \to \infty.$$  

Indeed, suppose there exists a sequence $t_n \to \infty$ such that $\|\varphi(t_n)\| \geq c > 0$. By the uniform boundedness of $\varphi(t)$ there exists a number $\tau > 0$ (not depending on $n$) such that $\|\varphi(t)\| \geq c/2$ for $t \in (t_n - \tau, t_n + \tau)$. This contradicts the integrability of $\|\varphi(t)\|^2$.

The concept of local smoothness is also useful in the proof of existence of "global" WO. Namely, Theorem 6 implies

**Corollary 7.** Suppose that the representation (1.9.3) holds, where the operator $G_0$ is $H_0$-bounded and $G$ is $H$-bounded. Suppose $\{\Lambda_n\}$ is a set of intervals whose union exhausts the core of the spectra of the operators $H_0$ and $H$ (up to a set of Lebesgue measure zero). If on each of the intervals $\Lambda_n$ the operator $G_0$ is $H_0$-smooth and $G$ is $H$-smooth, then the WO $W_\pm(H, H_0 ; \mathcal{E})$ and $W_\pm(H_0, H ; \mathcal{E})$ exist.

We emphasize that under the conditions of Corollary 7 the operators $H_0$ and $H$ may have nontrivial singular spectra. In particular, the eigenvalues of the operators $H_0$ and $H$ may not coincide.

3. An invariance principle (IP) holds in the theory of smooth perturbations; see §2.6. Suppose the function $\varphi$ is admissible (in the sense of Definition 2.6.2) on an open set $\Omega$ with respect to the selfadjoint operator $H_0$ and $H$. We set $\Omega_+ = \{\lambda \in \Omega : \pm \varphi(\lambda) > 0\}$. To justify the IP we need

**Lemma 8.** Suppose the interval $\Lambda \subset \Omega_+$, $\nu = \pm$", the operator $G$ is smooth relative to the selfadjoint operator $H$, and $f \in E(\Lambda)^{E(a)} \cap \mathcal{D}(H)$. Then

$$\lim_{t \to \pm \infty} \int_0^\infty \|G \exp(\pi isH - it\varphi(H))f\|^2 \, ds = 0. \quad (7)$$

**Proof.** Suppose $\{\omega_k\}$ is some orthonormal basis in $\Phi$, $G_t = G(H + i\lambda)^{-1}$, $f_t = (H + i\lambda)f$. Then

$$\|G \exp(\pi isH - it\varphi(H))f\|^2 = \sum_k \|\exp(\pi isH - it\varphi(H))f_t, G_t\omega_k\|^2$$

$$= \sum_k \left| \int_{\Lambda} \exp(\pi is\lambda - it\varphi(\lambda)) \frac{d(E(\lambda)f_t, G_t\omega_k)}{d\lambda} \right|^2.$$

Here we have used the fact that for $f \in E(\Lambda)^{E(a)}$ the integration can be restricted to $\Lambda$, and that, by Proposition 3.5, the function $(E(\lambda)f_t, G_t\omega_k)$ is absolutely continuous. From equalities (1.4.11) and (3.16) it follows that
4. SCATTERING FOR RELATIVELY SMOOTH PERTURBATIONS

d(E(\lambda)f_1, G_1^*(\omega_k), d\lambda = (F(\lambda), \omega_k), \text{ and hence the integral in (7) is}
\text{equal to}
\\sum_k \int_0^\infty ds \left| \int_A \exp(\pm is\lambda - \imath t\varphi(\lambda))(F(\lambda), \omega_k) d\lambda \right|^2.
\tag{8}

By Lemma 2.6.4 each term of the series over \(k\) tends to zero as \(t \to \pm \infty\).

At the same time, according to the Parseval equality and the estimate (3.15), the series (8) is majorized by the convergent numerical series
\[2\pi \sum_k \int_A \|F(\lambda), \omega_k\|^2 d\lambda = 2\pi \int_A \|F(\lambda)\|^2 d\lambda < \infty.\]

Therefore, by Lebesgue’s theorem the entire sum (8) tends to zero as \(t \to \pm \infty\). \(\Box\)

For smooth perturbations the IP can be formulated in the following manner.

\textbf{Theorem 9.} Under the conditions of Theorem 1 for the pair \(h_0 = \varphi(H_0), h = \varphi(H)\) the WO \(W_\pm(h, h_0; \mathcal{I})\) exist and equality (2.6.11) holds.

\textbf{Proof.} Let \(\Lambda\) be one of the component intervals of \(\Omega\). In view of Lemma 2.6.3 it suffices to verify that

\[\lim_{t \to \pm \infty} (W_\pm(H, H_0; \mathcal{I}) - I) \exp(-\imath t h_0) \mathcal{E}_0(\Lambda) = 0, \quad \Lambda \subset \Omega.\]

By equality (1) for this, in turn, it suffices to establish the estimate
\[\int_0^\infty (G_0 U_0(\pm s) \exp(-\imath t h_0) f_0, G U(\pm s) f) ds \leq \varepsilon(t) \|f\|, \quad f \in \mathcal{D}(H),\]

where \(f_0 \in \mathcal{E}_0(\Lambda) \mathcal{P}_0 \cap \mathcal{P}(H_0)\), and \(\varepsilon(t) \to 0\) as \(t\) tends to the corresponding infinity. Applying the Schwarz inequality and considering the definition (3.1) of \(H\)-smoothness of the operator \(G\), we estimate the left-hand side of (9) (cf. 2.1) by

\[(2\pi)^{1/2} \gamma_H(G) \|f\| \left(\int_0^\infty \|G_0 U_0(\pm s) \exp(-\imath t h_0) f_0\|^2 ds\right)^{1/2}.\]

By Lemma 8 this integral tends to zero as \(t \to \pm \infty\). This completes the verification of (9). \(\Box\)

In an altogether similar manner it can be verified that under the conditions of the “local” Theorem 6 there exist
\[\lim_{t \to \pm \infty} E(\lambda) \exp(\imath t h_0) E_0^{(a)}(\lambda) = W_\pm(H, H_0; \mathcal{I}, \mathcal{A}).\]

Here the function \(\varphi\) is admissible on the interval \(\Lambda\) and extended boundedly (in arbitrary manner) to \(\mathbb{R} \setminus \Lambda\). The projection \(E(\Lambda)\) on the left in (10) can be removed, for example, in the case of compactness of the operator \(\mathcal{I}^* \mathcal{I} - I\)

\[\text{(and, in particular, for } \mathcal{I} = I\text{). This follows from the standard computation (cf. the proof of Theorem 2.2.1)}\]

\[\|u(-\imath t) \mathcal{I} u_0(t) f_0 - W_\pm f_0\|^2 \]

\[= \|u_0(t) f_0\|^2 - 2 \Re\{E(\lambda) u(-\imath t) \mathcal{I} u_0(t) f_0, W_\pm f_0\} + \|W_\pm f_0\|^2 \quad (11)\]

\[= \|u_0(t) f_0\|^2 - \|W_\pm f_0\|^2 + o(1).\]

Here \(f_0 \in E_0^{(a)}(\Lambda) \mathcal{P}_0, W_\pm = W_\pm(H, H_0; \mathcal{I}, A), u(t) = \exp(-\imath t \varphi(H_0)), u(t) = \exp(-\imath t \varphi(H)).\) In (11) we have used the existence in (10) only of the weak limit. By Lemma 1.4.1 for \((\mathcal{I}^* \mathcal{I} - I)\mathcal{E}_0(\Lambda) \subset \mathcal{E}_\infty\) the first term on the right in (11) tends to \(\|f_0\|^2\), while \(\|W_\pm f_0\| = \|f_0\|\). Therefore, (10) is satisfied also without the projection \(E(\Lambda)\) on the left.

\section{6. Smoothness with respect to the full Hamiltonian}

In applications, as a rule, it is possible to directly verify smoothness only relative to the “unperturbed” operator \(H_0\). This impedes the use of the results of the preceding section. Here we present a stationary method of verifying \(H\)-smoothness based essentially on perturbation theory. This method makes it possible to reduce the proof of the existence and completeness of WO to verification of certain properties of the perturbation relative to the single operator \(H_0\). In connection with this the ideology of the exposition now changes somewhat. The full Hamiltonian \(H\) is no longer assumed to be given a priori, while a correct definition of it as a selfadjoint operator is a consequence of assumptions regarding the perturbation. We restrict ourselves to considering the case \(\mathcal{H}_0 = \mathcal{H}, \mathcal{I} = I\). Generalization to the case of a pair of spaces is possible, but requires invertibility of \(\mathcal{I}\).

Investigation of relatively compact perturbations is preceded by the study of the essentially simpler case of small operators \(V\). The considerations in this section have local character. Conditions for existence and completeness of “global” WO are derived from local results.

1. Let \(H_0\) be a selfadjoint operator in a Hilbert space \(\mathcal{H}\), and let \(V = G^* G_0\) be a perturbation satisfying conditions (1.9.6) (for some \(\theta_0 \in [1/2, 1]\) and \(\theta = 1 - \theta_0\)) and (1.9.7). Under the additional assumptions required below, the operators (1.9.18) exist for \(\text{Im} z \neq 0\). Therefore, by Theorem 1.10.3 there exists (and, of course, unique) a selfadjoint operator \(H\) corresponding to the sum \(H_0 + V\) in the sense of Definition 1.9.2. Relation (1.9.8) holds for the resolvent \(R(z)\) of this operator, so that the operator \(G\) is clearly \(H\)-bounded and, moreover, the product \(GR(z)G^*\) is well defined.

The study of \(R(z)\) is based on the resolvent identity (1.9.13) from which we obtain the required relation

\[GR(z)G^* = GR_0(z)G^* - (GR(z)G^*)(G_0 R_0(z)G^*), \quad \text{Im} z \neq 0.\]
We set
\[ B_0(z) = -G_0 R_0(z) G^* , \quad B^{(0)}(z) = GR_0(z) G^* , \quad B(z) = GR(z) G^* . \] (2)
Then (1) may be considered as an equation for the operator $B(z)$. Solving this equation, we find that
\[ B(z) = B^{(0)}(z)[I - B_0(z)]^{-1}, \quad \text{Im} \ z \neq 0, \] (3)
whereby the inverse operator exists by Theorem 1.9.5.

In the construction of scattering theory for the pair $H_0, H$ in this section we assume that $I = I$, but we consider local WO $W_0(H; H_0; \Lambda)$ connected with an arbitrary interval $\Lambda \subset \mathbb{R}$. We recall (see Part 3 of §2.3) that for $I = I$ the completeness of the WO $W_0(H; H_0; \Lambda)$ is equivalent to the existence of the “inverse” WO $W_0(H_0; H; \Lambda)$. Therefore, the completeness of all these WO follows from the existence of the WO $W_0(H_0; H; \Lambda)$ and $W_0(H_0; H; \Lambda)$ established below. It suffices to verify existence of the WO on a system of intervals $\Lambda_n$, whose union exhausts $\Lambda$ up to a set of measure zero.

We first consider the case of small perturbations.

**Theorem 1.** Suppose 1° the operator $G_0$ is $H_0$-smooth on $\Lambda$, 2° the operator-valued function $GR_0(z)G^*$ is uniformly bounded for $\text{Re} \ z \in \Lambda$, $\text{Im} \ z \neq 0$, and 3° there is the inequality
\[ \sup_{\text{Re} \ z \in \Lambda} \|G_0 R_0(z) G^*\| < 1. \] (4)
Then the WO $W_0(H; H_0; \Lambda)$ exist and are complete.

**Proof.** We note first of all that condition (4) guarantees the existence of the operators (1.9.18) for all $z$ with $\text{Re} \ z \in \Lambda$, $\text{Im} \ z \neq 0$. Therefore, the operator $H$ is well defined, and relation (3) holds for the corresponding operator $B(z)$ (see (2)). By conditions 2°, 3° both factors on the right in (3) are uniformly bounded for $\text{Re} \ z \in \Lambda$. Thus, the operator $B(z)$ is also bounded up to $\Lambda$ and hence, according to Theorem 3.10, the operator $G$ is $H$-smooth on $\Lambda$. Recalling further condition 1°, by Theorem 5.6 we find that all four WO $W_0(H; H_0; \Lambda)$ and $W_0(H_0; H; \Lambda)$ exist. □

For relatively compact perturbations we have

**Theorem 2.** Suppose 1° the operator $G_0$ is $H_0$-smooth on $\Lambda$, 2° the operator-valued functions $G_0 R_0(z) G^*$ and $GR_0(z) G^*$, holomorphic for $\text{Re} \ z \in \Lambda$, $\text{Im} \ z \neq 0$, are continuous in norm up to the “cut” along $\Lambda$, and 3° $(G_0 R_0(z) G^*)^n \in \mathfrak{S}_\alpha$ for $\text{Im} \ z \neq 0$ and some natural number $p$. Then the WO $W_0(H; H_0; \Lambda)$ exist and are complete.

**Proof.** By Lemma 1.10.5 and the corollary to it the Hamiltonian $H$ is well defined, $(I - B_0(z))^{-1}$ exists for $\text{Im} \ z \neq 0$, and relation (3) holds for the operators (2). To verify $H$-smoothness of the operator $G$ we apply Theorem 1.8.3 to the operator-valued function $I - B_0(z)$. By that theorem for some system of intervals $\Omega_n$ such that
\[ |\Lambda \setminus \Lambda_n| = 0, \quad \Lambda_n = \bigcup_n \Lambda_n, \] (5)
the operator-valued function $(I - B_0(z))^{-1}$ is continuous in $z$ (for $\text{Im} \ z \geq 0$ and $\text{Im} \ z \leq 0$) up to the “cuts” $\Lambda_n$. According to (3), the same is true for $B(z)$, so that the operator $G$ is $H$-smooth on $\Omega_n$. Further, using condition 1°, from Theorem 5.6 we find that the WO $W_0(H; H_0; \Lambda_n)$ and $W_0(H_0; H; \Lambda_n)$ exist. In view of (5) from this we obtain the existence of the WO $W_0(H; H_0; \Lambda)$ and $W_0(H_0; H; \Lambda)$. □

**Remark 3.** Condition 3° of this theorem can be replaced by the assumptions of existence of the operators $(I - B_0(z_\pm))^{-1}$ for two points $z_\pm \in \rho(H_0)$, $\pm \text{Im} \ z_\pm \geq 0$, and of compactness of the difference
\[ B_0(z) - B_0(z') = (z' - z) G_0 R_0(z) R_0(z') G^* \] (6)
for any $z, z' \in \rho(H_0)$.

**Proof.** According to Theorem 1.10.3, under the first of these assumptions a selfadjoint Hamiltonian $H$ satisfying Definition 1.9.2 exists. Moreover, according to Theorem 1.9.5, the operator-valued function $I - B_0(z)$ is invertible for all $z$ with $\text{Im} \ z \neq 0$. By the compactness of the operator (6), in considering this operator-valued function in the half plane $\pm \text{Im} z \geq 0$ Theorem 1.8.3 can be applied to the first factor in the product
\[ I - B_0(z) = [I - (B_0(z) - B_0(z_\pm))(I - B_0(z_\pm))^{-1}][I - B_0(z_\pm)]. \]
Therefore, the operator-valued function $(I - B_0(z))^{-1}$, as before, is continuous in $z$, $\text{Re} \ z \in \Lambda_n$, from above and below up to the cuts along $\Lambda_n$. As in the proof of Theorem 2, from this we obtain the existence of the operators $W_0(H; H_0; \Lambda)$ and $W_0(H_0; H; \Lambda)$. □

Theorem 2 (and Theorem 1) can be given a more effective character by using the concept of strong $H$-smoothness (see Part 2 of §4). Namely, combination of it with Theorem 4.7 immediately leads to the following result.

**Theorem 3.** Suppose that the operator $H_0$ has only absolutely continuous spectrum of constant multiplicity on the interval $\Lambda$. Suppose the operators $G_0$ and $G$ are strongly $H_0$-smooth (with some exponents $\alpha_0 > 0$ and $\alpha > 0$) on any compact subinterval of $\Lambda$, while the operator $(G_0 R_0(z) G^*)'$ is compact for $\text{Im} \ z \neq 0$ and some positive integer $1$. Then the WO $W_0(H; H_0; \Lambda)$ exist and are complete.

We recall that in accordance with agreement made at the start of the section it is here assumed that conditions (1.9.6) and (1.9.7) are satisfied. The assumption of the compactness of the operator $(G_0 R_0(z) G^*)'$ can, as before, be replaced by the conditions formulated in Remark 3.
Conditions for the existence and completeness of the "global" WO follow
from Theorem 4.

**Theorem 5.** Suppose the operators $G_0$ and $G$ are strongly $H_0$-smooth on a system of compact intervals $\Lambda_n$ such that
\[
\sigma(H_0) \cup \bigcup_n \Lambda_n = 0. \tag{7}
\]
Suppose again that the operator $(G_0 R_0 G^*)^{1/l}$ is compact for $\Im z \neq 0$ and some positive integer $l$. Then the WO $W_\perp(H, H_0)$ exist and are complete.

**Proof.** By Theorem 4, for the union $\tilde{\Lambda}$ of the intervals $\Lambda_n$ the WO $W_\perp(H, H_0; \tilde{\Lambda})$ and $W_\perp(H_0, H; \tilde{\Lambda})$ exist. It therefore suffices to verify that
\[
E_0(\tilde{\Lambda}) \mathcal{E}^{(a)}_0 = \mathcal{E}^{(a)}_0, \quad E(\tilde{\Lambda}) \mathcal{E}^{(a)} = \mathcal{E}^{(a)}.
\]
The first of these equalities follows directly from condition (7). To verify the second it is only necessary to additionally note that $\sigma^{(a)}(H) = \sigma^{(a)}(H_0)$ by Theorem 1.10.7. Since $\sigma^{(a)}(H) \subset \sigma^{(a)}(H_0)$ from this it follows that $\sigma^{(a)}(H) \setminus \tilde{\Lambda} = 0$. □

As explained in Part 3 of the preceding section, the $IP$ holds under the conditions of Theorems 1–5. Moreover, it is established in §§7.3 and 7.4 that under these conditions all the stationary representations of §§7.2 and 7.4 are realized. Properties of the scattering matrix are also discussed there.

### § 7. The Absolutely Continuous and Point Spectra of the Operator $H$

In this section we present sufficient conditions for the absence of a singular continuous component in the spectrum of the full Hamiltonian. These conditions impose also stringent restrictions on the structure of the point spectrum.

**I.** The question of the character of the spectrum for small perturbations can be resolved quite simply.

**Theorem 1.** Suppose the conditions of Theorem 6.1 are satisfied and $N(G) = \{0\}$. Then the spectrum of the operator $H$ on $\Lambda$ is absolutely continuous.

**Proof.** In the proof of Theorem 6.1 it was shown that the operator $G$ is $H$-smooth on $\Lambda$. Therefore, it is only necessary to use Proposition 3.5. □

For relatively compact perturbations the occurrence of a singular component is not excluded. Nevertheless, in complete analogy to Theorem 1 we establish

**Theorem 2.** Suppose the conditions of Theorem 6.2 are satisfied, and the operator $B_0(z)$ is defined by equality (6.2). We denote by $\mathcal{N}$ the set of those

\[\lambda \in \Lambda \text{ for which at least one of the equations} \quad f = B_0(\lambda \pm i 0) f \]

has a nontrivial solution. Then the set $\mathcal{N}$ is closed and has Lebesgue measure zero. Under the additional condition $N(G) = \{0\}$ the spectrum of the operator $H$ on the set $\Lambda_0 = \Lambda \setminus \mathcal{N}$ is absolutely continuous.

We further put in evidence facts regarding the operator-valued function $B(z) = G R(z) G^*$ established in the preceding section.

**Theorem 3.** Under the conditions of Theorem 6.2, the operator-valued function $B(z)$ is continuous in norm up to the cut along $\Lambda$ with the exception of the points of the set $\mathcal{N}$. Under the conditions of Theorem 6.4 this assertion remains in force with the replacement of ordinary continuity by Hölder continuity with exponent $\min\{\alpha_0, \alpha\}$.

2. For further investigation of the singular set $\mathcal{N}$ we shall need an assertion, in a certain sense converse to Theorem 4.7.

**Lemma 4.** Suppose the operators $G_j : \mathcal{H}_0 \to \mathfrak{H}$, $j = 1, 2$, are bounded relative to the operators $[H_0]^{\theta_j}$ where $\theta_j \in [0, 1]$ and $\theta_1 + \theta_2 \leq 1$, while the operator-valued function $G_1 R_0(z) G_2^*$ is continuous in norm for $\Re z \in \Lambda$, $\pm \Im z \geq 0$. Then for any $f_1, f_2 \in \mathfrak{H}$ the measure $(G_1 E_0(z) G_2^* f_2, f_1)$ is continuously differentiable, and

\[
2 \pi i \frac{d(G_1 E_0(\lambda) G_2^* f_2, f_1)}{d\lambda} = \lim_{z \to x + i 0} \frac{G_1 R_0(z - R_0(z)) G_2^* f_2, f_1}{(z - x)^{i 0}}. \tag{2}
\]

**Proof.** Let $X$ be an arbitrary bounded subinterval of $\Lambda$,

\[\tilde{G}_j = G_j ([H_0] + i \theta) \in \mathfrak{B},\]

and $\theta = \theta_1 + \theta_2$. On the right-hand side of the equality

\[
(G_1 R_0(z) G_2^* f_2, f_1) = (G_1 ([H_0] + i \theta) E_0(\Re X) R_0(z) G_2^* f_2, f_1)
\]

\[+ (G_1 R_0(z) G_2^* f_2, f_1) + (G_1, x R_0(z) G_2^* f_2, f_1), \tag{3}
\]

the first term is, obviously, analytic in $z$ in the strip $\Re z \in X$. From the conditions of the lemma it follows that the second term on the right in (3) is continuous as $z$ approaches the cut along $X$. Therefore, by Stone's formula (1.4.7) it follows that for any interval $(\lambda_0, \lambda_1) \subset X$

\[
(G_1, x E_0(\lambda_0, \lambda_1) G_2^* f_2, f_1) = \int_{\lambda_0}^{\lambda_1} \lim_{\epsilon \to 0} (G_1, x \delta(\mu + i \epsilon) G_2^* f_2, f_1) d\mu.
\]

This means that the function on the left is continuously differentiable with respect to $\lambda$, and its derivative is equal to the integrand on the right. The equality obtained is equivalent to (2). □

The set $\mathcal{N}$ can now be considered according to the scheme set forth in §1. We shall first establish an analogue of Lemma 1.3.
LEMMMA 5. Suppose the operator-valued functions \( G_0R_0(z)G^* \) and \( GR_0(z)G^* \) are continuous in norm in the half strips \( \Re z \in \Lambda, \pm \Im z \geq 0 \), while the element \( f' \) satisfies either equation (1)_+ or (1)_-. Then the derivative of the measure \( (GE_0(\cdot)G^*)f, f \) is equal to zero at the point \( \lambda \): \[
\frac{d(GE_0(\lambda)G^*)f, f}{d\lambda} = 0.
\] (4)

PROOF. Continuous differentiability on \( \Lambda \) of the measure \( (GE_0(\cdot)G^*)f, f \) follows from Lemma 4. Further, by the equation (1)_+\:
\[
\lim_{\varepsilon \to 0} (f + G_0R_0(\lambda + i\varepsilon)G^*f, GR_0(\lambda + i\varepsilon)G^*f) = 0.
\]
We take the imaginary part of this equality and consider the symmetry equality (1.9.7). Then
\[
\lim_{\varepsilon \to 0} (\delta_0(\lambda, \varepsilon)G^*f, G^*f) = 0.
\]
In view of (2), from this we obtain the desired equality (4). \( \square \)

LEMMMA 6. Suppose that, in addition to the conditions of Lemma 5, the operator-valued function \( G_0E_0(\cdot)G_0^* \) is continuously differentiable (in the weak sense) with respect to \( \lambda \in \Lambda \). Then both equations (1)_+ have solutions simultaneously, and these solutions coincide.

PROOF. It is necessary to show that under condition (4) \( B_0(\lambda + i\varepsilon)f = B_0(\lambda - i\varepsilon)f \). By the continuity of \( B_0(z) \) in \( z \) it suffices to show that
\[
\lim_{\varepsilon \to 0} (G_0\delta_0(\lambda, \varepsilon)G^*f, g) = 0
\]
for any \( g \in \Theta \). From (1.3.9), (1.4.11) it follows that the square of the modulus of the left-hand side in (5) is estimated by the quantity
\[
\frac{d(GE_0(\mu)G^*f, f)}{d\mu} \cdot \frac{d(GE_0(\mu)G_0g, g)}{d\mu}
\]
at the point \( \mu = \lambda \). The first factor here is continuous with respect to \( \mu \in \Lambda \) by Lemma 4, while the second is continuous by hypothesis. Moreover, according to Lemma 5, the first factor is equal to zero for \( \mu = \lambda \). This proves equality (5). \( \square \)

We shall further establish that the set \( \mathcal{N} \) is independent of the choice of factorization \( V = G^*G_0 \).

LEMMMA 7. Let \( V = G^*G_0 = G^*G_0 \). Suppose that the operator-valued functions \( B_0(z) = -G_0R_0(z)G^* \), \( \tilde{B}_0(z) = -G_0R_0(z)G^* \), and also \( G_0R_0(z)G^* \) and \( \tilde{G}_0R_0(z)G^* \) are continuous in norm up to the cut along \( \Lambda \). Then the sets \( \mathcal{N} \) and \( \tilde{\mathcal{N}} \) corresponding to \( B_0 \) and \( \tilde{B}_0 \) coincide.

PROOF. The solutions of the equations
\[
f = B_0(\lambda + i\varepsilon)f \quad \text{and} \quad \tilde{f} = \tilde{B}_0(\lambda + i\varepsilon)\tilde{f}
\]
are connected by the equalities
\[
f = \tilde{G}_0R_0(\lambda + i\varepsilon)f, \quad \tilde{f} = G_0R_0(\lambda + i\varepsilon)\tilde{f}.
\]
This follows from the identity
\[
(I + \tilde{B}_0(\lambda + i\varepsilon))\tilde{G}_0R_0(\lambda + i\varepsilon)G^* = \tilde{G}_0R_0(\lambda + i\varepsilon)G^*[I + B_0(\lambda + i\varepsilon)]
\]
and the analogous relation in which \( B_0 \) and \( \tilde{B}_0 \) change roles. \( \square \)

3. The next assertion is altogether analogous to Lemma 1.4. It may also be viewed as an extension of Proposition 1.9.6 to the case of eigenvalues lying on the continuous spectrum. The connection of solutions \( f \) of equation (1)_+ and eigenfunctions \( \psi \) of the operator \( H \) can be established again by equalities of the form (1.10.8). We must now assume that the exponential \( \alpha \) in Definition 4.5 of strong \( H_0 \)-continuity of the operator \( G \) is greater than \( 1/2 \). To a certain extent this condition plays the role of the assumption of Hölder continuity with exponent greater than \( 1/2 \) of the kernel \( v(\lambda, \mu) \) in the Friedrichs-Faddeev model.

LEMMMA 8. Suppose the conditions of Theorem 7.3 are satisfied and \( \alpha > 1/2 \). Then \( \mathcal{N} = \sigma^{(0)}(H) \cap \Lambda \) and the multiplicities of the eigenvalue \( \lambda \in \Lambda \) of the operator \( H_0 \) and of the eigenvalue \( \lambda \) of the operator \( B(\lambda + i0) \) coincide.

PROOF. We denote by \( \mathcal{F} \) the unitary mapping of \( L_{2}(\Lambda; h) \) onto \( L_{2}(\Lambda; h) \). Let \( f \neq 0 \) satisfy equation (1)_+ on \( X \subset \) a compact subinterval of \( \lambda \), \( \lambda \in \Lambda \), and \( g = (GE_0(X)f) \). Then \( \tilde{g} = \mathcal{F}g \in C^\infty(X; h) \) by definition of strong \( H_0 \)-smoothness of the operator \( G_0 \) on \( X \) and \( \tilde{g}(\lambda) = 0 \) by Lemma 5. For \( \alpha > 1/2 \) from this it follows that the vector-valued function \( (\mu - \lambda)^{-1}g(\mu) \) of the variable \( \mu \) belongs to the space \( L_{2}(X; h) \), and hence \( f = R_0(\lambda + i\varepsilon)g(\mu) \in D([H_0^0]^{\lambda}) \). We take the scalar product of equation (1)_+ with the element \( Gh \), where \( h \) is an arbitrary element of \( D(\mathcal{H}) \subset D([H_0^0]^{\lambda}) \). Then by the definition of \( \mathcal{F} \)
\[
(f, Gh) + (G_0\psi, Gh) = 0,
\]
and
\[
(f, Gh) = (H_0^0\psi, [H_0^0h] - \lambda(\psi, h)).
\]
By relation (1.9.10) equality (6) can now be rewritten in the form
\[
(\psi, (H - \lambda)h) = 0.
\]
For the selfadjoint operator \( H \) it follows from this that \( \psi \in D(H) \) and \( H\psi = \lambda\psi \). Moreover, \( \psi \neq 0 \), since, according to (1)_+ \( f = -G_0\psi \). This proves the inclusion \( \mathcal{N} \subset \sigma^{(0)}(H) \cap \Lambda \).

Conversely, suppose that \( H\psi = \lambda\psi \), \( \lambda \in \Lambda \) and \( \psi \in \mathcal{D}(H) \subset D([H_0^0]^{\lambda}) \). We set \( f = -G_0\psi \in \mathcal{H} \), and for arbitrary \( h \in \mathcal{H} \) we take the
scalar product of the equation $H\psi = \lambda \psi$ with the element $R_{0}(\lambda \mp i\epsilon)G_{0}\psi$. Then

$$(G_{0}R_{0}(\lambda \mp i\epsilon)(H_{0} - \lambda)\psi, \psi) = (G_{0}R_{0}(\lambda \mp i\epsilon)G^{*}f, h).$$

(7)

The strong $H_{0}$-smoothness of the operator $G_{0}$ ensures that

$$\lim_{\epsilon \to 0} \epsilon(G_{0}R_{0}(\lambda \pm i\epsilon)\psi, \psi) = 0.$$

Therefore, as $\epsilon \to 0$ the left-hand side of (7) tends to $(G_{0}\psi, \psi)$. The right-hand side tends to $-(B_{0}(\lambda \pm i0)f, h)$, since the operator $B_{0}(z) = -G_{0}R_{0}(z)G^{*}$ is continuous in norm as $z$ approaches the cut along $\Lambda$. Thus, from (7) it follows that $(f, h) = (B_{0}(\lambda \pm i0)f, h)$, i.e., $f$ satisfies both equations $(1)_{\pm}$. The element $f \neq 0$, since otherwise $(H_{0} - \lambda)\psi = 0$, and hence $\psi = 0$. This completes the verification of the inclusion $\sigma^{p}(H) \cap \Lambda \subset \mathcal{N}$.

This argument also establishes that the multiplicities of the eigenvalues of the operators $H_{0}$ and $B_{0}(\lambda \pm i0)$ coincide.

Combining this assertion with Theorem 2, we find

**Theorem 9.** Suppose the conditions of Theorem 6.4 are satisfied, $N(G) = 0$, and $\alpha > 1/2$. Then on the interval $\Lambda$ the operator $H$ has no singular continuous spectrum.

We note that in Lemma 8 the condition $\alpha > 1/2$ was used only to verify the inclusion $\mathcal{N} \subset \sigma^{p}(H) \cap \Lambda$ which is also necessary for the proof of absence of singular continuous spectrum. The reverse inclusion $\sigma^{p}(H) \cap \Lambda \subset \mathcal{N}$ is valid for any $\alpha > 0$. Therefore, compactness of the operators $B_{0}(\lambda \pm i0)$ implies that for any $\alpha > 0$ the eigenvalues $\lambda \in \Lambda$ of the operator $H$ have finite multiplicities.

Under the assumptions made the structure of the point spectrum of the operator $H$ cannot be arbitrary either. Namely, Theorem 9 admits the following supplement.

**Theorem 10.** Suppose the conditions of Theorem 9 are satisfied. Then the singular spectrum of the operator $H$ on the interval $\Lambda$ consists only of eigenvalues of finite multiplicities with no accumulation points inside $\Lambda$.

**Proof.** Assume otherwise. Let $H\psi_{n} = \lambda_{n}\psi_{n}$, where $\lambda_{n} \rightarrow \lambda_{0} \in \Lambda$ as $n \rightarrow \infty$, and let $\psi_{n}$ be the corresponding orthonormal sequence of eigenfunctions. Denote by $X$ any compact subinterval of $\Lambda$ containing all the points $\lambda_{n}$. As shown in the proof of Lemma 8, the elements $f_{n} = -G_{0}\psi_{n} = -G_{0}E_{0}(X)\psi_{n}$ satisfy equations $(1)_{\pm}$ for $\lambda = \lambda_{n}$. By the compactness of the operators $B_{0}(\lambda \pm i0)$ and their continuous dependence on $\lambda$, the sequence $f_{n}$ is compact in $\mathcal{N}$. We shall see that then the set of eigenfunctions $\psi_{n} = R_{0}(\lambda_{n} \pm i0)G^{*}f_{n}$ must also be compact in $\mathcal{N}$. Set $g_{n} = (G_{0}E_{0}(X))^{*}f_{n}$, $\tilde{g}_{n} = \mathcal{F}_{0}g_{n}$, $\psi_{n} = \mathcal{F}_{0}\psi_{n}$. It suffices to verify the compactness of the set of vector-valued functions $\tilde{\psi}_{n}(\lambda) = (\lambda - \lambda_{n} \pm i0)^{-1}\tilde{g}_{n}(\lambda)$ in the space $L_{2}(X, h)$. By the compactness of $f_{n}$ in $\mathcal{N}$ this set is clearly compact in the space $L_{2}(X \setminus (\lambda_{0} - \epsilon, \lambda_{0} + \epsilon) ; h)$ for any $\epsilon > 0$. At the same time $\mathcal{F}_{0}(\lambda_{n}) = 0$ according to Lemma 5, so that strong $H_{0}$-smoothness on $X$ of the operator $G$ implies the uniform in $n$ estimate

$$|\tilde{\psi}_{n}(\lambda)| \leq C|\lambda - \lambda_{n}|^{\alpha}.$$

For $2\alpha > 1$ this makes it possible to estimate the norms of $\tilde{\psi}_{n}$ in $L_{2}(\lambda_{0} - \epsilon, \lambda_{0} + \epsilon) ; h)$ by a quantity $c(\epsilon)$, independent of $n$ and tending to zero as $\epsilon \to 0$. Thus, the set $\{\tilde{\psi}_{n}\}$ is compact in $\mathcal{N}$, which contradicts the orthonormality of these functions.

It turns out that under slightly refined assumptions Theorem 10 remains in force also for arbitrary $\alpha > 0$. This result will be presented in Volume 2 in application to the Schrödinger operator.
CHAPTER 5

The General Scheme in Stationary Scattering Theory

In this chapter we find conditions on the perturbation that make it possible to realize the scheme of constructing stationary scattering theory presented in §2.7. Roughly speaking, these conditions consist in a factorization of the perturbation \( V = G^* G_0 \) into two factors \( G_0 \) and \( G \), smooth in a certain sense relative to the operators \( H_0 \) and \( H \), respectively. Here smoothness weaker than the Kato smoothness introduced in §4.3 is sufficient.

Section 1 is devoted to an exposition of properties of operators relatively smooth in a weak sense. In §2 precise conditions are presented that make it possible to justify the stationary scheme of §2.7, and corresponding justifications are given. The connection between the stationary and the time-dependent approach under these assumptions is discussed in §3. The invariance principle is also considered there. Effective sufficient conditions for an operator to be an integral operator (see Part 3 of §1.5) in the corresponding direct decomposition are indicated by means of the concept of \( H \)-smoothness in §4. These results are used in §5 in justifying the formal representations of §2.8 for the scattering matrix. The construction of complete isometric WO is equivalent to the theorem of decomposition with respect to certain special "eigenvectors" of the operator \( H^{(a)} \). This point of view is developed in §6. Finally, in §7 we consider scattering for relatively compact perturbations and in §8 present a local version of the theory.

In a weak sense an arbitrary Hilbert-Schmidt operator is smooth relative to any selfadjoint operator which explains the existence of the WO for trace class perturbations. Consideration of trace class perturbations is, however, put off until the following Chapter 6. A comparison of the scheme presented here with the smooth theory of Chapter 4 is presented in Part 1 of §7.3.

§1. Weak smoothness

In this section we apply without additional clarifications the notation and conventions introduced at the beginning of §4.3. In particular, for an \( H \)-bounded operator \( G: \mathcal{D} = \mathcal{D}(H) \rightarrow \mathcal{H} \) the adjoint operator \( G^* \) always exists if it is understood as a mapping of \( \mathcal{H} \) into \( \mathcal{D}^* \).
As explained in Part 3 of §1.4, for a selfadjoint operator \( H \) the operator-valued function \( \delta(\lambda, \varepsilon) = \delta_\mu(\lambda, \varepsilon) \) does not have even a weak limit as \( \varepsilon \to 0 \), for any \( \lambda \). At the same time for a suitable bordering operator \( G \) the weak limit of the product \( \text{G} \delta(\lambda, \varepsilon) G^* \) may exist. We adopt

**Definition 1.** An \( H \)-bounded operator \( G \) is called \( H \)-smooth in the weak sense if for a.e. \( \lambda \in \mathbb{R} \) there exists

\[
\lim_{\varepsilon \to 0} \text{G}\delta(\lambda, \varepsilon) G^*.
\]

It is important that the concept of \( H \)-smoothness admits a number of equivalent formulations. The next result can be verified in an altogether simple manner.

**Lemma 2.** An operator \( G \) is \( H \)-smooth in the weak sense if and only if any of the following two inequalities are satisfied:

\[
\|G\delta(\lambda, \varepsilon) G^*\| \leq C(\lambda), \quad \text{a.e. } \lambda \in \mathbb{R},
\]

\[\varepsilon^{1/2} \|G\text{R}(\lambda \pm i\varepsilon)\| \leq C(\lambda), \quad \text{a.e. } \lambda \in \mathbb{R}.
\]

**Proof.** The estimates (2) and (3) are equivalent by definition (1.4.5). The estimate (2) follows directly from (1). We establish the converse. Consider in \( \mathfrak{D} \) a dense set \( \mathfrak{D} \) of finite linear combinations of elements of some basis \( \varphi_i \), \( i = 1, 2, \ldots \). By relation (1.4.11) the quantity \( \delta(\lambda, \varepsilon) \varphi_i \) has a limit as \( \varepsilon \to 0 \) on a set \( \Lambda_{ij} \) full measure. Deleting it from \( \mathfrak{D} \), we obtain a set of full measure on which the quantities \( \delta(\lambda, \varepsilon) \varphi, \varphi^* \) have limits as \( \varepsilon \to 0 \) for all \( \varphi, \varphi^* \in \mathfrak{D} \). We further delete from it the set of measure zero on which (2) is violated for \( G \). In view of the uniform boundedness of \( \|G\delta(\lambda, \varepsilon) G^*\| \) with respect to \( \varepsilon \) and the equality \( \mathfrak{D} = \mathfrak{D} \) the limit (1) exists on the set of full measure obtained. □

According to Definition (4.3.4), for a Kato \( H \)-smooth operator \( G \) the estimate (2) is satisfied for all \( \lambda \in \mathbb{R} \), where the constant on the right-hand side does not depend on \( \lambda \). Thus, on the basis of Lemma 2 any Kato \( H \)-smooth operator \( G \) is \( H \)-smooth in the weak sense.

Together with Lemma 2 there is an altogether similar assertion formulated in terms of the spectral family (in place of the operators \( \delta(\lambda, \varepsilon) \)).

**Lemma 3.** The following three assertions are equivalent:

\[
\begin{align*}
\text{w}\lim_{\eta \to 0} (2\eta)^{-1} \|E(\lambda + \eta) - E(\lambda - \eta)\| G^*, \quad \text{a.e. } \lambda \in \mathbb{R}, \quad (4) \\
\|GE((\lambda - \eta, \lambda + \eta)) G^*\| \leq C(\lambda) \eta, \quad \text{a.e. } \lambda \in \mathbb{R}, \quad (5) \\
\|GE((\lambda - \eta, \lambda + \eta))\| \leq C(\lambda) \eta^{1/2}, \quad \text{a.e. } \lambda \in \mathbb{R}. \quad (6)
\end{align*}
\]

**Proof.** The estimates (5) and (6) are equivalent, since the left-hand side of (5) is equal to the square of the left-hand side of (6). Inequality (5) follows directly from (4). To prove the converse, it is necessary to use the fact that the measure \( (E(\cdot) G^* \varphi, G^* \psi) \) is differentiable for any \( \varphi, \psi \in \mathfrak{D} \). The set of points \( \lambda \) where this derivative exists depends on \( \varphi, \psi \). As in the proof of Lemma 2, by restricting attention to elements of \( \mathfrak{D}, \mathfrak{D} = \mathfrak{D} \), we find that this derivative exists on a common set of points \( \lambda \). Together with the uniform estimate (5) this ensures the existence of the weak limit (4). □

The estimate (6) is reminiscent of Definition (4.3.3) of Kato smoothness although it differs from (4.3.5) in the dependence on \( \lambda \) of the constant in the right-hand side. It turns out that, similarly to Kato smoothness, weak smoothness can also be defined in terms of the spectral family. This follows from the next assertion.

**Lemma 4.** The conditions of Lemmas 2 and 3 are equivalent to one another.

**Proof.** We first derive the existence of the limit (1) from condition (4). By the spectral theorem for any \( f, g \in \mathfrak{D} \)

\[
(\delta(\lambda, \varepsilon) G^* f, G^* g) = \varepsilon \pi^{-1} \int_0^{\pi} ((\mu - \lambda)^2 + \varepsilon^2)^{-1} d(E(\mu) G^* f, G^* g).
\]

By condition (4) all the functions \( E(\mu) G^* f, G^* g \) have (symmetric) derivatives on a common set \( \Lambda \) of full measure (independent of \( f, g \)). Therefore, according to (Fatou's) Theorem 1.2.7, for \( \lambda \in \Lambda \) and \( \varepsilon \to 0 \) there exist the limits of the Poisson integrals (7). Since \( \Lambda \) does not depend on \( f, g \), this implies that the limit (1) exists on \( \Lambda \).

Conversely, suppose that for some \( \lambda \in \mathbb{R} \) the estimate (3) holds. We shall verify inequality (6) for this \( \lambda \). By the spectral theorem, for any \( \varepsilon > 0 \)

\[
\|E(\lambda - \varepsilon, \lambda + \varepsilon) G^* f\| \leq (\varepsilon^2 + \varepsilon)^{1/2} \|G(\lambda + i\varepsilon) f\|.
\]

For \( g = G^* f \) the right-hand side can be estimated by \( (\varepsilon^2 + \varepsilon)^{1/2} \|G(\lambda + i\varepsilon) f\| \). Therefore, according to (3), we have the inequality

\[
\|E(\lambda - \varepsilon, \lambda + \varepsilon) G^* f\| \leq C(\varepsilon^2 + \varepsilon)^{1/2} \varepsilon^{-1/2} \|f\|.
\]

For \( \varepsilon = \eta \) from this we obtain (6). □

Thus, on the basis of Lemmas 2–4 the concept of weak \( H \)-smoothness can be effectively formulated in any one of the forms (1)–(6). We further note that by (1.4.11) the limits (1) and (4) for a.e. \( \lambda \) are equal to one another. Limits of the form (1) and (4) exist also if \( \delta(\lambda, \varepsilon) \) is bounded on the left and right by different \( H \)-smooth operators.

**Lemma 5.** Suppose the operators \( G_j, j = 1, 2 \), are weakly smooth relative to \( H \). Then for a.e. \( \lambda \in \mathbb{R} \)

\[
\text{w}\lim_{\eta \to 0} G_j \delta(\lambda, \varepsilon) G^*_j = \text{w}\lim_{\eta \to 0} (2\eta)^{-1} G_j [E(\lambda + \eta) - E(\lambda - \eta)] G_j^*, \quad (8)
\]

and both these limits exist.

**Proof.** In view of (1.4.11) it is only necessary to verify the existence of both sides of (8). As in Lemmas 2, 3, for this it suffices to verify the uniform
boundedness with respect to $\varepsilon$ and $\eta$ of the operators in (8). By definition (1.4.5) from the estimate (3) it follows that

$$\pi \|G_{\delta}(\lambda, \varepsilon)G^{*}_{2}\| \leq \varepsilon \|G_{1}R(\lambda + i\varepsilon)\| \|G_{2}R(\lambda + i\varepsilon)\| \leq C(\lambda).$$

In an entirely similar way, by (6)

$$\|G_{1}[E(\lambda + \eta) - E(\lambda - \eta)]G^{*}_{2}\| \leq \|G_{1}E((\lambda - \eta, \lambda + \eta))\| \|G_{2}E((\lambda - \eta, \lambda + \eta))\| \leq C(\lambda)\eta.$$

We further consider limits of vector-valued functions.

**Lemma 6.** Suppose the operator $G$ is weakly smooth relative to $H$. Then for any $f \in \mathcal{H}$ there exist the equal limits

$$w-\lim_{\varepsilon \to 0} G\delta(\lambda, \varepsilon) f = w-\lim_{\eta \to 0} (2\eta)^{-1} g[I(\lambda + \eta) - I(\lambda - \eta)]f, \quad a.e. \lambda \in \mathbb{R},$$

(9) (the corresponding set of full measure depends on the element $f$).

**Proof.** In view of equality (1.4.11) it is again necessary to verify only the existence of both limits in (9). For this, in turn, it suffices to demonstrate (cf. the proofs of Lemmas 2 and 3) the uniform boundedness with respect to $\varepsilon$ and $\eta$ of the elements in (9). In the inequality

$$\pi^{2}\|G\delta(\lambda, \varepsilon)f\|^{2} \leq (\varepsilon \|G_{R}(\lambda - \varepsilon^{2})\|)(\varepsilon \|R(\lambda + i\varepsilon)\|)^{2}$$

the first factor on the right is bounded uniformly with respect to $\varepsilon$ by (3), while the second is uniformly bounded by (1.4.12). Similarly,

$$\|G\delta((\lambda - \eta, \lambda + \eta))f\| \leq \|G\delta((\lambda - \eta, \lambda + \eta))\| \|E((\lambda - \eta, \lambda + \eta))f\|.\quad (10)$$

According to (6), the first factor here can be estimated by $C(\lambda)\eta^{-1/2}$. At points $\lambda$ where the function $(E(\lambda)f, f)$ is differentiable

$$(E((\lambda - \eta, \lambda + \eta))f, f) \leq C(\lambda, f)\eta,$$

and therefore also the second factor on the right-hand side of (10) does not exceed $C(\lambda, f)\eta^{-1/2}$. □

We note that for an $H$-smooth operator $G$ there exist not only the symmetric derivatives (4), (9) but also the usual weak derivatives

$$\frac{dG\delta(\lambda)G^{*}}{d\lambda}, \quad \frac{dG\delta(\lambda)f}{d\lambda}.$$

The definition of weak $H$-smoothness can also be formulated locally for an arbitrary Borel set $\Lambda$. Namely, an $H$-bounded operator $G$ is called weakly $H$-smooth on $\Lambda$ if at least one of (and then all the remaining) relations (1)–(6) is satisfied for a.e. $\lambda \in \Lambda$. In this case the limits (8), (9) also exist for a.e. $\lambda \in \Lambda$. It is not hard to see (cf. Theorem 4.3.10) that under the condition $|\Lambda| = 0$ for a weakly $H$-smooth operator $G$ on the set $\Lambda$ the product $GE(\Lambda)$ is (globally) weakly $H$-smooth. It is also clear that weak smoothness of the operator $G$ is equivalent to weak smoothness of the operators $GE(\Lambda)$ for all bounded intervals $\Lambda$.

**§2. Justification of the stationary method**

This section is related to §2.7. Here we present precise formulations and their proofs omitted in §2.7. Parts 1–3 of this section correspond to the parts of §2.7 with the same number. In Part 4, we discuss conditions for isometricity of the stationary WO and conditions for their completeness.

1. Suppose, as always, that $H_{0}, H$ are selfadjoint operators in Hilbert spaces $\mathcal{H}_{0}, \mathcal{H}$ and $\mathcal{J}: \mathcal{H}_{0} \to \mathcal{H}$ is a bounded identification. In this part we establish that Definition 2.7.2 of the WO $\mathcal{H}_{0} = \mathcal{H}(H_{0}, H; \mathcal{J})$ is correct. First we enumerate a number of auxiliary facts.

Suppose the form $a_{\pm}(f_{0}, f; \lambda)$ is defined by relation (2.7.4). From the existence of that limit we obtain the estimate

$$|a_{\pm}(f_{0}, f; \lambda)|^{2} \leq \|\lambda\|^{2} \frac{\partial^{2} d(E(\lambda)f_{0}, f_{0})}{d\lambda} \frac{\partial^{2} d(E(\lambda)f, f)}{d\lambda}, \quad a.e. \lambda \in \mathbb{R}.\quad (1)$$

To prove it, it suffices to pass to the limit as $\varepsilon \to 0$ in the inequality

$$|\varepsilon(JR_{0}(z))\varepsilon, R(z)f\|^{2} \leq \|\lambda\|^{2} |\varepsilon(R_{0}(z))\varepsilon, R(z)f\|^{2}$$

and consider relation (1.4.11). The next result can be verified in just as simple a manner.

**Lemma 1.** For any $f_{0} \in \mathcal{H}_{0}$, $f \in \mathcal{H}$, Borel sets $\Lambda_{0}, \Lambda \subset \mathbb{R}$, and $\mathcal{X} = \Lambda_{0} \cap \Lambda$

$$a_{\pm}(E_{0}(\Lambda_{0})f_{0}, E(\Lambda)f; \lambda) = 0, \quad a.e. \lambda \in \mathbb{R}\setminus \mathcal{X}.\quad (3)$$

Moreover, if the limit (2.7.4) exists, then there also exists

$$a_{\pm}(E_{0}(\Lambda_{0})f_{0}, E(\Lambda)f; \lambda) = \chi_{\mathcal{X}}(\lambda) a_{\pm}(f_{0}, f; \lambda), \quad a.e. \lambda \in \mathbb{R}.\quad (4)$$

**Proof.** We use inequality (2) with $f_{0}$, $f$ replaced by $E_{0}(\Lambda_{0})f_{0}, E(\Lambda)f$. According to (1.3.13), (1.4.11) the limit as $\varepsilon \to 0$ and a.e. $\lambda \in \mathbb{R}$ of its right-hand side is equal to

$$\|\lambda\|^{2} \chi_{\mathcal{X}}(\lambda) \frac{\partial^{2} d(E_{0}(\lambda)f_{0}, f_{0})}{d\lambda} \frac{\partial^{2} d(E(\lambda)f, f)}{d\lambda}.$$

For $\lambda \in \mathbb{R}\setminus \mathcal{X}$ this expression is equal to zero which proves (3).

Further, by (3) it suffices to verify relation (4) on the set $\mathcal{X}$. We use the equality

$$a_{\pm}(E_{0}(\Lambda_{0})f_{0}, E(\Lambda)f; \lambda) = - a_{\pm}(E_{0}(\mathbb{R}\setminus \Lambda_{0})f_{0}, E(\Lambda)f; \lambda) - a_{\pm}(f_{0}, E(\mathbb{R}\setminus \Lambda)f; \lambda) + a_{\pm}(f_{0}, f; \lambda).$$

(4)
which follows directly from the sesquilinearity (linear in the first argument and antilinear in the second) of the form $a_{\pm}$. It remains to note that for a.e. $\lambda \in \mathcal{X}$ the first two terms on the right are equal to zero by (3).

According to Lemma 1.3.6 the singular (relative to some selfadjoint operator $H$) component $f^{(s)}$ of the element $f$ can be written in the form $f^{(s)} = E(Z_\lambda) f$, where $Z_\lambda$ is a Borel set of Lebesgue measure zero. Therefore, it follows from (3), (4) that condition (2.7.4) is equivalent to the same condition for any of the pairs $\{f^{(a)}, f\}$, $\{f^{(a)}, f^{(s)}\}$, $\{f^{(s)}, f^{(s)}\}$; for all four pairs the value of the form $a_{\pm}$ for a.e. $\lambda \in \mathcal{R}$ is the same.

The derivatives on the right-hand side of (1) are equal to zero for a.e. $\lambda$ in the complements of the cores of the spectra $\delta_0$ and $\delta$ (see Definition 1.3.8) of the operators $H_0$ and $H$. Hence by (1) for a.e. $\lambda \in \mathcal{R}(\delta_0 \cap \delta)$ the limit (2.7.4) exists and is equal to zero.

Suppose now that on the set $\mathcal{L}_0 \times \mathcal{L}$, where $\mathcal{L}_0 = \mathcal{H}_0$, $\mathcal{L} = \mathcal{H}$, the limit (2.7.4) exists for a.e. $\lambda \in \mathcal{R}$. According to Definition 2.7.2 the stationary WO $\mathcal{U}_\pm$ can be introduced in terms of the sesquilinear form (2.7.5) defined on $\mathcal{L}_0 \times \mathcal{L}$. It follows directly from (1) that the function (2.7.4) is absolutely integrable, the integral (2.7.5) is finite, and by (1.3.11)

$$|\langle \mathcal{U}_\pm(f,0) \rangle| \leq \|\mathcal{P} \| \|\mathcal{P} \| \|f\|$$

Therefore, to the form (2.7.5) there corresponds a bounded operator $\mathcal{U}_\pm = \mathcal{U}_\pm(H, H_0; 2)$, with $\|\mathcal{U}_\pm\| \leq \|\mathcal{P}\|$ and the relations (2.7.6) are satisfied. The conditions of Definition 2.7.2 are symmetric with respect to the operators $H_0$ and $H$. Thus, the operators $\mathcal{U}_\pm(H, H_0; 2)$ and $\mathcal{U}_\pm(H_0, H; 3')$ exist simultaneously, and, according to (2.7.5), the equality (2.7.9) holds. We further note that, according to (1), the integral (2.7.5) is actually taken only over the set $\delta_0 \cap \delta$.

It is convenient to base further study of the operator $\mathcal{U}_\pm$ on consideration of an auxiliary operator $\mathcal{U}_\pm(\varepsilon)$. We define this operator in terms of the sesquilinear form

$$\langle \mathcal{U}_\pm(\varepsilon)f, 0 \rangle = \pi^{-1} \int_{-\infty}^{\infty} (3R_0(\lambda \pm \varepsilon)f) f \, d\lambda,$$

defined on $\mathcal{H}_0 \times \mathcal{H}$. According to (2), the right-hand side of (5) can be estimated by the quantity

$$\|\mathcal{P}\| \left( \int_{-\infty}^{\infty} \pi^{-1} \varepsilon ||R_0(\lambda \pm \varepsilon)f||^2 \, d\lambda \int_{-\infty}^{\infty} \pi^{-1} \varepsilon ||R(\lambda \pm \varepsilon)f||^2 \, d\lambda \right)^{1/2}.$$

By equality (1.4.6) from this we obtain the uniform in $\varepsilon$ estimate

$$\|\mathcal{U}_\pm(\varepsilon)\| \leq \|\mathcal{P}\|,$$

and we obtain the expression (5) as $\varepsilon \to 0$ can be found in the case where one of the elements $f_0$ or $f$ is absolutely continuous.
Theorem 4. Suppose the conditions of Definition 2.7.2 are satisfied. Then the bounded operator $\mathcal{A}_\pm = \mathcal{A}(H; H_0; \mathcal{S})$ is well defined and $\|\mathcal{A}_\pm\| \leq \|\mathcal{A}\|$. For the operator $\mathcal{A}_\pm$ relations (2.7.6)–(2.7.8) hold. This weak time-dependent abelian WO $\mathcal{A}_\pm(H; H_0; \mathcal{S})$ also exists and coincides with $\mathcal{A}_\pm$; $\mathcal{A}_\pm(H_0; H; \mathcal{S})$ exists simultaneously with $\mathcal{A}_\pm$, and (2.7.9) is satisfied.

Conditions ensuring the existence of the limit (2.7.4) are naturally formulated in terms of the "perturbation" $V = H^2 - 2H_0$. We assume that there is the factorization (1.9.2) where $G_0$ is bounded relative to $H_0$, and $G$ is bounded relative to $H$. As explained in §1.9, in a precise sense it is assumed that equality (1.9.3) is satisfied on elements $f_0 \in \mathcal{D}(H_0)$, $f \in \mathcal{D}(H)$. According to Proposition 1.9.1, we then have the resolvent identity (1.9.4). With consideration of the factorization of $V$ the first equality of (2.7.10) can now be rewritten in the form

$$\pi^{-1} e^{3R_0(\lambda \pm i\epsilon)f_0}, R(\lambda \pm i\epsilon)f$$

$$= (\delta(\lambda, i\epsilon)f_0, f) + (G_0R_0(\lambda \pm i\epsilon)f_0, G\delta(\lambda, \epsilon)f).$$

By (1.4.11) for any $f_0 \in \mathcal{H}_0$, $f \in \mathcal{K}$ and a.e. $\lambda \in \mathbb{R}$ the first term on the right has a limit as $\epsilon \to 0$. We therefore have

Lemma 5. The limit (2.7.4) exists for the pair $f_0$, $f$ if the vector-valued functions

$$G_0R_0(\lambda \pm i\epsilon)f_0, \quad G\delta(\lambda, \epsilon)f$$

have limits as $\epsilon \to 0$. Here (either) one of these limits may be weak if the other is strong.

Using the second equality of (2.7.10), the roles of the operators $H_0$ and $H$ can be interchanged here.

Lemma 5'. The assertion of Lemma 5 is preserved if one of the vector-valued functions

$$G_0\delta(\lambda, \epsilon)f_0, \quad GR(\lambda \pm i\epsilon)f$$

has a weak limit and the other has a strong limit.

In particular, if the conditions of Lemma 5 and 5' are satisfied on dense sets of elements $f_0$ and $f$, then there exist the WO $\mathcal{A}_\pm(H; H_0; \mathcal{S})$.

2. We proceed to the justification of the representation (2.7.11). This requires the same type of assumptions as in Lemmas 5 or 5', though they are somewhat stronger.

Lemma 6. Suppose for elements $f_0$ of some dense (linear) set $\mathcal{M}_0$ in $\mathcal{H}_0$

$$\lim_{\epsilon \to 0} G_0R_0(\lambda \pm i\epsilon)f_0, \quad \text{a.e. } \lambda \in \mathbb{R}. \quad (10)$$

Moreover, suppose that the operator $G$ is weakly $H$-smooth. Then the WO $\mathcal{A}_\pm$ exist, and the representation (2.7.5) for their sesquilinear forms holds on $\mathcal{M}_0 \times \mathcal{K}$.

Proof. According to Lemmas 1.6 and 5, the conditions of Definition 2.7.2 are satisfied for $\mathcal{L}_0 = \mathcal{M}_0$, $\mathcal{L} = \mathcal{K}$. \qed

In an altogether similar fashion by means of Lemmas 1.6 and 5' we obtain

Lemma 6'. Suppose for elements $f$ of some set $\mathcal{M}$ dense in $\mathcal{K}$ there exists the limit (4.3.14), and the operator $G_0$ is weakly $H_0$-smooth. Then the WO $\mathcal{A}_\pm$ exists, and the representation (2.7.5) holds on $\mathcal{M}_0 \times \mathcal{M}$.

To prove (2.7.11) we start from relation (2.7.12), valid under the conditions of Definition 2.7.2. We first give a precise meaning to relation (2.7.13).

Lemma 7. Under the conditions of Lemma 6, for $f_0 \in \mathcal{M}_0$

$$w\lim_{\epsilon \to 0} G\delta(\lambda, \epsilon)f_0 = w\lim_{\epsilon \to 0} \pi^{-1} e^{GR(\lambda \pm i\epsilon)R_0(\lambda \pm i\epsilon)f_0}, \quad (11)$$

and the limits in both sides exist for a.e. $\lambda \in \mathbb{R}$.

Proof. By virtue of equality (2.7.12), it is only necessary to verify the existence of the limits in (11). The existence of the limit on the left for any $f_0 \in \mathcal{H}_0$ follows directly from Lemma 1.6. We consider the right-hand side of (11). The scalar product of the element under the limit sign with an arbitrary $\varphi \in \mathcal{M}$ is equal to

$$\pi^{-1} e^{3R_0(\lambda \pm i\epsilon)f_0, R(\lambda \pm i\epsilon)h}, \quad (12)$$

where $h = G^*\varphi \in \mathcal{D}^*$. We now use condition (10) and note that the weak limit of $G\delta(\lambda, \epsilon)h$ exists for any $h$. Therefore, by Lemma 5 the limit as $\epsilon \to 0$ of the expression (12) exists for $f_0 \in \mathcal{M}_0$, arbitrary $\varphi \in \mathcal{M}$, and a.e. $\lambda \in \mathbb{R}$. The corresponding set of full measure may, of course, be assumed to be independent of the choice of $\varphi$ in the set $D$ of linear combinations of some basis in $\mathcal{M}$. Therefore, to prove the existence of the limit on the right-hand side of (11) it remains to obtain the estimate

$$e\|GR(\lambda \pm i\epsilon)R_0(\lambda \pm i\epsilon)f_0\| \leq C(\lambda), \quad f_0 \in \mathcal{M}_0, \quad \text{a.e. } \lambda \in \mathbb{R}.$$
and as $\varepsilon \to 0$, according to (10), they have a strong limit, while the second factors, according to (11), have a common weak limit.

Under the conditions of Lemma 6 it is possible to set $f = \mathcal{U}_\pm g_0$ in the representation (2.7.5). Now (2.7.11) is a direct consequence of (2.7.14). Namely, we have

**Theorem 8.** Under the conditions of Lemma 6 for $f_0, g_0 \in \mathcal{M}_n$ the representation (2.7.11) holds, and the limit under the integral sign exists for a.e. $\lambda \in \mathbb{R}$.

According to Definition 2.7.2, the last assertion of Theorem 8 implies that the WO $\mathcal{U}_\pm^{(0)} = \mathcal{U}_\pm(H_0, H_\lambda; \mathfrak{I})$ exists for the auxiliary triple $H_0$, $H_\lambda$, $\mathfrak{I}$. The integral (2.7.11) is here equal to $(\mathcal{U}_\pm^{(0)} f_0, g_0)$. Thus, Theorem 8 admits the following reformulation.

**Theorem 9.** Under the conditions of Lemma 6 the stationary WO $\mathcal{U}_\pm (H_0, H_\lambda; \mathfrak{I})$ exists and equality (2.7.16) holds.

By changing the roles of the operators $H_0$ and $H$ and considering equality (2.7.9), we obtain the following assertion.

**Theorem 9'.** Under the conditions of Lemma 6 the stationary WO $\mathcal{U}_\pm (H, H\lambda; \mathfrak{I}) = \mathcal{U}_\pm (H, H; \mathfrak{I})$, and the WO on both sides exist.

3. We shall now establish the existence of strong time-dependent WO. In this part we restrict ourselves to considering abelian WO (2.2.9).

**Theorem 10.** Under the conditions of Lemma 6 there exists the strong, time-dependent abelian WO $\mathfrak{A}_\pm (H, H_\lambda; \mathfrak{I})$ which coincides with $\mathcal{U}_\pm (H, H_\lambda; \mathfrak{I})$.

**Proof.** According to Theorems 4 and 9 under the conditions of Lemma 6 there exist the weak time-dependent abelian WO $\tilde{\mathfrak{A}}_\pm (H, H_\lambda; \mathfrak{I})$ and $\tilde{\mathfrak{A}}_\pm (H_\lambda, H_0; \mathfrak{I})$ which coincide with the corresponding stationary WO. Moreover, (2.7.16) gives equality

$$\tilde{\mathfrak{A}}_\pm (H, H_\lambda; \mathfrak{I}) = \tilde{\mathfrak{A}}_\pm (H_\lambda, H_0; \mathfrak{I}).$$

The existence of $\tilde{\mathfrak{A}}_\pm (H, H_\lambda; \mathfrak{I})$ is now a consequence of Theorem 2.2.1 which is valid also for abelian WO. \(\square\)

The existence of the time-dependent WO $\mathfrak{A}_\pm (H, H_\lambda; \mathfrak{I})$ is discussed in the next section.

4. We here consider conditions for the isometricity of the stationary WO $\mathcal{U}_\pm = \mathcal{U}_\pm (H, H_\lambda; \mathfrak{I})$. We start from Theorem 8. For $\mathfrak{M}_0 = \mathfrak{M}$, $\mathfrak{I} = I$ the isometricity of $\mathfrak{U}_\pm$ on $\mathfrak{M}_0^{(a)}$ follows directly from relation (2.7.11) by (1.3.11), (1.4.11).

$$\varepsilon \left\| R_\lambda (\lambda \pm i \varepsilon) f_0 \right\| \sim \varepsilon \left\| R_\lambda (\lambda \pm i \varepsilon) f_0 \right\|, \quad \text{a.e. } \lambda \in \mathbb{R},$$

for any $f_0 \in \mathcal{M}_n$. We estimate the absolute value of the difference of the left- and right-hand sides of (14) by

$$e^{1/2} \left\| (\mathfrak{I}^* - I) R_\lambda (\lambda \pm i \varepsilon) f_0 \right\| = 0, \quad \forall f_0 \in \mathfrak{M}_0, \quad \text{a.e. } \lambda \in \mathbb{R}.$$

It is easy to see that for $\mathfrak{I}^* - I \leq 0$ condition (15) is also necessary for (14). We formulate the result.

**Proposition 11.** Under the conditions of Lemma 6 conditions (15) are sufficient for the isometricity of the operators $\mathcal{U}_\pm$ on $\mathfrak{M}_0^{(a)}$ (and for $\| \mathfrak{I} \| \leq 1$ they are also necessary).

**Corollary 12.** The stationary WO $\mathcal{U}_\pm$ is isometric on $\mathfrak{M}_0^{(a)}$ if under the conditions of Lemma 6 the operator $\mathfrak{I}^* - I$ is compact or at least condition (2.1.9) holds.

**Proof.** It suffices to show that

$$\varepsilon \left\| R_\lambda (\lambda \pm i \varepsilon) f_0 \right\| = 0, \quad \text{a.e. } \lambda \in \mathbb{R}$$

On the basis of (1.4.9) the limit of $\langle R_\lambda (\lambda \pm i \varepsilon) f_0, \varphi \rangle$ exists on a set of full measure not depending on the choice of linear combination $\varphi$ of elements of some basis in $\mathfrak{M}_0$. On that same set of points $\lambda$ the limit of $e^{1/2} (R_\lambda (\lambda \pm i \varepsilon) f_0, \varphi)$ is obviously equal to zero. To prove (16) it remains to consider the estimate (1.4.12). \(\square\)

Finally, interchanging the roles of the operators $H_0$ and $H$, we obtain

**Proposition 11'.** If the conditions of Lemma 6' are satisfied and

$$\lim_{\varepsilon \to 0} \left\| (\mathfrak{I}^* - I) R_\lambda (\lambda \pm i \varepsilon) f_0 \right\| = 0, \quad \forall f_0 \in \mathfrak{M}, \quad \text{a.e. } \lambda \in \mathbb{R},$$

then the WO $\mathcal{U}_\pm (H, H_\lambda; \mathfrak{I})$ is isometric.

We next combine Propositions 11 and 11'.

**Corollary 12.** If the conditions of both Lemmas 6 and 6' are satisfied as well as conditions (15) and (17), then the WO $\mathcal{U}_\pm (H, H_\lambda; \mathfrak{I})$ is isometric and complete.
§3. Connection with the time-dependent approach. IP

1. In contrast to the abelian WO \( \mathfrak{A}_1(H, H_0; \mathfrak{J}) \) (see Part 3 of §2) the existence of the strong, time-dependent WO \( \mathcal{W}_a(H, H_0; \mathfrak{J}) \) does not follow directly from results regarding stationary WO. However, according to Theorem 2.9, the matter reduces to proof of the existence of the WO \( \mathcal{W}_a(H, H_0; \mathfrak{J}) \) and \( \mathcal{W}_a(H_0, H_0; \mathfrak{J}^*) \). Consideration of the weak WO is elementary and does not depend on the constructions of §2. In this respect one uses the following

**Lemma 1.** Suppose the operator \( H \) is selfadjoint in \( \mathcal{H} \) and the operator \( G: \mathcal{H} \to \phi \) is bounded relative to \( H^2 \). We suppose that for the elements \( f \) of a dense linear manifold \( \mathfrak{M} \) in \( \mathcal{H} \) the limit (4.3.16) exists for a.e. \( \lambda \in \mathbb{R} \). Then for some linear manifold \( D \) in \( \mathcal{H}^{(a)} \) consisting of vectors with compact support

\[
\int_{-\infty}^{+\infty} \|G(t)g\|^2 dt < \infty, \quad g \in D.
\]

**Proof.** For any \( f \in \mathfrak{M} \) we denote by \( F(\lambda) \) the value of the limit (4.3.16), we set \( X_{N,n} = X_{N,n}(f) = \{ \lambda : |\lambda| \leq n, \|F(\lambda)\| \leq N \} \) and for \( D \) we take the set of linear combinations of all elements of the form \( g = E^{(a)}(X_{N,n})f \) (for all possible \( f \in \mathfrak{M} \), \( n \) and \( N \)). Similarly to the proof of Lemma 2.5.3 it can be seen that for any \( n \)

\[
\lim_{N \to +\infty} |(-n, n) \setminus X_{N,n}| = 0.
\]

From this it follows that for each \( f \in \mathfrak{M} \) the element \( Pf \) can be approximated by elements \( E^{(a)}(X_{N,n})f \in D \). Thus, \( \mathcal{D} = \mathcal{H}^{(a)} \).

Let \( \{ \varphi_p \} \) be an orthonormal basis in \( \phi \). By the spectral theorem and relations (1.3.13), (1.4.11) for \( g \in D \) there is the representation

\[
(GU(t)g, \varphi_p) = \int_{-\infty}^{+\infty} \exp(-i\lambda t) \frac{d(GE(\lambda)g, \varphi_p)}{d\lambda} d\lambda = \int_{X_{N,n}} \exp(-i\lambda t)(F(\lambda), \varphi_p) d\lambda.
\]

We now apply the Parseval equality and sum over the index \( p \). We then find that

\[
\int_{-\infty}^{+\infty} \|GU(t)g\|^2 dt = 2\pi \int_{X_{N,n}} \|F(\lambda)\|^2 d\lambda.
\]

Thus, by the construction of the set \( X_{N,n} \) the integral (1) converges (and does not exceed \( 4\pi N^2 n^2 \)).

In considering the weak WO \( \mathcal{W}_a(H, H_0; \mathfrak{J}) \) it suffices to assume that relation (1.9.3) is satisfied only on elements \( f_0 \in \mathcal{D}(H_0^2) \) and \( f \in \mathcal{D}(H^2) \), while the operators \( G_0 \) and \( G \) are bounded relative to \( H_0^2 \) and \( H^2 \).

---

**Theorem 2.** Suppose that on linear manifolds \( \mathfrak{M}_0, \mathfrak{M} \) dense in \( \mathcal{H}_0, \mathcal{H} \) for a.e. \( \lambda \in \mathbb{R} \)

\[
\exists \lim_{\epsilon \to 0} G_0 g_0(\lambda, \epsilon) f_0, \quad \lim_{\epsilon \to 0} G \delta(\lambda, \epsilon) f, \quad f_0 \in \mathfrak{M}_0, \quad f \in \mathfrak{M}.
\]

Then the weak time-dependent WO \( \mathcal{W}_a(H, H_0; \mathfrak{J}) \) exists.

**Proof.** Since the norm of the operator (1.2.1) does not exceed \( ||\mathfrak{M}|| \) it suffices to establish the existence of the limit of \( (W(t) f, f) \) for \( f_0 \in \mathfrak{M}_0, \quad f \in \mathfrak{M} \). According to the Schwarz inequality, the integral (4.5.1) can be estimated in terms of

\[
\left( \int_{t_0}^{t_1} \int_{t_0}^{t_1} \|G_0 U_0(s) f_0\|^2 ds \int_{t_0}^{t_1} \|G U(s) f\|^2 ds \right)^{1/2}.
\]

By Lemma 1 this expression tends to zero as \( t_1, t_2 \to \pm \infty \) which proves the existence of the desired limit.

According to Lemma 1.6, the conditions of Theorem 2 are satisfied for \( \mathfrak{M}_0 = \mathcal{H}_0, \mathfrak{M} = \mathcal{H} \), and hence the weak WO \( \mathcal{W}_a \) exist if the operators \( G_0 \) and \( G \) are, respectively, weakly \( H_0 \) and \( H \) smooth. The assertion of Theorem 2 remains in force if in place of (1.9.2) the representation (4.5.3) holds, and conditions (2) are satisfied on \( \mathfrak{M}_0, \mathfrak{M} \) for all \( G_0, G \). We further note that if (2) holds the same conditions are satisfied also for the collection \( H, H_0, \mathfrak{J}^* \). This must be the case, since the WO \( \mathcal{W}_a(H, H_0; \mathfrak{J}) \) and \( \mathcal{W}_a(H_0, H; \mathfrak{J}^*) \) exist simultaneously. Finally, combining Theorem 2 with Theorem 2.4 and Lemmas 2.5 or 2.5, we establish

**Theorem 3.** Suppose the conditions of Lemmas 2.5 or 2.5' are satisfied for both signs for elements \( f_0 \) and \( f \) from dense sets in \( \mathcal{H}_0 \) and \( \mathcal{H} \). Then the WO \( \mathcal{W}_a(H, H_0; \mathfrak{J}) \) and \( \mathcal{W}_a(H_0, H; \mathfrak{J}^*) \) exist and coincide.

2. We now present sufficient conditions for the existence of the strong WO (2.1.1). We shall again assume without special mention that the operators \( G_0 \) and \( G \) are bounded relative to the operators \( H_0 \) and \( H \) respectively. We first give a semi-effective condition for the existence of the WO \( \mathcal{W}_a(H, H_0; \mathfrak{J}) \).

**Theorem 4.** Suppose the conditions of Lemma 2.6 are satisfied and the weak WO \( \mathcal{W}_a(H_0, H; \mathfrak{J}^*) \) exists. Then the strong WO \( \mathcal{W}_a(H, H_0; \mathfrak{J}) \) exists and coincides with the stationary WO \( \mathcal{W}_a(H, H_0; \mathfrak{J}) \).

**Proof.** By Theorem 2.9 the stationary WO \( \mathcal{W}_a(H_0, H; \mathfrak{J}^*) \) exists and coincides with \( \mathcal{W}_a(H, H_0; \mathfrak{J}^*) \). At the same time from Theorem 3 it follows that the WO \( \mathcal{W}_a(H, H_0; \mathfrak{J}) = \mathcal{W}_a(H, H_0; \mathfrak{J}) \) exist. Equality (2.7.16) for the stationary WO gives relation (2.2.3). Therefore, the existence of the strong WO \( \mathcal{W}_a(H, H_0; \mathfrak{J}) \) is a consequence of Theorem 2.2.1. \( \Box \)
In the important special case \( \mathcal{N}_0 = \mathcal{N} \), \( J = I \) the assumption of the existence of \( \mathcal{W}_\pm(H_0, H_0; J^*) \) here drops out. We shall present a more general assertion of this type.

**Theorem 5.** Suppose in addition to the conditions of Lemma 2.6, that relation (2.1.9) is satisfied for any bounded set \( \Lambda \). Then the assertion of Theorem 4 holds, and the WO \( \mathcal{W}_\pm(H, H_0; J) \) is isometric on \( \mathcal{M}_0^{(8)} \).

**Proof.** The existence of the WO \( \mathcal{W}_\pm(H_0, H_0; J^*) \) (and its equality to the projection \( P_\pm \)) is a consequence of Lemma 1.4.1. Therefore, the WO \( \mathcal{W}_\pm(H, H_0; J) \) exists by Theorem 4. Its isometricity follows from Proposition 2.1.3. □

It is possible to give concrete conditions for the existence of the WO \( \mathcal{W}_\pm(H_0, H_0; J^*) \) in the general case as well. As compared with Lemma 2.6, only minor additional assumptions are required. The next assertion is rather convenient, although the verification of the existence of \( \mathcal{W}_\pm(H_0, H_0; J^*) \) in it is somewhat artificial.

**Theorem 6.** Suppose that both conditions (2.10) are satisfied on a set dense in \( \mathcal{N}_0 \), and, moreover, the operator \( G \) is \( |H|^{1/2} \)-bounded and

\[
\exists \omega \lim_{\omega \to 0} B(\lambda + i\omega) =: B(\lambda + i0), \quad \lambda \in \mathbb{R}, \quad B(z) = GR(z)G^*. \quad (3)
\]

Then the strong WO \( \mathcal{W}_\pm(H, H_0; J) \) exists and coincides with the stationary WO \( \mathcal{W}_\pm(H, H_0; J) \).

**Proof.** Under condition (3) the operator \( G \) is, of course, weakly \( H \)-smooth. Therefore, by Theorem 4 it is only necessary to establish the existence of the weak WO \( \mathcal{W}_\pm(H_0, H_0; J^*) \). We start from Theorem 2. For an \( |H|^{1/2} \)-bounded operator \( G \) it follows immediately from the resolvent identity (1.9.4) that \( J : \mathcal{D}(H_0) \to \mathcal{D}(H^{1/2}) \). Therefore, in the form

\[
(\text{sgn} |H|^{1/2} J_{0}, |H|^{1/2} \varphi) - (3H_0^2 \varphi, \varphi) = (G_0 f_0, G \varphi)
\]

equality (1.9.3) extends to arbitrary \( f_0 \in \mathcal{D}(H_0), \varphi \in \mathcal{D}(H^{1/2}) \). We now compute the "perturbation"

\[
H_0^2 J^* - 3^* \partial H_0 = 3^* (H^2 - \partial H_0) - (3^* H - H^2 J^*) J = 3^* V - V^* J
\]

\[
= (GJ) G_0 - G_0 (GJ).
\]

In a precise sense (5) must, of course, be understood as equality of the corresponding sesquilinear forms on the elements \( f_0, g_0 \in \mathcal{D}(H_0) \). This is justified by appeal to (4) where for \( g_0 \in \mathcal{D}(H_0) \) it is possible to set \( \varphi = \lambda g_0, \varphi \in \mathcal{D}(H^{1/2}) \). Equality (5) gives a representation of the form (4.5.3) for the collection \( H_0, H_0, J^* \). By conditions (2.10) \pm \ an assumption of the type (2) for the factor \( G_0 \) holds. It remains to verify it for the factor \( \tilde{G}_0^* = GJ \).

To this end it suffices (cf. Lemma 1.2) to obtain the estimate

\[
\|GJ \delta_0(\lambda, e) f_0\| \leq C(\lambda), \quad \lambda \in \mathbb{R}, \quad f_0 \in \mathcal{M}_0.
\]

An \( |H|^{1/2} \)-bounded operator \( G \) can be applied to both sides of (1.9.4). This gives the equality

\[
GJ \delta_0(\lambda, e) f_0 = G(\lambda z) + B(z) G_0 R_0(x).
\]

Evaluating the operator (7) on the element \( R_0(x) f_0 \), we find that

\[
\|GJ \delta_0(\lambda, e) f_0\| \leq \pi^{-1} (\lambda \|GR(z)\|) \|3^* \| (e^{1/2} \|R_0(x) f_0\|) + \|B(z)\| \|G_0 \delta_0(\lambda, e) f_0\|, \quad \lambda = \lambda + i\epsilon.
\]

Now (6) follows from (2.10) and (3) if we consider (1.4.12). □

The \( |H|^{1/2} \)-boundedness of the operator \( G \) in Theorem 6 can be relaxed somewhat. Namely, if for some auxiliary operator \( h \geq \epsilon > 0 \) equality (1.9.1) holds for \( H_0 = \theta = 1/2 \), then it suffices to require of \( G \) that it be \( \lambda \)-bounded. Analogous remarks apply also to other assertions of this chapter, where the assumptions of \( |H|^{1/2} \)-boundedness (or \( |H|^{1/2} \)-boundedness) of some operators arise.

3. By combining existence conditions for time-dependent WO with completeness conditions for stationary WO, we obtain completeness (or \( J \)-completeness) criteria for the WO \( \mathcal{W}(H, H_0; J) \) and also existence criteria for the "inverse" WO \( \mathcal{W}(H_0, H; J^*) \).

**Theorem 7.** Suppose the conditions of Lemma 2.6 and 2.6' as well as relations (2.1.9) and (2.1.7) are satisfied. Then the WO \( \mathcal{W}(H, H_0; J) \) and \( \mathcal{W}(H_0, H; J^*) \) exist, are isometric (respectively on \( \mathcal{M}_0^{(8)} \) and \( \mathcal{M}_0^{(8)} \)), and are complete.

**Proof.** The existence and isometricity of \( \mathcal{W}(H, H_0; J) \) follow from Theorem 5. Its completeness is equivalent to the isometricity of the WO

\[
\mathcal{W}(H, H_0; J) = \mathcal{W}_\pm(H_0, H; J^*) \mathcal{W}_\pm(H, H_0; J)
\]

which follows directly from Proposition 2.1.1. The existence of the WO \( \mathcal{W}(H_0, H; J^*) \) is now ensured by Corollary 2.3.10. In view of equality (2.3.10) this WO is isometric and complete together with \( \mathcal{W}(H, H_0; J) \). □

In the next assertion conditions for the existence of the WO \( \mathcal{W}(H_0, H; J^*) \) are obtained from Theorem 6 by interchanging the roles of the operators \( H_0, H \).

**Theorem 8.** Suppose that, in addition to the conditions of Theorem 6, both limits (4.3.14) exist on a dense set in \( \mathcal{N} \), the operator \( G_0 \) is \( |H_0|^{1/2} \)-bounded, and

\[
\exists \omega \lim_{\omega \to 0} G_0 R_0(\lambda + i\omega) G_0^*.
\]
Then the strong WO $W_\pm(H, H_0; 3)$ and $W_\pm(h, H_0; 3)$ exist and are, respectively, 3 and 3'-complete.

**Proof.** By Theorem 3.2.4 it is only necessary to verify the existence of the WO. The WO $W_\pm(H, H_0; 2)$ exists by Theorem 6. For the WO $\bar{W}_\pm(H_0, H; 3')$ the assumptions (4.3) and (8) play the role of conditions (2.10) and (3). Therefore, by Theorem 6 the WO $W_\pm(H_0, H; 3')$ also exists. $\square$

4. Finally, we discuss the invariance principle (IP). Consider any function $\varphi$ admissible in the sense of Definition 2.6.2 on an open set $\Omega$. It is assumed that $\varphi$ is admissible with respect to the operator $H$. We recall that $\Omega_\pm = \{ \lambda \in \Omega : \pm \varphi(\lambda) > 0 \}; \Omega$ is the set of compactly supported (relative to $E(\cdot)$) vectors. The following auxiliary assertion is analogous to Lemma 1; in particular, the set $D$ in it is the same as in Lemma 1.

**Lemma 9.** Under the conditions of Lemma 1 on some linear manifolds $D^{(\pm)} \subset \mathbb{C}$ dense in $E^{(\pm)}(\Omega_\pm)$\ $\mathcal{F}$

$$\lim_{t \to \pm \infty} \int_0^\infty \| G \exp(\pm \lambda t H + it \varphi(H))g \|^2 ds = 0, \quad g \in D^{(\pm)}.$$ (9)

Moreover, for any $g \in D$ the integral (9) can be estimated uniformly with respect to $t$.

**Proof.** We denote by $D^{(\pm)}$ the set of elements $g \in D$ belonging to the same subspace $E(\Omega_\pm)$ for any finite union $\Omega_\pm$ of the component intervals $\Omega_\pm$. In the notation of the proof of Lemma 1 the integral (9) admits the representation

$$\sum_{\lambda \in D^{(\pm)}} \int_0^\infty ds \left| \int_{X_{\lambda, n}} \exp(\pm is \lambda + it \varphi(\lambda)) (F(\lambda), \varphi, d\lambda \right|^2.$$ (10)

For $g \in D^{(\pm)}$ each term of the sum (10) tends to zero as $t \to \pm \infty$ by Lemma 2.6.4. At the same time, according to the Parseval equality, the terms of the series (10) are estimated uniformly with respect to $t$ by

$$2\pi \int_{X_{\lambda, n}} |(F(\lambda), \varphi)|^2 d\lambda.$$

This series converges, since on the set $X_{\lambda, n}$ the function $F(\lambda)$ is clearly square integrable. This justifies termwise passage to the limit in (10).

Finally, for any $g \in D$ a uniform estimate of the integral (9) can be established by duplicating the proof of Lemma 1. $\square$

We first establish the IP for weak WO. We set $h_0 = \varphi(H_0)$, $h = \varphi(H)$, $u_0(t) = \exp(-ith_0 t)$, $u(t) = \exp(-ith)$. We suppose that the function $\varphi$ is admissible on $\Omega$ with respect to both operators $H_0$ and $H$; $D_0$ is the dense set in $E_0^{(\pm)}(\Omega_\pm)$ constructed in Lemma 9 (but with respect to the operator $H_0$).

**Theorem 10.** Under the conditions of Theorem 2 the weak WO $\bar{W}_\pm(h_0, h_0; 3)$ exist, and

$$\bar{W}_\pm(h_0, h_0; 3)E_0(\Omega_\pm) = \bar{W}_\pm(H, H_0; 3)E_0(\Omega_\pm), \quad \nu = +.$$

**Proof.** Suppose in relation (4.5) $f_\nu = u_0(t_0)g_0$, $f = u(t)g$, where $g_0 \in D_0$, $g \in D$. Since by Theorem 2 the WO $\bar{W}_\pm(H, H_0; 3) = \bar{W}_\pm$ exists, it is possible to set $t_0 = 0$, $t_2 = \pm \infty$ in this relation. Using the intertwining property, we then obtain the equality

$$\bar{W}_\pm g_0, g - (3u_0(t)g_0, u(t)g)$$

$$= i \int_0^{\pm \infty} (G_0 U_0(s)u_0(t)g_0, GU(s)u(t)g) ds.$$ (12)

By the Schwarz inequality the integral on the right does not exceed

$$\left( \int_0^{\pm \infty} \| G_0 U_0(s)u_0(t)g_0 \|^2 ds \right)^{1/2} \left( \int_0^{\pm \infty} \| GU(s)u(t)g \|^2 ds \right)^{1/2}.$$

By Lemma 9 the first factor here tends to zero as $t \to \pm \infty$, while the second is bounded uniformly with respect to $t$. The desired assertion now follows from equality (12). $\square$

Justification of the IP for strong WO is based on Theorem 2.2.1 or on a corollary of it.

**Theorem 11.** Under the conditions of Theorems 5 or 6 the strong WO $W_\pm(\varphi(H), \varphi(H_0); 3)$ exist and satisfy equality (2.6.11).

**Proof.** Suppose, for example, that the conditions of Theorem 6 are satisfied. For the collection $H_0$, $H$, $3'$ the conditions of Theorem 2 are then satisfied. Hence, by Theorem 10 the weak WO $\bar{W}_\pm(h_0, h_0; 3')$ exists and is connected with $\bar{W}_\pm(H_0, H_0; 3')$ by equalities of the form (11). We now use equality (2.2.3) which is valid under the conditions of Theorem 6. From this and Theorem 10, we obtain the relation

$$\bar{W}_\pm(h_0, h_0; 3)\bar{W}_\pm(h_0, h_0; 3') = \bar{W}_\pm(h_0, h_0; 3'),$$

(13)

Therefore, by Theorem 2.2.1 the strong WO $W_\pm(h_0, h_0; 3)$ exist. Equalities (11) imply (2.6.11).

Under the conditions of Theorem 5 the weak WO $\bar{W}_\pm(h_0, h_0; 3) = \bar{W}_\pm(h_0, h_0; 3)$ are isometric on $\mathcal{F}_0^{(\pm)}$. Under condition (2.1.9) these WO are also strong by Corollary 2.2.2. $\square$

§ 4. Integral operators in direct decompositions

Formal representations for the scattering matrix are given in terms of the kernels of certain operators which are interpreted as integral operators (see Part 3 of § 1.5). It is important here that with the help of the concept of weak smoothness a reasonable meaning can be ascribed to the values of the kernels on the diagonal.
1. We shall discuss conditions that some operator $A$ be an integral operator in a decomposition of a Hilbert space into a direct integral, diagonal for a selfadjoint operator $H$. More precisely, in connection with applications in scattering theory in place of $A$ in the decomposition (1.5.6) we consider the operator $PAP$ where $P = \rho \|H\|$. We recall that in (1.5.6) $\sigma$ is the core of the spectrum of the operator $H$. We denote by $\mathcal{F}$ the mapping of $\mathcal{H}$ onto the direct integral (1.5.6) which acts unitarily on $\mathcal{H}\{\lambda\}$ and is equal to zero on $\mathcal{H}\{\sigma\}$. As explained in Part 4 of §1.5, the operator $A$ need not be assumed to be bounded. It suffices to assume that $A$ acts boundedly from $\mathcal{D} = \mathcal{D}(H)$ to $\mathcal{D}^\prime$. Then in $\mathcal{H}, A$ acquires an operator meaning only on being bordered on both sides by the resolvents of the operator $H$.

From the viewpoint of scattering theory, Definition 1.5.2 is too broad. The fact of the matter is that the values of the kernel $a(\mu, \nu)$ of an integral operator are fixed by it only for a.e. $(\mu, \nu)$ with respect to two-dimensional Lebesgue measure on $\sigma \times \sigma$. At the same time, to derive formula representations of the scattering matrix one needs to be able to ascribe a reasonable meaning to the kernel on the direct product $\Lambda \times \Lambda$, where $|\sigma\Lambda| = 0$. For such kernels it is possible, in particular, to define values on the diagonal even though it has two-dimensional measure zero. Precisely in terms of weak $H$-smoothness a definition of the kernel can be given suited for scattering theory. Namely, we here consider kernels of operators of the form

$$
A = G^\sigma G, \tag{1}
$$

where $\mathcal{A}$ is a bounded operator in $\mathcal{B}$, and $G : \mathcal{H} \rightarrow \mathcal{B}$ is $H$-smooth in the weak sense.

With the help of the operator $G$ we construct for a.e. $\lambda \in \sigma$ a bounded mapping $Z(\lambda) = Z(\lambda; G)$ of the space $\mathcal{B}$ into $h(\lambda)$. We denote by $\Lambda_\sigma$ the set of $\lambda \in \sigma$ for which the limits (1.1) and (1.4) exist, while $\Lambda_\sigma(\lambda) = \Delta(\lambda; G)$ is their common value. Let $D$ be a dense set in $\mathcal{B}$ of linear combinations of elements $\varphi$, of some basis, and suppose the set $\Lambda_\sigma$ of full measure in $\sigma$ consists of those $\lambda \in \sigma$ for which all the functions $(\mathcal{F} G^\sigma \varphi)\lambda$ are defined. On elements $\varphi \in D$ the operator $Z(\lambda; G)$ for $\lambda \in \Lambda_\sigma$ can be defined by the equality

$$
Z(\lambda; G)\varphi = (\mathcal{F} G^\sigma \varphi)\lambda. \tag{2}
$$

The function (2) belongs to $h(\sigma)$ if $G \in \mathcal{B}$. In the general case it is necessary to bear in mind the extension of the mapping $\mathcal{F}$, indicated in Part 4 of §1.5, to the entire class $\mathcal{H}\{\lambda\}$ (or at least to $\mathcal{D}\{\lambda\}$). From (1.5.7) it follows that

$$
\langle Z(\lambda; G)\varphi, Z(\lambda; G)\psi \rangle = \langle \Delta(\lambda; G)\varphi, \psi \rangle \tag{3}
$$

for $\lambda \in \Lambda = \Lambda_\sigma \cap \Lambda_\sigma$. Since the operator $\Delta(\lambda; G)$ is bounded, it follows from this that on a set $\Lambda$ of full measure in $\sigma$ the operator $Z(\lambda; G)$ is also bounded and extends by continuity to all of $\mathcal{B}$. Moreover, according to (3),

$$
Z^\ast(\lambda; G)Z(\lambda; G) = \Delta(\lambda; G). \tag{4}
$$

We point out that the definition given of the operator $Z(\lambda)$ depends on the basis $\{\varphi\}$ because of the fact that the set $\Lambda$ on which $Z(\lambda)$ is defined depends on it. If a mapping $\widetilde{Z}(\lambda)$ is constructed for $\lambda \in \Lambda$ with respect to another basis $\{\tilde{\varphi}\}$, then for $\lambda \in \Lambda \cap \Lambda$ we necessarily have $Z(\lambda) = \tilde{Z}(\lambda)$.

With the help of the mapping $Z(\lambda)$ it is possible to "compute" the limits (1.9).

**Lemma 1.** Suppose the operator $G$ is weakly smooth relative to $H$. Then for any $f \in \mathcal{H}$ the weak derivative

$$
\frac{d(GE(\lambda)f)}{d\lambda} = Z^\ast(\lambda; G)\tilde{Z}(\lambda), \quad a.e. \lambda \in \sigma. \tag{5}
$$

**Proof.** It suffices to verify that for $\varphi \in D$

$$
\frac{d(GE(\lambda)\varphi, \varphi)}{d\lambda} = (\tilde{Z}(\lambda), Z(\lambda; G)\varphi).
$$

This equality follows directly from definition (2) and relation (1.5.7).

We note that for a.e. $\lambda \in \mathbb{R}\setminus \sigma$ the limits (1.1) and (1.4) exist and are equal to zero for any $H$-bounded operator $G$ and, in particular, for $G = I$. By extending $Z(\lambda; G)$ by zero to $\mathbb{R}\setminus \sigma$, it is natural to assume that this mapping is defined for a.e. $\lambda \in \mathbb{R}$. Equivalences (4) and (5) hereby hold for a.e. $\lambda \in \mathbb{R}$.

2. We can now introduce

**Definition 2.** Suppose the operator $A : \mathcal{D} \rightarrow \mathcal{D}^\prime$ can be represented in the form (1). The kernel of the operator $PAP$ in the decomposition (1.5.6) is the operator-valued function

$$
a(\mu, \nu) = Z(\mu; G)w^\ast Z^\prime(\nu; G). \tag{6}
$$

The kernel (6) is defined on the square $\Lambda \times \Lambda$, where the set $\Lambda$ is the set of full measure in $\sigma$ consisting of points $\lambda$ for which the operator $Z(\lambda; G)$ is defined. The set $\Lambda$ does not depend on the operator $\mathcal{A}$. The values of the kernel $a(\mu, \nu)$ are bounded mappings of $h(\nu)$ into $h(\mu)$. The value $a(\lambda, \lambda)$ of the kernel on the diagonal is hereby defined for $\lambda \in \Lambda$. Sometimes under the conditions of Definition 2 we also call for brevity the operator $A$ an integral operator and call $a(\mu, \nu)$ its kernel.

Definitions 2 and 1.5.2 are, of course, consistent with one another.

**Lemma 3.** Suppose the operator $A$ has the form (1), where the operator $G$ is weakly $H$-smooth. Then $A$ is an integral operator, and the function (6) is its kernel in the sense of Definition 1.5.2.

**Proof.** It suffices to show that for the operator-valued function (6) equality (1.5.11) holds for any $f, g \in \mathcal{H}\{\lambda\} \cap D$. Indeed, according to Lemma 1

$$
(Af, g) = \int_0^1 \frac{d}{d\mu}(E(\mu)G^\ast wZGf, g) d\mu = \int_0^1 \langle \mathcal{A} Gf, Z^\ast(\mu; G)\tilde{g}(\mu) \rangle d\mu. \tag{7}
$$
Similarly, for a fixed $\mu \in \tilde{\sigma}$
\[
(\mathcal{G} f, Z^*(\mu; G)\tilde{g}(\mu)) = \int_{\delta} \frac{d}{d\nu}(GE(\nu)f, \mathcal{A}^*Z^*(\mu; G)\tilde{g}(\mu)) \, d\nu
\]
\[
= \int_{\delta} (Z^*(\nu; G)\tilde{f}(\nu), \mathcal{A}^*Z^*(\mu; G)\tilde{g}(\mu)) \, d\nu
\]
\[
= \int_{\delta} (a(\mu, \nu)\tilde{f}(\nu), \tilde{g}(\mu)) \, d\nu,
\]
where $a(\mu, \nu)$ is given by equality (6). It remains to substitute this expression into (7). □

Under the conditions of Definition 2 the form $(\mathcal{A}f, g)$ also satisfies the representation (1.5.11) with the order of integrations interchanged. Therefore, the operator $\mathcal{A}^*$ is also an integral operator in the sense of Definition 1.5.2.

We recall that for any integral operator (in the sense of Definition 1.5.2) the sesquilinear form of its kernel can be recovered by equality (1.5.13) for a.e. $(\mu, \nu) \in \tilde{\sigma} \times \tilde{\sigma}$. For the operators (1) this assertion can be refined.

**LEMMA 4.** Suppose the operator $A$ has the form (1), where the operator $G$ is weakly H-smooth, and the set $D$ consists of linear combinations of some basis $(\varphi_i)$ in $\mathcal{F}$. Then on some measurable square $\Lambda \times \Lambda$ of full measure in $\delta \times \tilde{\delta}$ for any $\varphi, \psi \in D$
\[
(a(\mu, \nu)\tilde{\varphi}(\nu), \tilde{\psi}(\mu)) = \left(\mathcal{A}^* \frac{d}{d\nu}GE(\nu)\varphi, \frac{d}{d\mu}GE(\mu)\psi\right).
\]

**PROOF.** Suppose $\Lambda_{\varphi}$ is the set of $\lambda \in \delta$ for which equality (5) holds. Then for $\nu \in \Lambda_{\varphi}$, $\mu \in \Lambda_{\psi}$ relation (8) follows directly from definition (6). For any $\varphi, \psi \in D$ relation (8) is satisfied on the square $\Lambda \times \Lambda$ where $\Lambda = \bigcap \Lambda_{\varphi}$. The set $\Lambda$ has full measure in $\delta$, since $|\delta \setminus \Lambda_{\varphi}| = 0$ by Lemma 1. □

By Lemma 4 for elements $\varphi, \psi \in D$ the sesquilinear form
\[
(a(\mu, \nu)\tilde{\varphi}(\nu), \tilde{\psi}(\mu))
\]
can be recovered by equality (1.5.13) on a measurable square $\Lambda \times \Lambda$ of full measure in $\delta \times \tilde{\delta}$. From this it follows that the kernel $a(\mu, \nu)$ itself is uniquely defined on $\Lambda \times \Lambda$ and, in particular, does not depend on the factorization (1). This assertion is stronger than Lemma 1.5.3 since now the kernel $a(\mu, \nu)$ is constructed on a set of full measure having the structure of a direct product.

We further present a version of Lemma 4 corresponding to the representation (1.5.9). We recall that in terms of this representation the relation (1.5.13) assumes the form (1.5.14) where $(\mathcal{F} \Psi_{\nu}(\lambda) = \chi_{\nu}(\lambda)) \frac{d}{d\nu}$. We shall first establish an assertion similar to Lemma 1.6.

**LEMMA 5.** Suppose the operator $G$ is weakly H-smooth. Then for a.e. $\lambda \in X_m$ the operator-valued function $GE(\lambda)\mathcal{F}^*\Psi_m$ is bounded and the operator is weakly differentiable.

**PROOF.** It suffices to verify the estimate
\[
\|GE((\lambda - \eta, \lambda + \eta))\mathcal{F}^*\Psi_m\| \leq C(\lambda)|\eta|.
\]

By the definition of $\Psi_m$
\[
\|E((\lambda - \eta, \lambda + \eta))\mathcal{F}^*\Psi_m\| = \int_{-\kappa}^{\kappa} \|\Psi_{m}(\mu)\|^2 \, d\mu \leq 2\kappa|\eta|^2.
\]
which together with (1.6) proves (9). □

The next assertion is obtained from Lemma 4 by passing to the representation (1.5.9). It extends equality (1.5.14) to a measurable rectangle of full measure which does not depend on the choice of the pair $x, y$.

**LEMMA 6.** Suppose the operator $A$ has the form (1), where the operator $G$ is weakly H-smooth. Then on some measurable rectangle in $X_m \times X_m$ of full measure for any $x \in h_1$, $y \in h_2$
\[
(a(\mu, \nu)x, y) = \left(\mathcal{A}^* \frac{d}{d\nu}GE(\nu)\mathcal{F}^*\psi, \frac{d}{d\mu}GE(\mu)\mathcal{F}^*\varphi\right).
\]

**PROOF.** On elements $f = \mathcal{F}^*\psi, \varphi = \mathcal{F}^*\varphi$ equality (8) takes the form (10). It may be assumed that (10) is satisfied on a common measurable rectangle of full measure in $X_m \times X_m$ if $x$ and $y$ are formed from finite linear combinations of elements of some basis in $h_1$ and $h_2$. Moreover, by Definition 2 the operators $a(\mu, \nu) \in h_1 \otimes h_2$ are bounded on a rectangle of full measure, and by Lemma 5 the operators $dGE(\lambda)\mathcal{F}^*\Psi_m/\lambda$ are bounded for a.e. $\lambda \in \delta$. Therefore, equality (10) extends by continuity to all $x \in h_1$, $y \in h_2$. □

The scheme described generalizes naturally to finite sums of the form
\[
A = \sum_{i, j=1}^{r} A_{ij}, \quad A_{ij} = G_i^* \mathcal{A}^*_j G_j,
\]
where the operators $\mathcal{A}^*_j$ are bounded in $\mathcal{H}$ and $G_j : \mathcal{F} \rightarrow \mathcal{H}$ are weakly H-smooth. The kernel $a(\mu, \nu)$ of the operator $A$ hereby obtained decomposes into the sum of the kernels of the individual terms:
\[
a(\mu, \nu) = \sum_{i, j=1}^{r} a_{ij}(\mu, \nu), \quad a_{ij}(\mu, \nu) = Z(\mu; G_j)A_{ij}^* Z(\nu; G_j).
\]

We note that this more general case can be reduced (cf. Remark 4.5.3) to the case of operators of the form (1). For this it is necessary to introduce the new auxiliary space $\mathcal{F}^{(r)} = \mathcal{F} \otimes \mathcal{C}^r$ and set $G^{(r)} = \{G_1, G_2, \ldots, G_f\}$. The operator $G$ turns out to be weakly H-smooth by Lemma 1.5.

3. In view of Lemma 1.6 under the conditions of Definition 2 relation (1.5.13) (see also its form (8)) can be rewritten in the form
\[
(a(\mu, \nu)\tilde{f}(\nu), \tilde{g}(\mu)) = \lim_{\nu \rightarrow \nu_0, \epsilon \rightarrow 0} \lim_{\delta(\nu, \epsilon)) \rightarrow G(\mu, \varphi, \delta(\mu, \epsilon))
\]
\[
a.e. \mu \in \delta, \quad a.e. \nu \in \delta.
\]

The limit here is understood as an iterated limit whereby the limits on $\epsilon$ and $\eta$ in (13) can be interchanged. It turns out that under minor additional assumptions the iterated limit in (13) can be replaced by a double limit. Moreover, the case of operators $A$ depending on an additional parameter can be considered. We shall write the corresponding formulas for the kernels $a_{ij}$ of the operators $A_{ij}$ (see (11), (12)).
Lemma 7. Suppose that the operators $G_1$ and $G_j$ are weakly $H$-smooth and $a_{ij}$ is the kernel of the operator $A_{ij} = G_i^* A_j G_j$. (1) Suppose there exists (any) one of the strong limits
\[ \lim_{\varepsilon \to 0} G_j \delta(\lambda, \varepsilon) f, \quad \lim_{\varepsilon \to 0} G_j \delta(\lambda, \varepsilon) g, \quad \text{a.e. } \lambda \in \delta. \] (14)

Then for a.e. $\mu \in \delta$ and a.e. $\nu \in \delta$ there exists the double limit
\[ \lim_{\varepsilon \to 0, \eta \to 0} (A_{ij}(\mu, \eta)f, \delta(\mu, \varepsilon)g) = (a_{ij}(\mu, \nu)\tilde{f}(\nu), \tilde{g}(\mu)). \] (15)

(2) Suppose both limits (14) exist, $A_{ij}(\kappa) = G_i^* A_j(\kappa) G_j$, and $A_{ij}(\kappa)$ converges to $A_{ij}$ as $\kappa \to 0$. Then for a.e. $\mu \in \delta$ and a.e. $\nu \in \delta$ there exists the triple limit
\[ \lim_{\varepsilon \to 0, \eta \to 0, \kappa \to 0} (A_{ij}(\kappa) \delta(\nu, \eta)f, \delta(\mu, \varepsilon)g) = (a_{ij}(\mu, \nu) \tilde{f}(\nu), \tilde{g}(\mu)). \] (16)

Proof. Let us clarify, for example, the first assertion. By Lemma 1.6 the second of the limits of (14) exists with $\varepsilon$-limit replaced by $w$-limit. Thus, for $\varepsilon \to 0$ one of the factors in $(\delta(\lambda) G_j \delta(\nu, \eta)f, G_i \delta(\mu, \varepsilon)g)$ has a strong limit while the second has a weak limit. The double limit of this scalar product therefore exists. Equality (15) itself follows now from (13).

This lemma extends naturally to a.e. $\mu \in \mathbb{R}$ and a.e. $\nu \in \mathbb{R}$. Here, as explained at the end of Part 1, both parts of (15) and (16) are equal to zero if at least one of the variables $\mu$ or $\nu$ does not belong to $\delta$.

§5. The scattering matrix

This section is related to §2.8. In Parts 1 and 2 we justify, respectively, the representations for the stationary scattering operator and matrix presented without proof in Parts 1 and 2 of §2.8.

1. Representations for the form $\langle E(X) \mathcal{H}_- f_\omega, \mathcal{H}_+ g_\omega \rangle$ are justified according to the scheme of Part 2 of §2. Indeed, all the representations of Part 1 of §2.8 are derived directly from relation (2.8.1). Therefore, it suffices only to justify this relation. In analogy to (2.7.14) relation (2.8.1) decomposes into two asymptotic equalities of the type (2.13). More precisely, the first of them coincides with the first equality of (2.13) for the upper sign, while the second has the form
\[ \langle G_0 R_0(\lambda - i\varepsilon) f_\omega, G_i(\lambda, \varepsilon) \mathcal{H}_+ g_\omega \rangle \sim \pi^{-\frac{1}{2}} e(\lambda) G_0 R_0(\lambda - i\varepsilon) f_\omega, GR(\lambda - i\varepsilon) \mathcal{H}_+ g_\omega \rangle. \]

Therefore, relation (2.8.1) follows again from Lemma 2.7. We emphasize, however, that (2.8.1) requires both conditions (2.10) while Part 2 of §2 condition (2.10) was used only with "its own sign."

To derive the representation (2.8.2) it is now only necessary to set $f = \mathcal{H}_- f_\omega$ in equality (2.7.8) (for the lower sign) and consider (2.8.1). The remaining relations of Part 1 of §2.8 are derived from (2.8.2) by identity transformations. Thus, in analogy to Theorem 2.8 we have

Theorem 1. Suppose on a set $\mathcal{M}$ dense in $\mathcal{H}$ both conditions (2.10) and (2.10) are satisfied and the operator $G$ is weakly $H$-smooth. Then for $f_\omega, g_\omega \in \mathcal{M}$ the representations (2.8.2), (2.8.6) hold as well as the representations obtained from (2.8.2) by consideration of the identities (2.8.3).

Actually the integrals (2.8.2), (2.8.6) are taken, of course, only over the set $\delta \cap X$.

2. The representations of Part 2, §2.8 for the scattering matrix $S(\lambda) = S(\lambda; H, H_0; \delta)$ can be derived from (2.8.6). The representation (2.8.7) for the sesquilinear form of $S(\lambda)$ is a direct consequence of (2.8.6) and is hence satisfied under the same conditions.

The representation (2.8.9) requires that there exist at least the weak limit $T_0(\lambda + i\varepsilon)$ of the operators (2.8.4), acting from $\mathcal{D}_0 = \mathcal{D}(H_0)$ to $\mathcal{D}_0^*$. Moreover, it is required that a kernel in the sense of Definition 4.2 can be ascribed to the operator $T_0^0(\lambda + i\varepsilon) T_0^0$ in the decomposition (2.4.2). It is hereby necessary to assume that the corresponding operators $Z_0$ are defined by equalities of the form (4.2) from a unitary mapping $\mathcal{F}_0$ of the space $\mathcal{H}_0^{(0)}$ onto the direct integral (2.4.2) (the mapping $\mathcal{F}_0$ is extended to $\mathcal{H}_0^{(0)}$ by zero).

Suppose condition (3.3) is satisfied. Then there exists
\[ T_\pm(\lambda + i\varepsilon) := w-lim \lim_{\varepsilon \to 0} T_\pm(\lambda + i\varepsilon) = \tilde{V}_\pm - G_0^* B(\lambda + i\varepsilon) G_0, \] (1)

where
\[ \tilde{V}_+ = \tilde{G}_0 G_0, \quad \tilde{V}_- = G_0^* \tilde{G}_0, \quad \tilde{G}_0 = G_0. \] (2)

We shall see that (1), (2) give a representation of the form (4.11) for $T_0(\lambda + i\varepsilon)$. The next assertion can be established in analogy to the proof of Theorem 3.6.

Lemma 2. Suppose the operator $G_0$ is weakly $H_0$-smooth, while $G$ is $H_0^{1/2}$-bounded and condition (3.3) holds. Then the operator $G_3$ is weakly $H_0$-smooth.

Proof. According to (3.7)
\[ ||G_3 R_0(z)|| \leq ||G R(z)|| + ||B(z)|| ||G_0 R_0(z)||. \]

From (3.3) and the estimate (1.3) for the pairs $H_0, G_0$ and $H, G$ we now obtain
\[ \varepsilon^{1/2} ||G_3 R_0(\lambda + i\varepsilon)|| \leq C(\lambda), \quad \text{a.e. } \lambda \in \mathbb{R}. \]

This verifies definition (1.3) of weak $H_0$-smoothness of the operator $G_3$. Thus, by Definition 4.2 under the conditions of Lemma 2 to the operator $T_0 T_0^0(\lambda + i\varepsilon) T_0^0$ in the decomposition (2.4.2) there corresponds a kernel $t_\omega(\mu, \nu; \lambda + i\varepsilon)$. According to (4.12)
\[ t_\omega(\mu, \nu; \lambda + i\varepsilon) = Z_0(\mu; G_0) Z_0^*(\nu; G_0) - Z_0(\mu; G_0) B(\lambda + i\varepsilon) Z_0^*(\nu; G_0). \]
A similar expression is valid also for the kernel \( t_-(\mu, \nu; \lambda + i0) \). Here the points \( \lambda, \mu, \nu \) run through some set of full measure in \( \delta_0 \) independently of one another. Hence, for a.e. \( \lambda \in \delta_0 \) the bounded operator \( t_-(\lambda, \lambda; \lambda + i0) \) in \( H_0(\lambda) \) is well defined.

It is now not hard to justify the desired representations (2.8.9)\( _{\pm} \) which in a precise sense are realized in the form of the relations

\[
S(\lambda) = u_+(\lambda) - 2\pi i Z_0(\lambda; G_0) Z_0^*(\lambda; G_0),
\]

\[
S(\lambda) = u_-(\lambda) - 2\pi i Z_0(\lambda; G_0) G_0 B(\lambda + i0) Z_0^*(\lambda; G_0).
\]

We recall that \( u_\pm(\lambda); \delta_0(\lambda) \rightarrow \delta_0(\lambda) \) is the family of operators into which the operator \( Z_\pm(H_0; H_0; T^2) \) commuting with \( H_0 \) goes over in the decomposition (2.4.2).

**Theorem 3.** Suppose on a dense set \( \mathcal{M}_0 \) in \( H_0^g \) both conditions (2.10)\( _{\pm} \) and (2.10)\( _{-} \) hold, the operator \( G_0 \) is weakly \( H_0 \)-smooth, and the operator \( G \) is \( |H|^1/2 \)-bounded and (3.3) is satisfied. Then for the stationary scattering matrix for a.e. \( \lambda \in \delta_0 \) both representations (2.8.9)\( _{\pm} \) or (3)\( _{\pm} \) hold.

**Proof.** We start from the representation (2.8.7), which holds by Theorem 1. In view of Lemma 1.5.1 it suffices to verify relation (2.8.8) for \( f_0, g_0 \in \mathcal{M}_0 \) and a.e. \( \lambda \in R \). We use Lemma 4.7. According to (1), the operator \( T_+T_0F_0 \) consists of two terms. Under conditions (2.10)\( _{\pm} \) it follows from (4.15) that

\[
\lim_{\varepsilon \to 0, \eta \to 0} \left( V_+ (\mu, \eta) f_0, \delta(\mu, \varepsilon) g_0 \right) = (Z_0(\mu; G_0) \tilde{f}_0(\mu), Z_0^*(\mu; \tilde{g}_0(\mu))).
\]

An analogous equality can be obtained with \( V_+ \) replaced by \( V_- \). Moreover, under conditions (2.10)\( _{\pm} \) and (3.3) it follows from (4.16) that

\[
\lim_{\varepsilon \to 0, \eta \to 0} \left( G_0^* B(\lambda + i0) G_0 \delta(\mu, \eta) f_0, \delta(\mu, \varepsilon) g_0 \right) = (B(\lambda + i0) Z_0(\mu; G_0) \tilde{f}_0(\mu), Z_0^*(\mu; \tilde{g}_0(\mu))).
\]

Setting now in these equalities \( \lambda = \mu = \nu, \varepsilon = \eta = \kappa \) and recalling the definition of the kernel \( t_+ \), we obtain the desired relation (2.8.8). \( \square \)

According to Theorem 3.6, under the conditions of Theorem 3 there exist the strong time-dependent W.O., which coincide with the stationary operators. Therefore, relations (2.8.9)\( _{\pm} \) or (3)\( _{\pm} \) may also be considered as representations for the time-dependent scattering matrix. For W.O. \( W_+ \) \( (H_0; H_0; T) \) isometric on \( \mathcal{M}_0^{\delta_0} \) we have \( \nu_\delta(\lambda) = 1(\lambda) \), so that (2.8.9)\( _{\pm} \) reduces to (2.10)\( _{\pm} \).

3. We shall make special consideration of the case \( \mathcal{M} = \mathcal{M}_0 \), \( J = I \), where \( T_+ = T_- \) and hence the representations (3)\( _{\pm} \) coincide, i.e.,

\[
S(\lambda) = I - 2\pi i Z_0(\lambda; G_0) Z_0^*(\lambda; G) - Z_0(\lambda; G_0) G_0 B(\lambda + i0) Z_0^*(\lambda; G_0). \tag{4}
\]

By symmetry of the perturbation we have

\[
Z_0(\lambda; G_0) Z_0^*(\lambda; G) = Z_0(\lambda; G) Z_0^*(\lambda; G_0).
\]

Under minor additional assumptions the representation for \( S(\lambda) \) can be written in terms of the operator \( \tilde{B}(\lambda) = G_0 R(\lambda) G^* \).

which is sometimes more convenient. We shall now suppose that for \( \theta_0 + \theta = \pi \) the operators \( G_0 \) and \( G \) are, respectively, \( |H_0|^{3/2} \)- and \( |H_0|^{3/2} \)-bounded, and therefore the operator (5) is well defined for \( \text{Im} \lambda \neq 0 \).

**Theorem 4.** Suppose the conditions of both Lemmas 6 and 6' are satisfied. Suppose also that for a.e. \( \lambda \) there exist the weak limits \( \tilde{B}(\lambda + i0) \) of the operators \( \tilde{B}(\lambda + i0) \) as \( \varepsilon \to 0 \) on elements \( f_0 \) of a dense set \( \mathcal{M}_0 \) for both signs

\[
\exists_{\varepsilon \to 0} s\text{-lim} G R_0(\lambda \pm i0) f_0, \tag{6}
\]

Then the strong time-dependent W.O. \( W_+ \) \( (H_0; H_0; T) \) also exist, and for the corresponding scattering matrix the representation

\[
S(\lambda) = I - 2\pi i Z_0(\lambda; G) (I - \tilde{B}(\lambda + i0)) Z_0^*(\lambda; G_0) \tag{7}
\]

holds for a.e. \( \lambda \).

The existence and completeness of the W.O. \( W_+ \) \( (H_0; H_0; T) \) follow from Corollary 2.12 and Theorem 3.5. Theorem 1 shows that on \( \mathcal{M} \times \mathcal{M}_0 \) the representation (2.6.7) holds. Let us evaluate the limit of its right-hand side as \( \varepsilon \to 0 \). Since \( G_0 \) and \( G \) are \( H_0 \)-smooth, the operators \( Z_0(\lambda; G_0) \) and \( Z_0^*(\lambda; G) \) are well defined for a.e. \( \lambda \). On the basis of Lemma 4.7 from relations (2.10)\( _{\pm} \) and (6) it now follows that

\[
\lim_{\varepsilon \to 0} (T(\lambda \pm i0) \delta_0(\lambda, \varepsilon) f_0, \delta_0(\lambda, \varepsilon) g_0) = (I - \tilde{B}(\lambda + i0)) Z_0(\lambda; G_0) \tilde{f}_0(\lambda), Z_0^*(\lambda; G) \tilde{g}_0(\lambda)). \tag{7}
\]

For the proof of (7) it remains to use Lemma 1.5.1. \( \square \)

Theorem 4 can be formulated considerably more briefly if it is assumed that in the factorization (1.9.2) the factors \( G_0 \) and \( G \) are connected by the relation

\[
G_0 = \mathcal{V} G, \quad \mathcal{V} = \mathcal{V}^* \in \mathcal{B} \tag{8}
\]

Then (1.9.2) takes the form

\[
V = G^* \mathcal{V} G. \tag{9}
\]

The perturbation (9) is automatically symmetric formally. The assumption (8) makes it possible to formulate all conditions in terms of the operator \( G \) alone. Representations (4) and (7) hereby coincide. Namely, we have
5. THE GENERAL SCHEME IN STATIONARY SCATTERING THEORY

THEOREM 4'. Suppose that relations (8) and (9) are satisfied, where the operator \( G \) is \( [H_0]^{1/2} \)-bounded. Suppose the operator \( G \) is weakly \( H_0 \)-smooth, conditions (6), (3.3) hold, and also on the elements \( f \) of a dense set \( \mathcal{M} \)

\[
\exists \lim_{\varepsilon \to 0} GR(\lambda \pm i\varepsilon)f.
\]

Then WO \( W_0^\varepsilon(H, H_0) \) exist and are complete, and

\[
S(\lambda) = I - 2\pi i Z_0^\varepsilon(\lambda; G) \mathcal{V}[I - B(\lambda + i0)\mathcal{V}]Z_0^\varepsilon(\lambda; G), \text{ a.e. } \lambda. \tag{10}
\]

In conclusion we note that, with the agreement adopted at the end of Part 1 of §4, for the operators (4), (7), and (10) we have \( S(\lambda) = I \) for \( \lambda \in \mathbb{R} \setminus \sigma_0 \). The same is true also under the conditions of Theorem 3, if we set \( u_\pm(\lambda) = I \) for \( \lambda \in \mathbb{R} \setminus \sigma_0 \).

§6. THE DECOMPOSITION THEOREM

Here we show how to construct a spectral representation of the operator \( H^{(a)} \) with the help of the WO. Moreover, we establish the connection between decompositions into direct integrals corresponding to the operators \( H_0^{(a)} \) and \( H^{(a)} \).

1. Let \( \mathcal{U}: \mathcal{H}_0^{(a)} \to \mathcal{H} \) be a bounded operator satisfying the intertwining property

\[
E(X)\mathcal{U} = \mathcal{U} E_0(X) \tag{1}
\]

and such that \( R(\mathcal{U}) \subset \mathcal{H}_0^{(a)} \), \( \mathcal{H}_0^{(a)} \subset N(\mathcal{U}) \). Suppose that the mapping \( \mathcal{F}_0 \) of the space \( \mathcal{H}_0^{(a)} \) onto the direct integral (2.4.2) is fixed; as always, this mapping is extended by zero to \( \mathcal{H}_0^{(a)} \). Denote by \( \mathcal{A} \) the operator of multiplication by \( \lambda \) in \( \mathcal{H}_0^{(a)} \); its spectral projection \( E_\lambda(X) \) is multiplication by the characteristic function of the set \( X \). According to (1) the operator

\[
\mathcal{F} = \mathcal{F}_0\mathcal{A}^*: \mathcal{H} \to \mathcal{H}_0^{(a)} \tag{2}
\]

satisfies the intertwining property

\[
\mathcal{F} E(X) = E_\lambda(X) \mathcal{F}. \tag{3}
\]

Since \( \mathcal{F}^* \mathcal{F}_0 = P_0 \), the operator \( \mathcal{U} \) can be recovered from \( \mathcal{F} \) by the equality

\[
\mathcal{U} = \mathcal{F} \mathcal{F}_0. \tag{4}
\]

If the operators \( \mathcal{U}: \mathcal{H}_0^{(a)} \to \mathcal{H}^{(a)} \) and, hence, \( \mathcal{F}: \mathcal{H}^{(a)} \to \mathcal{H}_0^{(a)} \) are unitary, then \( \mathcal{F} \) realizes the transition to the spectral representation of the operator \( H^{(a)} \), whereby \( \mathcal{H}_0^{(a)} \) is isometric and complete. According to (3), for \( u = h(\lambda) \) the "element" \( \mathcal{F} u \) can be interpreted as a generalized "eigenvector" of the operator \( H \) corresponding to the "eigenvalue" \( \lambda \).

Suppose now that \( \mathcal{U} = \mathcal{U}_\pm \) is a wave operator. We suppose that the conditions of Definition 2.7.2 are satisfied, and hence by Theorem 2.4 the stationary WO \( \mathcal{U}_\pm = \mathcal{U}_\pm(H, H_0; 3) \) is well defined. For \( \mathcal{F}_\pm = \mathcal{F}_0 \mathcal{U}_\pm \) we obtain an explicit expression in terms of the spectral decomposition of the operator \( H_0 \) and the resolvent of the operator \( H \). Let the conditions of Lemma 2.6' be satisfied. Then the representation (2.7.5) for the sesquilinear form of the WO \( \mathcal{U}_\pm \) is valid on the set \( \mathcal{H}_0^{(a)} \times \mathcal{M} \). Considering the expression (2.7.10) for the integrand and relations (1.3.11), (1.4.11), we write (2.7.5) in the form

\[
(\mathcal{U}_\pm f_0, f) = (\mathcal{F}_0 f_0, f) - \int_0^\infty \lim_{\varepsilon \to 0} (V \delta_0(\lambda, \varepsilon) f_0, R(\lambda \pm i\varepsilon) f) d\lambda, \tag{5}
\]

where \( V = G^* G_0 \). Under the conditions of Lemma 2.6' the operator \( Z_0(\lambda; G_0) \) is well defined for a.e. \( \lambda \). Therefore, according to Lemmas 1.6 and 4.1, the integrand in (5) is equal to

\[
(\tilde{f}_0(\lambda), Z_0(\lambda; G_0) GR(\lambda \pm i0) f), \quad \tilde{f}_0 = \mathcal{F}_0 f_0,
\]

where \( GR(\lambda \pm i0) f \) denotes the strong limit of \( GR(\lambda \pm i\varepsilon) f \) as \( \varepsilon \to 0 \), and \( \tilde{f}_0 = \mathcal{F}_0 f_0 \). Substituting this expression into (5) and using (4), we find that

\[
(\tilde{f}_0, \mathcal{F}_0 f)_{\mathcal{H}_0^{(a)}} = (\tilde{f}_0, \mathcal{F}_0^* f)_{\mathcal{H}_0^{(a)}} \int_{\sigma_0} (\tilde{f}_0(\lambda), Z_0(\lambda; G_0) GR(\lambda \pm i0) f) \, d\lambda.
\]

Since \( \tilde{f}_0 \in \mathcal{H}_0^{(a)} \) is arbitrary, from this it follows that

\[
(\mathcal{F}_0 f_0, f) = (\mathcal{F}_0^* f)(\lambda) - Z_0(\lambda; G_0) GR(\lambda \pm i0) f, \quad \text{a.e. } \lambda \in \sigma_0. \tag{6}
\]

We note that under the assumptions made the vector-valued function (6) is well defined for a.e. \( \lambda \in \sigma_0 \) and is square-integrable over \( \sigma_0 \).

If the WO \( \mathcal{U}_\pm \) is isometric and complete the mapping \( \mathcal{F}_\pm: \mathcal{H}^{(a)} \to \mathcal{H}_0^{(a)} \) turns out to be unitary. We present a summary of our considerations. The theorem formulated below differs from Lemma 2.6' and Corollary 2.12, in essence, only by a change of the point of view.

THEOREM 1. Under the conditions of Lemma 2.6' we define on the set \( \mathcal{M} \) the operator \( \mathcal{F}_\pm: \mathcal{H} \to \mathcal{H}_0^{(a)} \) by equality (6). Then the operator \( \mathcal{F}_\pm \) is bounded, \( \mathcal{F}_\pm f = 0 \) for \( f \in \mathcal{H}_0^{(a)} \), the intertwining property (3) is satisfied, and \( \mathcal{F}_\pm \) is connected with the stationary WO \( \mathcal{U}_\pm \) by equalities (2) and (4), where \( \mathcal{U} = \mathcal{U}_\pm \). Under the conditions of Corollary 2.12 the operator \( \mathcal{F}_\pm \) unitarily maps \( \mathcal{H}^{(a)} \) onto \( \mathcal{H}_0^{(a)} \) and hence realizes a spectral representation of the operator \( H^{(a)} \).

2. In §7.1 we shall need to construct in the decomposition (2.4.2) the kernels \( \tilde{a}(\mu, \nu) \) of operators \( \tilde{A} = \mathcal{U}^* \mathcal{A} \mathcal{U} \), where the bounded \( \mathcal{U} \) satisfies the intertwining property (1). Regarding \( \mathcal{A} \), it is assumed that

\[
A = G_0^* G_1 \tag{7}
\]
for a weakly $H$-smooth operator $G_i$ and $\mathcal{A} \in \mathcal{B}(\mathcal{H})$. Then, by verifying, for example, condition (1.3), we find that the operator $G_i \mathcal{A}$ is weakly $H_\delta$-smooth. Therefore, the kernel $\hat{a}(\mu, \nu)$ is well defined by Definition 4.2 on a measurable square of full measure. Here

$$\hat{a}(\mu, \nu) = Z(\mu; G_i, \mathcal{A})Z^*(\nu; G_i),$$

where according to (4.2)

$$Z(\lambda; G_i)\varphi = (\mathcal{F}G_i(\lambda))\varphi, \quad \mathcal{F} = \mathcal{F}\mathcal{A}^*, \quad \varphi \in D.$$  

Our purpose here is to find a connection of the operators (9) with the operators $Z_0$ corresponding to the mapping $\mathcal{F}_0$.

To shorten notation it will sometimes be convenient for us to use the formal notation $\mathcal{Z}_0(\lambda)$, defined by the equality $\mathcal{Z}_0(\lambda)f = (\mathcal{F}_0f)(\lambda)$. The mapping $\mathcal{Z}_0(\lambda)$ is defined on $\mathcal{H}$, where $\mathcal{Z}_0(\lambda)G_i = Z_0(\lambda; G_i)$. The formal mapping $\mathcal{Z}(\lambda), \mathcal{Z}(\lambda)f = (\mathcal{F}f)(\lambda)$ is introduced in a similar way.

We now suppose that $\mathcal{F} = \mathcal{F}_\pm$ is the stationary WO for the pair $H_0, H$ and the identification $\mathcal{I}$, and hence the mapping $\mathcal{F} = \mathcal{F}_\pm$ is given by equality (6). For the corresponding operator $\mathcal{Z}_\pm(\lambda)$ we then have the formal relation

$$\mathcal{Z}_\pm(\lambda) = \mathcal{Z}_0(\lambda)(\mathcal{I}^* - V^* R(\lambda \pm i0)),$$

This equality can be given a precise meaning within the framework of assumptions of the same type as in Theorem 1.

**Theorem 2.** Suppose the conditions of Lemma 2.6 are satisfied, the operators $G$ and $G_i$ are bounded relative to $|H|^\theta$ and $|H_0|^\theta$, where $\theta + \theta_1 \leq 1,$

$$\exists \lim_{t \to 0} GR(\lambda \pm it)G_i^* =: GR(\lambda \pm i0)G_i^*,$$

and, moreover, the operator $G_i \mathcal{J}$ is weakly $H_0$-smooth. Then the operator (9) has the representation

$$Z_\pm(\lambda; G_i) = Z_0(\lambda; G_i \mathcal{J}) - Z_0(\lambda; G_i)(GR(\lambda \pm i0)G_i^*), \quad \text{a.e. } \lambda \in \delta_0.$$  

3. In a representation diagonal for the operator $H^{(a)}_0$ any operator commuting with it (and, in particular, the scattering operator) acts as multiplication by an operator-valued function. An analogous assertion holds for the WO. We shall now assume that there are given unitary mappings $\mathcal{F}_0$ and $\mathcal{F}$ of the spaces $\mathcal{H}_0$ and $\mathcal{H}$ onto the respective direct integrals $\mathcal{S}_0(\lambda)$ and $\mathcal{S}(\lambda)$, where these mappings are extended by zero to $\mathcal{H}_0(\delta)$ and $\mathcal{H}(\delta)$.

**Proposition 3.** Suppose for a bounded operator $\mathcal{U}: \mathcal{H}_0 \to \mathcal{H}$ the intertwining property (1) holds. Then the operator

$$\mathcal{F}_0(\mathcal{U})^*: \mathcal{S}_0(\lambda) \to \mathcal{S}(\lambda).$$

$\S 7$. RELATIVELY COMPACT PERTURBATIONS

acts as multiplication by an operator-valued function $u(\lambda): h_0(\lambda) \to h(\lambda)$.

Moreover,

$$\|\mathcal{U}\| = \ess \sup_{\lambda \in \delta_0} |u(\lambda)|$$

and for a.e. $\lambda \in \delta_0 \setminus \delta$ the mappings $u(\lambda)$ must be zero. If the operator $\mathcal{U}$ unitarily maps $\mathcal{H}_0(\delta)$ onto $\mathcal{H}(\delta)$, then the operators $u(\lambda): h_0(\lambda) \to h(\lambda)$ are unitary for a.e. $\lambda \in \delta_0 \setminus \delta$.

**Proof.** If the operators $H_0^{(a)}$ and $H^{(a)}$ are unitarily equivalent, then it may be assumed that $h_0(\lambda) = h(\lambda)$ for a.e. $\lambda \in \delta_0 \setminus \delta$. The operator (12) then commutes with multiplication by $\lambda$ in the direct integral $\mathcal{S}_0(\lambda) = \mathcal{S}(\lambda)$. Thus (see Part 1 of §1.5) it acts as multiplication by $u(\lambda)$ in $h_0(\lambda) = h(\lambda)$. The general case reduces to that considered, since by Theorem 2.1.6 the restrictions of the operators $H_0$ and $H$ to the subspaces

$$(\mathcal{F}_0 \ominus \mathcal{N}(\mathcal{Z})) \cap \mathcal{H}_0(\delta) \quad \text{and} \quad \mathcal{K}(\mathcal{Z}) \cap \mathcal{H}_0(\delta),$$

respectively, are unitarily equivalent.

$\S 7$. Scattering for relatively compact perturbations

We explained in §4.6 that in the case $\mathcal{H}_0 = \mathcal{H}$, $\mathcal{I} = I$ within the framework of smooth perturbations all the properties of the perturbation required to construct scattering theory can be derived from suitable conditions formulated only with respect to the free Hamiltonian. This method will now be applied in a broader situation. We first present a general assertion (Theorem 1) giving conditions for the existence and completeness of the WO and guaranteeing the validity of the stationary representation for the scattering matrix. Theorem 1 combines the smooth and trace class versions, but, just as other assertions of this chapter, it has semieffective character. The more concrete Theorem 2, which is directly applicable (see below Part 4 of §6.4) under assumptions of trace class type, follows directly from Theorem 1.

1. We suppose that in the factorization (1.9.2) the factors $G_0$ and $G$ are, respectively, $|H_0|^\theta$- and $|H|^\theta$-bounded, where $\theta_0 + \theta_1 = 1, \theta_0 \in [1/2, 1]$. By the way, this condition can be relaxed in the standard way—see Part 5 of §1.9. Suppose for Im $\tau \neq 0$ the operator (1.9.18) exists and is bounded. By Theorem 1.10.3 there then exists a unique operator $H$ satisfying Definition 1.9.2. The representation (1.9.15) holds for its resolvent. Below we shall use the notation of (4.6.2) and (5.5). All these operators are well defined, and, according to the identity (1.9.14),

$$I - B(z) = (I - B_0(z))^{-1}. \quad (1)$$

In this section various conditions are imposed at once for both signs "±" which ensures the existence of both WO $W_\pm$.  

THEOREM 1. Suppose for a.e. $\lambda \in \mathbb{R}$ and $\varepsilon \to 0$ the weak limits of the operators $B^{(0)}(\lambda \pm i\varepsilon)$ exist, the operator $G_0$ is weakly $H_0$-smooth, and for the elements $f_0$ of some set $\mathcal{M}$ dense in $\mathcal{M}$ conditions (2.10) and (5.6) are satisfied. Suppose, in addition, that for a.e. $\lambda \in \mathbb{R}$ and $\varepsilon \to 0$ there exist the limits in norm of the operators $B_0(\lambda \pm i\varepsilon)$ whereby

$$\exists (I - B_0(\lambda \pm i\varepsilon))^{-1} \in \mathcal{B}, \quad \text{a.e. } \lambda \in \mathbb{R}. \quad (2.10)$$

Then the WO $W_{\pm}(H, H_0)$ exist and are complete, and for the corresponding scattering matrix the representation

$$S(\lambda; H, H_0) = I - 2\pi iZ_0(\lambda; G)(I - B_0(\lambda + i\varepsilon))^{-1}Z_0^*(\lambda; G_0) \quad (3.10)$$
holds for a.e. $\lambda \in \delta_0$.

**Proof.** We start from Theorem 5.4. The conditions in it concerning the operator $H_0$ are also assumed now. Therefore, it is only necessary to verify the conditions formulated with respect to $H$. By (1), from condition (2) it follows that the limits (in norm) of the operators $\tilde{B}(\lambda \pm i\varepsilon)$ exist. Similarly, by equality (4.6.3) the weak limits of the operators $\tilde{B}(\lambda \pm i\varepsilon)$ exist, so that the operator $G$ is clearly weakly $H$-smooth. Further, according to the resolvent identity,

$$GR(z) f_0 = (I - B_0^*(z))^{-1} GR_0(z) f_0. \quad (4.6)$$

By hypothesis (5.6) for $f_0 \in \mathcal{M}$ the element $GR_0(\lambda \pm i\varepsilon)f_0$ has a strong limit as $\varepsilon \to 0$, and the inverse operator converges in norm. From this it follows that the vector-valued function (4) has a strong limit. Thus, all the conditions of Theorem 5.4 are satisfied. Therefore, the WO $W_{\pm}(H, H_0)$ exist and are complete, while for the scattering matrix the representation (5.7) holds. According to (1), the latter can be rewritten in the form (3).

Under the additional condition (5.8) this theorem can be formulated more briefly.

**Theorem 1'.** Suppose for a.e. $\lambda \in \mathbb{R}$ and $\varepsilon \to 0$ the limits in norm $B^{(0)}(\lambda \pm i\varepsilon)$ of the operators $B^{(0)}(\lambda \pm i\varepsilon)$ exist and

$$\exists (I - \mathcal{Y}B^{(0)}(\lambda \pm i\varepsilon))^{-1} \in \mathcal{B}, \quad \text{a.e. } \lambda \in \mathbb{R}, \quad (5.8)$$

and also condition (5.6) is satisfied. Then the WO $W_{\pm}(H, H_0)$ exist and are complete and

$$S(\lambda; H, H_0) = I - 2\pi iZ_0(\lambda; G)(I + \mathcal{Y}B^{(0)}(\lambda + i\varepsilon))^{-1}\mathcal{Y}Z_0^*(\lambda; G) \quad \text{a.e. } \lambda \in \delta_0. \quad (6.8)$$

We note that the operators contained in (6) are connected by the relation

$$Z_0^*(\lambda; G)Z_0(\lambda; G) = B^{(0)}(\lambda + i\varepsilon) - B^{(0)}(\lambda - i\varepsilon), \quad (7.8)$$

which follows directly from (4.4). Equality (6) is well defined on a set of full measure, where the operators $Z_0(\lambda; G)$ are defined and the operators

By the equality $(B^{(0)}(\lambda \pm i\varepsilon))^* = B^{(0)}(\lambda \pm i\varepsilon)$, together with (5) there also exist the operators

$$(I + B^{(0)}(\lambda \pm i\varepsilon))^* \quad \text{a.e. } \lambda \in \mathbb{R}. \quad (8.8)$$

Therefore, (6) can further be rewritten in the form

$$S(\lambda; H, H_0) = I - 2\pi iZ_0(\lambda; G)\mathcal{Y}(I + B^{(0)}(\lambda \pm i\varepsilon))^{-1}Z_0^*(\lambda; G) \quad \text{a.e. } \lambda \in \delta_0. \quad (9.8)$$

2. Practical application of Theorem 1 is, of course, impeded by the difficulty of verifying condition (2). One of the effective conditions for its validity is given by Theorem 4.6.2 where it is required that the operator-valued function $B_0(z)$ be continuous up to the real axis and $B^{(0)}_0(z) \in \mathcal{S}_\infty$ for some positive integer $l$. We recall that condition (2) was derived from this by means of Theorem 1.8.3 on the inversion of an operator-valued function. We now present another effective version based on Theorem 1.8.5.

We note that under the conditions of the preceding assertion the existence and boundedness of the operator $(1.9.18)$ follow from Lemma 1.10.5 and Corollary 1.10.6. Therefore, as in Part 1, the selfadjoint operator $H$ is well defined.

**Theorem 2.** Suppose in the conditions of Theorem 1 assumption (2) is replaced by the following. The operator-valued function $B_0(z)$ has angular limit values in norm as $z \to \lambda \pm i\varepsilon$ for a.e. $\lambda$. Moreover, $B_0(z) \in \mathcal{S}_p$ for some positive integers $l$ and $p$ and $B^{(0)}_0(z)$ has angular limit values as $z \to \lambda \pm i\varepsilon$ in $\mathcal{S}_p$ for a.e. $\lambda$. Then all the conclusions of Theorem 1 hold.

**Proof.** Under the assumptions made regarding the existence of the angular limit values of $B_0(z)$ and $B^{(0)}_0(z)$ the condition (2) follows directly from Theorem 1.8.5.

Under the assumption (5.8) this theorem can also be formulated more briefly.

**Theorem 2'.** Suppose the operator-valued function $B^{(0)}(z)$ has angular values in norm as $z \to \lambda \pm i\varepsilon$ for a.e. $\lambda$. Moreover, suppose that $(B^{(0)}(z))^\dagger \in \mathcal{S}_p$ for some positive integers $l$ and $p$ and $(B^{(0)}(z))^\dagger$ has angular limit values as $z \to \lambda \pm i\varepsilon$ in $\mathcal{S}_p$ for a.e. $\lambda \in \mathbb{R}$. Let additionally condition (5.6) be satisfied. Then all the conclusions of Theorem 1' hold.

§8. A local version of the stationary scheme

Here we briefly discuss an extension of the results of the preceding sections to local WO.

1. Let $\Lambda$ be an arbitrary Borel set on the spectral axis. The stationary WO $\mathcal{W}_{\pm}(H, H_0; \lambda) =: \mathcal{W}_{\pm} (\lambda, \Lambda)$ is the operator defined by the sesquilinear
we have $H\tilde{J} - 3H_0 = \tilde{G}^*\tilde{G}_0$. Suppose condition (2.10)$_\pm$ for $\tilde{G}_0$ is satisfied for a.e. $\lambda \in \mathbb{R}$, while the operator $\tilde{G}$ is weakly $H$-smooth (on the entire axis). Then, according to (2), the form $(\mathcal{V}_+^\pm (\mathfrak{J}, \Lambda)f_0, \mathcal{V}_+^\pm (\mathfrak{J}, \Lambda)g_0)$ can be represented by the right-hand side of (2.7.11)$_\pm$, where the role of $\mathfrak{J}$ is played by $\tilde{J}$. By Lemma 2.1, applied to the pair $H_0, \tilde{H}_0$ and the identification $\mathfrak{J}^*E(X)\mathfrak{J}$, this representation can be rewritten in the form

$$(\mathcal{V}_+^\pm (\mathfrak{J}, \Lambda)f_0, \mathcal{V}_+^\pm (\mathfrak{J}, \Lambda)g_0) = \int_{\lambda \in \mathbb{R}} \lim_{E\to 0} \pi^{-1} E(\mathfrak{J}) \mathcal{R}_0(\lambda \pm iE)f_0, \mathcal{R}_0(\lambda \pm iE) g_0 d\lambda.$$ 

For $X_0 = \Lambda, \mathfrak{J} = \mathbb{R}$ this coincides with the expression on the left obtained by a consistent taking into account of locality (see Part 1).

In some cases the conditions regarding the operators $\tilde{G}_0$ and $\tilde{G}$ can be derived from analogous assumptions formulated in terms of $G_0$ and $G$ on the set $\Lambda$. For example, this is possible for $X_0 = \mathfrak{J} = \mathbb{R}$ and certain conditions concerning the structure of the set $\Lambda$ (for example, if $\Lambda$ is a finite union of intervals).
CHAPTER 6

Scattering for Perturbations of Trace Class Type

In abstract scattering theory consideration of trace class perturbations occupies a central place. This is connected with the unitary invariance of the conditions for the existence and completeness of the WO formulated here.

In the construction of scattering theory for trace class perturbations we use two approaches in parallel. The first of them, the stationary approach, is based on the verification in §1 of conditions of the preceding chapter. Trace class scattering theory thus fits into the general stationary scheme of Chapter 5. Discussion of the basic result of the trace class theory—the Kato-Rosenblum theorem—constitutes §2. Here a generalization of this theorem to a pair of spaces is presented. The second approach, expounded in §3, gives a direct proof of the existence of the limits in the time-dependent definition of the WO. This proof is comparatively brief, but can hardly be considered transparent.

The next sections §§4, 5 are devoted to various generalizations of the Kato-Rosenblum theorem. These generalizations are needed, in particular, for application of trace class methods to the theory of differential operators. As an example, in §6 we consider the perturbation of the operator of multiplication by an operator of Fourier type. In §7 we give additional information valid for a one-dimensional perturbation. Finally, in §8 we describe the machinery of double Stieltjes integral operators which is convenient, for example, for the investigation of the difference of functions of two selfadjoint operators.

Within the framework of trace class assumptions the stationary approach makes it possible to justify the formula representations of §§2.7 and 2.8. Detailed consideration of them and also the study of properties of the scattering matrix are, however, put off to Chapter 7.

§1. Weak smoothness of Hilbert-Schmidt operators

In this section we show that for any selfadjoint operator $H$ and any Hilbert-Schmidt operator the operator-valued function $GE(\lambda)G^*$ is differentiable in the trace norm for a.e. $\lambda \in \mathbb{R}$. Of course, from this it follows that in the weak sense $G$ is smooth relative to $H$. Moreover, as it becomes clear, in the Hilbert-Schmidt class the product $GR(\lambda \pm i\varepsilon)G^*$ has limit values as
$\varepsilon \to 0$ for a.e. $\lambda \in \mathbb{R}$. The trace class theory can thus be incorporated into the stationary scheme of the preceding chapter.

1. Here we consider some special properties of the ideals $\mathcal{G}_p$, $1 \leq p < \infty$, consisting of compact operators with a finite quantity (1.6.10).

**Lemma 1.** Any operator $A \in \mathcal{G}_p$ can be represented in the form $A = TB$ (or $A = BT$) where $B \in \mathcal{G}_p$ and $T \in \mathcal{G}_\infty$.

**Proof.** We start from the Schmidt decomposition (1.6.3) of an operator $A \in \mathcal{G}_p$. The sequence of its s-numbers $s_n \in l_p$ can be written (see, for example, the textbook [24]) in the form $s_n = \sigma_n \tau_n$ where, as before, $\sigma_n \in l_p$ and $\tau_n \to 0$ as $n \to \infty$. It is clear that for the operators

$$B = \sum_n \sigma_n (\cdot, \varphi_n) \varphi_n, \quad T = \sum_n \tau_n (\cdot, \varphi_n) \psi_n$$

we have $TB = A$. □

**Lemma 2.** If $A \in \mathcal{G}_p$ and for any $T \in \mathcal{G}_\infty$ the product $AT \in \mathcal{G}_p$ (or $TA \in \mathcal{G}_p$), where $p < \infty$, then necessarily $A \in \mathcal{G}_p$.

**Proof.** If it is a priori that $A \in \mathcal{G}_\infty$, then, as in Lemma 1, it is possible to use the representation (1.6.3) for $A$ and take $T$ in the form $T = \sum_n \tau_n (\cdot, \varphi_n) \varphi_n$. The singular numbers of $AT$ are equal to $s_n \tau_n$. It remains to note (again see [24]) that $s_n \in l_p$ if $s_n \tau_n \in l_p$ for any sequence $\tau_n \to 0$.

To prove that $A \in \mathcal{G}_\infty$, we establish the compactness of its modulus $|A|$. In view of the equality $s_n(\mathcal{G}_\infty) = \sigma_n |A|$, for any $T \in \mathcal{G}_\infty$, if $|A| \notin \mathcal{G}_\infty$, then for some $\gamma > 0$ the subspace $\mathcal{H}_\gamma = E_{\gamma}(\mathcal{G}_\infty)$ is infinite-dimensional. Therefore, there exists a compact operator $T: \mathcal{H}_\gamma \to \mathcal{G}_\infty$ not lying in the class $\mathcal{G}_\infty$. By hypothesis, for $A^* = E_{\sigma}(\mathcal{G}_\infty)$ the product $A^*T: \mathcal{H}_\gamma \to \mathcal{G}_\infty$ belongs to $\mathcal{G}_\infty$. Therefore, from the invertibility of $A^*$ in $\mathcal{H}_\gamma$ it follows that $T \in \mathcal{G}_\infty$. Thus, the assumption $|A| \notin \mathcal{G}_\infty$ has led to a contradiction. □

We further present two elementary assertions regarding convergence in the classes $\mathcal{G}_p$.

**Lemma 3.** Suppose $A \in \mathcal{G}_p$ and $T_n \to 0$ as $n \to \infty$. Then $\|T_n A\| \to 0$ as $n \to \infty$.

**Proof.** For any $\varepsilon > 0$ there exists a finite-dimensional operator $A_\varepsilon$ such that $\|A - A_\varepsilon\| < \varepsilon$. Since $\|T_n\| \leq C$, from this it follows that

$$\|T_n A\| \leq C \|T_n A_\varepsilon\|.$$ 

It thus suffices to consider finite-dimensional or even one-dimensional operators $A$. For $A = (\cdot, f)$

$$\|T_n A\| = \|f\| \cdot \|T_n g\| \to 0,$$

since, by hypothesis, $T_n \to 0$ as $n \to \infty$. □

In the proof of the next assertion the concept of a transformer is useful. These are operators which themselves act in spaces of operators.

**Lemma 4.** Suppose $A_n \in \mathcal{G}_p$, $\|A_n\| \leq C < \infty$ and $A_n \to A$ as $n \to \infty$. Then $A \in \mathcal{G}_p$ and for any compact operators $T$ and $Y$

$$\lim_{n \to \infty} \|T(A_n - A)Y\| = 0.$$

**Proof.** Consider the transformers $\Gamma_n$ acting from the space $\mathcal{G}_\infty$ to the space $\mathcal{G}_p$ and defined by the equality $\Gamma_n Y = T A_n Y$. Set also $\Gamma Y = T A Y$. The desired assertion is equivalent to the strong convergence of $\Gamma_n$ to the transformer $\Gamma$. According to the Banach-Steinhaus theorem it suffices to establish the uniform boundedness of the norms of the transformers $\Gamma_n: \mathcal{G}_\infty \to \mathcal{G}_p$ and the strong convergence of $\Gamma_n$ on the set of finite-dimensional operators $Y$ dense in $\mathcal{G}_\infty$. The first of these assertions follows from the estimate

$$\|\Gamma_n Y\| = \|T A_n Y\| \leq \|T\| \|A_n\| \|Y\|$$

and the uniform boundedness of $\|A_n\|$. In the proof of the second assertion the operator $Y$ may be assumed to be one-dimensional, i.e., $Y = (\cdot, f)$. In this case $\Gamma_n Y = (\cdot, f) T A_n g$. Since $A_n g \to A g$ and $T \in \mathcal{G}_\infty$, it follows that $T A_n g \to T A g$ as $n \to \infty$. Thus, as $n \to \infty$

$$\|(\Gamma_n - \Gamma) Y\| = \|f\| \cdot \|T(A_n - A)g\| \to 0,$$

i.e., $\Gamma_n \to \Gamma$. Finally, the inclusion $A \in \mathcal{G}_p$ follows from $T A Y \in \mathcal{G}_p$ on the basis of Lemma 2. □

2. We shall establish the differentiability of the spectral measure $E(\cdot)$ of the selfadjoint operator $E(\cdot)$, bordered by any Hilbert-Schmidt operators. We recall that by Lebesgue's theorem the monotone function $(E(\lambda), f)$ for any $f \in \mathcal{H}$ is differentiable for a.e. $\lambda \in \mathbb{R}$. At the same time $\|E(\lambda + \varepsilon) - E(\lambda - \varepsilon)\| = 1$ for $\lambda \in \sigma(H)$ and any $\varepsilon > 0$. Therefore the spectral measure cannot have a weak (operator) derivative at any point of the spectrum. The situation changes if $E(\cdot)$ is bordered by Hilbert-Schmidt operators.

**Theorem 5.** Suppose $H$ is a selfadjoint operator in a Hilbert space $\mathcal{H}$ and $G: \mathcal{H} \to \mathcal{G}$ is a Hilbert-Schmidt operator. Then for a.e. $\lambda \in \mathbb{R}$ the operator-valued function $GE(\lambda)G^*$ is differentiable in the trace norm, the operator-valued function $G \delta(\lambda, \varepsilon)G^*$ has a limit in $\mathcal{G}_\infty$ as $\varepsilon \to 0$, and

$$\lim_{\varepsilon \to 0} G \delta(\lambda, \varepsilon)G^* = \frac{dGE(\lambda)G^*}{d\lambda}.$$ (1)

**Proof.** On the basis of Lemma 1 we represent the operator $G$ in the form $T G$, where $G_1 \in \mathcal{G}_2$ and $T \in \mathcal{G}_\infty$. By Lemma 4 in considering $GE(\lambda)G^*$ it suffices to establish weak differentiability of the operator-valued function $G_1 E(\lambda)G_1^*$ and the estimate

$$\|G_1 E((\lambda - \varepsilon, \lambda + \varepsilon))G_1^*\|_1 \leq C(\varepsilon)\varepsilon$$ (2)
for a.e. \( \lambda \in \mathbb{R} \). Suppose \( D \) is a dense set in \( \mathcal{G} \) consisting of finite linear combinations of elements of some basis in \( \mathcal{G} \). Then (cf., for example, the proof of Lemma 5.1.2) on some set of full measure the functions \( E(\lambda)G_{i}^{*}f, G_{i}^{*}g \) are differentiable if \( f, g \in D \). Therefore, if (2) is satisfied the operator-valued function \( G_{i}E(\lambda)G_{i}^{*} \) is weakly differentiable.

Since \( G_{i}E(\cdot)G_{i}^{*} \geq 0 \), the estimate (2) is equivalent to

\[
\text{Tr} G_{i} E((\lambda - \varepsilon, \lambda + \varepsilon)) G_{i}^{*} \leq C(\lambda)\varepsilon.
\]

The last inequality is obviously valid at all points \( \lambda \) at which the monotonically increasing function \( \text{Tr} G_{i} E(\lambda)G_{i}^{*} \) is differentiable. By Lebesgue's theorem such \( \lambda \) form a set of full measure.

The operator-valued function \( G\delta(\cdot, \cdot)G^{*} \) can be considered in an entirely similar way with the help of Theorems 1.2.5 or 1.2.7. Namely, according to either of them the limits of \( (\delta(\lambda, \varepsilon)G_{i}^{*}f, G_{i}^{*}g) \) for \( f, g \in D \) and \( \varepsilon \to 0 \) exist on a common set of full measure. The role of (2) is played by the estimate

\[
\|G_{i}\delta(\lambda, \varepsilon)G_{i}^{*}\|_{1} \leq C(\lambda), \tag{3}
\]

which is equivalent by the inequality \( \delta(\lambda, \varepsilon) \geq 0 \) to the analogous estimate for the scalar function

\[
\text{Tr} G_{i} \delta(\lambda, \varepsilon)G_{i}^{*} = \pi^{-1} \varepsilon \int_{-\infty}^{\infty} (\mu - \lambda)^{2} + \varepsilon^{2} \int_{-\infty}^{\infty} \text{Tr} G_{i} E(\mu)G_{i}^{*}.
\]

Since the function \( \text{Tr} G_{i} E(\mu)G_{i}^{*} \) is differentiable for a.e. \( \mu \), on the basis of Theorems 1.2.5 or 1.2.7 the left-hand side here has a limit and hence is bounded with respect to \( \varepsilon \) for a.e. \( \lambda \). From this we obtain the estimate (3) and hence also the existence of a limit in the norm of \( \mathcal{G}_{i} \) of the left-hand side of (1). Equality (1) itself is, of course, a corollary of relation (1.4.11). \( \square \)

**Corollary 6.** For arbitrary operators \( G_{j} \in \mathcal{G}_{j}, j = 1, 2 \), the operator-valued function \( G_{i}E(\lambda)G_{i}^{*} \) is differentiable in the trace norm for a.e. \( \lambda \in \mathbb{R} \). In the same norm for a.e. \( \lambda \in \mathbb{R} \) there also exists the limit of \( G_{i}\delta(\lambda, \varepsilon)G_{i}^{*} \) as \( \varepsilon \to 0 \), and an equality of the form (1) is preserved.

**Proof.** It is only necessary to apply the "polarization" identity (cf. (1.3.1))

\[
4G_{i}E\delta G_{i}^{*} = (G_{i} + G_{i})E(G_{i} + G_{i})^{*} - (G_{i} - G_{i})E(G_{i} - G_{i})^{*} + i(G_{i} - G_{i})E(G_{i} - G_{i})^{*} + i(G_{i} + G_{i})E(G_{i} + G_{i})^{*}
\]

for \( E(\lambda) \) and an analogous relation for \( \delta(\lambda, \varepsilon) \). \( \square \)

**Corollary 7.** For \( G \in \mathcal{G}_{i} \) and any \( f \in \mathcal{H} \) for a.e. \( \lambda \in \mathbb{R} \) there exist the strong limits on both sides of (5.1.9) and this relation itself is satisfied.

**Proof.** It suffices to use Corollary 6 for \( G_{1} = G \) and any operator \( G_{2} \in \mathcal{G}_{2} \) for which \( G_{2}f = f \). \( \square \)

According to Theorem 5, any Hilbert-Schmidt operator is weakly smooth (in the sense of Definition 5.1.1) relative to any selfadjoint operator \( H \).

Moreover, as compared with the general results of §5.1, for \( G \in \mathcal{G}_{2} \) the limits (5.1.8) and (5.1.9) exist in a stronger sense.

3. For applications in scattering theory we need to establish that each of the two terms \( GR(\lambda \pm i\varepsilon)G^{*} \) composing the left-hand side of (1) has separately a limit as \( \varepsilon \to 0 \). The proof of this fact uses the same considerations as in the proof of Theorem 5, but is considerably more complicated. We shall here need the elementary

**Lemma 8.** Suppose the function \( f(z) \) is holomorphic in the upper half-plane and \( |f(z)| \geq 1 \) for \( \text{Im} z > 0 \). Then \( f(z) \) has limit values \( f(\lambda \pm i0) \) for a.e. \( \lambda \in \mathbb{R} \).

**Proof.** We apply Theorems 1.2.2 and 1.2.3 (for \( p = \infty \)) to the holomorphic and (upper) bounded function \( g(z) = f(z)^{-1} \). By the first of them the limit values \( g(\lambda \pm i0) \) exist for a.e. \( \lambda \). By the second these limit values are nonzero on a set of full measure. On this set the limits \( f(\lambda \pm i0) \) also exist. \( \square \)

**Theorem 9.** Suppose \( H \) is a selfadjoint operator in a Hilbert space \( \mathcal{H} \) and \( G: \mathcal{H} \to \mathcal{G} \) is a Hilbert-Schmidt operator. Then for a.e. \( \lambda \in \mathbb{R} \) the operator-valued function \( GR(\lambda \pm i\varepsilon)G^{*} \) has a limit as \( \varepsilon \to 0 \) in the Hilbert-Schmidt norm.

**Proof.** Let again \( G = TG_{1} \), where \( G_{1} \in \mathcal{G}_{2} \) and \( T \in \mathcal{G}_{\infty} \). By Lemma 4 it suffices to establish for a.e. \( \lambda \) the weak convergence as \( \varepsilon \to 0 \) of the operator-valued function

\[
B(\lambda \pm i\varepsilon) = G_{i}R(\lambda \pm i\varepsilon)G_{i}^{*}
\]

and the estimate

\[
\|B(\lambda \pm i\varepsilon)\|_{2} \leq C(\lambda). \tag{4}
\]

We again denote by \( D \) a dense set in \( \mathcal{G} \) of linear combinations of some basis. According to Theorem 1.2.5, on a common set of full measure for \( f, g \in D \) there exist the limits of \( (R(\lambda \pm i\varepsilon)G_{i}^{*}f, G_{i}^{*}g) \). Therefore, the weak convergence of \( B(\lambda \pm i\varepsilon) \) follows from the estimate (4).

It suffices to carry out the proof of (4) for the upper sign. According to (1.7.9), we have the equality

\[
\|B\|_{2}^{2} = \text{Tr}(B^{*}B) \leq \text{Det}(I + B^{*}B). \tag{5}
\]

We now use relation (1.7.13) and the fact that

\[
1 \leq \text{Det}(I + A_{1}) \leq \text{Det}(I + A_{2})
\]

for \( 0 \leq A_{1} \leq A_{2} \). By means of the condition

\[
iB^{*} - iB = 2\pi iG_{i}RR^{*}G_{i}^{*} \geq 0
\]

for a.e. \( \lambda \in \mathbb{R} \). Moreover, as compared with the general results of §5.1, for \( G \in \mathcal{G}_{2} \) the limits (5.1.8) and (5.1.9) exist in a stronger sense.
we then find that

$$1 \leq \det(I + B^*B) \leq \det((I + iB^*)(I - iB)) = |\det(I - iB)|^2.$$  

(6)

From (5) and (6) it follows that

$$\|B(z)\|_2 \leq |\det(I - iB(z))|.$$  

(7)

We now consider the function $f(z) = \det(I - iB(z))$ holomorphic in the upper half-plane. According to (6), $|f(z)| \geq 1$, so that by Lemma 8 for a.e. $\lambda$ there exist finite limit values $f(\lambda \pm i0)$. On this set of full measure the estimate (4) follows from (7).

We note that in the lower half-plane the role of (7) is played by the inequality

$$\|B(z)\|_2 \leq |\det(I + iB(z))|.$$

The next two assertions are obtained in full analogy to the derivation of Corollaries 6 and 7 from Theorem 5.

**Corollary 10.** For arbitrary $G_j \in \Theta_1$, $j = 1, 2$, the operator-valued function $G_j R(\lambda \pm i\varepsilon) G^*_j$ has a limit in $\Theta_2$ as $\varepsilon \to 0$ for a.e. $\lambda \in \mathbb{R}$.

**Corollary 11.** For $G \in \Theta_2$ and any $f \in \mathcal{H}$ the vector-valued function $GR(\lambda \pm i\varepsilon)f$ has a strong limit in $\Theta$ as $\varepsilon \to 0$ for a.e. $\lambda \in \mathbb{R}$.

**Remark 12.** In Theorem 9 and the corollaries to it there exist the angular (not only radial) limit values.

4. The proof of Theorem 5 is, of course, considerably simpler than the proof of Theorem 9. In connection with this, we note that Theorem 9 can be derived from Theorem 5 if we use the following assertion regarding Cauchy integrals of vector-valued functions.

**Theorem 13.** Let $\mathcal{H}$ be a Hilbert space, $f \in L_1(\mathbb{R}, \mathcal{H})$, and

$$F(z) = \int_{-\infty}^{\infty} f(\mu)(\mu - z)^{-1} d\mu.$$

Then for a.e. $\lambda \in \mathbb{R}$ the function $F(\lambda \pm i\varepsilon)$ has a limit in $\mathcal{H}$ as $\varepsilon \to 0$.

A proof of this assertion can be found in the work [87] of Asano.

To derive Theorem 9 we apply Theorem 13 to the vector-valued function $f(\lambda) = dG \varepsilon(\lambda) G^* d\lambda$ which, according to Theorem 5, takes values in the Hilbert space $\Theta_2 = \Theta_2(\Theta)$. Here

$$\|f\|_{L_1(\mathbb{R}, \Theta_2)} = \int_{-\infty}^{\infty} \|dG \varepsilon(\lambda) G^*\|_{\Theta_2} d\lambda$$

$$\leq \int_{-\infty}^{\infty} \frac{d}{d\lambda} \text{Tr}G \varepsilon(\lambda) G^* d\lambda$$

$$\leq \|G\|_2^2 < \infty.$$  

Therefore, by Theorem 13 the function $F(\lambda \pm i\varepsilon) = GR(\lambda \pm i\varepsilon) G^*$ has a limit in $\Theta_2$ as $\varepsilon \to 0$ for a.e. $\lambda \in \mathbb{R}$.

Theorem 13 does not generalize to the case of Banach spaces $\mathcal{H}$, so that it cannot be applied for $\varepsilon = \mathfrak{C}_p$ if $p \in (1, 2)$. Actually, as shown by S. N. Naboko [70], the limit in $\Theta_1$ of the operator $GR(\lambda \pm i\varepsilon) G^*$ may not exist on any set of positive measure. In addition he showed that the operator $GR(\lambda \pm i\varepsilon) G^*$ has a limit in $\Theta_p$ for a.e. $\lambda$ for any $p > 1$.

**§2. The Kato-Rosenblum theorem. “Negative” results**

Suppose $H_0$ and $H$ are selfadjoint operators in Hilbert spaces $\mathcal{H}_0$ and $\mathcal{H}$ and $\mathcal{H}_0 \to \mathcal{H}$ is a bounded identification. We recall that the precise definition of the perturbation $V = H^3 - 3H_0$ was given in Part 1 of §1.9.

1. The original result of trace class scattering theory is the Kato-Rosenblum theorem pertaining to the case $\mathcal{H}_0 = \mathcal{H}$, $\mathcal{H}$.

**Theorem 1.** Suppose $H_0$ and $H$ are selfadjoint operators in a Hilbert space $\mathcal{H}$ and their difference is of trace class $H - H_0 \in \Theta_1$. Then the WO $W_\pm(H, H_0)$ exist.

We put aside the proof of this result, since it is a special case of Theorem 3.

**Corollary 2.** For $H - H_0 \in \Theta_1$ the WO $W_\pm(H_0, H)$ also exist, and all four WO $W_\pm(H, H_0)$ and $W_\pm(H_0, H)$ are complete. In particular, the operators $H_0^{(a)}$ and $H^{(a)}$ are unitarily equivalent.

**Proof.** The WO $W_\pm(H_0, H)$ exist, since the operators $H_0$ and $H$ enter the conditions of Theorem 1 in a symmetric manner. Completeness of the WO now follows from Corollary 2.3.9.

In addition to the simplicity and elegance of the formulation, merits of the Kato-Rosenblum theorem are the unitary invariance and the practical verifiability of the condition $V = H - H_0 \in \Theta_1$. Moreover, in terms of the classes $\Theta_1$ this condition cannot be relaxed. In this regard see Part 2.

Theorem 1 admits a direct extension to operators acting in different Hilbert spaces. The identification may hereby be any bounded operator. The following assertion was obtained by Pearson.

**Theorem 3.** Suppose that $H_0$ and $H$ are selfadjoint operators in $\mathcal{H}_0$ and $\mathcal{H}$, $\mathcal{H}_0 \to \mathcal{H}$, and $V \in \Theta_1$. Then the WO $W_\pm(H, H_0; \mathcal{H})$ exist.

**Corollary 4.** For $H^3 - 3H_0 \in \Theta_1$ the WO $W_\pm(H_0, H; \mathcal{H})$ also exist, and the WO $W_\pm(H, H_0; \mathcal{H})$ and $W_\pm(H_0, H; \mathcal{H})$ are, respectively, $\mathcal{H}$-complete (see Definition 3.2.3) and $\mathcal{H}$-complete.

**Proof.** Since $H^3 - 3H_0 \in \Theta_1$, the WO $W_\pm(H_0, H; \mathcal{H})$ exist by Theorem 3. Now the $\mathcal{H}$- and $\mathcal{H}$-completeness of the WO follow from Theorem 3.2.4.
It is clear that for \( \mathcal{H}_0 = \mathcal{H} \), \( J = I \) Theorem 3 and Corollary 4 reduce to Theorem 1 and Corollary 2, respectively. Since in Theorem 3 there are no assumptions regarding the identification \( J \) (aside from its boundedness), it is applicable also in those cases where the WO \( W_{\pm}(H, H_0 ; J) \) are clearly not isometric and (or) not complete.

With the help of Theorem 1.9 Theorem 3 can be derived without difficulty from the general results of Chapter 5. We present the corresponding argument.

**Proof of Theorem 3.** We write the operator \( V \in \mathfrak{S}_p \) in the form \( V = G^* G \), where \( G_0 \in \mathfrak{S}_1 \) and \( G \in \mathfrak{S}_2 \). According to Corollary 1.11, the condition (5.2.10) is now satisfied on all elements \( f_0 \in \mathcal{M}_0 =: \mathcal{M} \). By Theorem 1.9 the operator-valued function \( B(z) \) in (5.3.3) has a limit in the Hilbert-Schmidt norm. Therefore, condition (5.3.3) is satisfied even in a stronger sense than needed. The existence of the WO \( W_{\pm}(H, H_0 ; J) \) now follows from Theorem 5.3.6. \( \square \)

The proof of Theorem 3 presented may be considered to belong to the stationary approach. Indeed, time-dependent considerations were immediately reduced to the elementary Lemma 5.3.1, and the conditions of it were expressed in terms of resolvents. The central feature is the proof of equality (2.7.16) obtained by purely stationary means.

The results of §5.3 show also that under the conditions of Theorem 3 the invariance principle holds. Namely, by Theorem 5.3.11 we have

**Theorem 5.** Suppose \( H^2 - 2H_0 \in \mathfrak{S}_1 \) and the function \( \varphi \) is admissible (see Definition 2.6.2) for both operators \( H_0 \) and \( H \). Then the WO \( W_{\pm}(\varphi(H), \varphi(H_0) ; J) \) exist and equalities (2.6.11) hold.

2. In the remainder of the section we shall discuss from various points of view the result of Theorem 1 on invariance of the absolutely continuous component under trace class perturbations. The results presented here do not properly belong to scattering theory, so that the proofs are omitted. Without further stipulation we suppose that \( \mathcal{H}_0 = \mathcal{H} \), \( J = I \). The next assertion demonstrates the sharpness of the condition \( V \in \mathfrak{S}_p \) of Theorem 1.

**Theorem 6.** Let \( H_0 \) be an arbitrary selfadjoint operator. For any \( p > 1 \) and \( \varepsilon > 0 \) there exists a selfadjoint operator \( V = V_{p, \varepsilon} \) such that \( V \in \mathfrak{S}_p \), \( \| V \|_p < \varepsilon \), and the operator \( H = H_0 + V \) has purely point spectrum.

Theorem 6 is due to H. Weyl, von Neumann, and Kuroda. A proof of it can be found in the monograph [11]. We note that in Theorem 6 the class \( \mathfrak{S}_p \), \( p > 1 \), can be replaced by any symmetrically normed ideal (see Part 5 of §1.6) not coinciding elementwise with \( \mathfrak{S}_1 \).

Under the conditions of Theorem 6 the operator \( H \) has no absolutely continuous part. At the same time it is admitted that the operator \( H_0 \) be absolutely continuous. Thus, for any \( p > 1 \) under perturbations of the class \( \mathfrak{S}_p \) the absolutely continuous part of the spectrum may vanish. Moreover, Theorem 6 makes it possible to render the norm of the perturbation in \( \mathfrak{S}_p \) arbitrarily small. Of course, it is clear that in this circumstance the WO \( W_{\pm}(H, H_0) \) do not exist provided that the absolutely continuous part of \( H_0 \) is nontrivial.

Under the conditions of Theorem 6 the operator \( H \) also has no singular continuous part. Thus, for perturbations of the class \( \mathfrak{S}_p \), \( p > 1 \), the continuous component (the sum of the absolutely and singular continuous parts) may vanish entirely. This result can naturally be compared with H. Weyl's theorem that for a compact difference \( H - H_0 \) the essential spectra of the operators \( H_0 \) and \( H \) coincide. The apparent contradiction of these two results is resolved by the fact that under the assumptions of Theorem 6 the set of eigenvalues of the operator \( H \) is dense on \( a^{(\text{ess})}(H_0) \). We see that the continuous component is considerably less stable than the essential spectrum.

3. As compared with the absolutely continuous spectrum, the singular component is still less stable—it is not preserved even under trace class perturbations. This follows from the next assertion, which supplements Theorem 6.

**Theorem 7.** Let \( H_0 \) be any selfadjoint operator with purely singular spectrum. Then for any \( e > 0 \) there exists a selfadjoint trace class operator \( V = V_\varepsilon \) such that \( \| V \|_1 < \varepsilon \) and the operator \( H = H_0 + V \) has purely point spectrum.

Theorem 7 was established in the work of Carey and Pincus [89]. In a certain sense the singular component is not stable under any perturbations. Namely, Howland proved the following result [103].

**Theorem 8.** Let \( V \) be any bounded, selfadjoint operator. Suppose that the singular parts of the operators \( H_0 \) and \( H = H_0 + V \) are unitarily equivalent for any selfadjoint operator \( H_0 \). Then necessarily \( V = 0 \).

In special circumstances the singular continuous component may possess particular stability. Thus, in the Friedrichs-Faddeev model the singular continuous spectrum is preserved (it is absent for both operators \( H_0 \) and \( H \)) if the kernel of the perturbation is sufficiently smooth (see Corollary 4.2.2). Here the assumption of smoothness lies at the heart of the matter. Without it a singular continuous component can appear (or vanish) even under perturbations of rank 1. For details in this regard see §7. Thus, conditions for the preservation of the singular continuous spectrum can hardly be formulated in unitarily invariant terms.

Of course, the point spectrum is the most unstable component. The fact of the matter is that the eigenvalues of a selfadjoint operator, generally speaking, are displaced under arbitrarily small one-dimensional perturbations. At the same time in special circumstances this component may also possess a
6. SCATTERING FOR PERTURBATIONS OF TRACE CLASS TYPE

certain stability. Thus, according to Theorem 4.1.7, for sufficiently small coupling constants $\epsilon$ the Hamiltonian $H_\epsilon = H_0 + \epsilon V$ in the Friedrichs-Faddeev model has no eigenvalues. Consequently, the point component is preserved (is absent).

4. In perturbation theory Lebesgue measure plays a distinguished role. We shall clarify this fact.

In §1.3 the concepts of absolute continuity, singularity, etc. were formulated with respect to Lebesgue measure. Of course, all these concepts can be reformulated with respect to an arbitrary countably additive and countably finite Borel measure $m$ on $\mathbb{R}$. The next assertion shows that Theorem 1 does not admit substantial generalization to measures different from Lebesgue measure.

**Theorem 9.** Let $V \neq 0$ be any bounded, selfadjoint operator, and let $m$ be a nonzero Borel measure on $\mathbb{R}$. Suppose that the absolutely continuous parts of the operators $H_0$ and $H = H_0 + V$ with respect to $m$ are unitarily equivalent for any selfadjoint operator $H_0$. Then the measure $m$ is absolutely continuous with respect to Lebesgue measure. Moreover, the absolutely continuous parts (with respect to Lebesgue measure) of the operators $H_0$ and $H$ are unitarily equivalent.

Theorem 9 was obtained in the work of Howland [103] already mentioned. We note further that, as shown in [103], under the conditions of that theorem the operator $V$ is necessarily compact.

§3. Time-dependent proofs

We here present still another proof of Theorem 2.3. It reproduces the original arguments of Pearson. This proof does not depend on the considerations of the preceding chapter and is based on a direct investigation of the time-dependent limits (2.1.1). In this way in Part 3 we also obtain substantial estimates for the WO and the scattering operator that make it possible to establish their continuous dependence on the operator $H$. Moreover, a time-dependent proof of the IP is given in Part 4.

1. For trace class perturbations the proof of the existence of the weak WO $\tilde{W}_2(H, H_0; \mathcal{A})$ presents no difficulties. It suffices to use the convergence for $G \in \mathcal{S}_1$ of integrals of the form (5.3.1). This fact, in turn, follows from Corollary 1.7 and Lemma 5.3.1.

The time-dependent proof of the existence of strong WO $W_2(H, H_0; \mathcal{A})$ is based on an effective upper bound for the integral (5.3.1). It is here required that the element $g$ belong to the set $\mathcal{A} = \mathcal{A}_H$ with a finite quantity $\tau_0(g)$ (see definition (2.5.2)). We recall that according to Lemma 2.5.3 this set is dense in the absolutely continuous subspace $\mathcal{A}^{(a)}$. The appropriate estimate of the integral (5.3.1) is given in the next lemma, due to M. Rosenblum.

**Lemma 1.** Suppose $H$ is a selfadjoint operator in $\mathcal{H}$ and $G: \mathcal{H} \rightarrow \mathcal{G}$ is a Hilbert-Schmidt operator. Then for any $f \in \mathcal{A}_H$ there is the inequality

$$\int_{-\infty}^{\infty} \|GU(t)f\|^2 \, dt \leq 2\pi^2 \|f\|^2. \tag{1}$$

**Proof.** Using a Schmidt decomposition of the form (1.6.3) for $G$, we find that

$$GU(t)f = \sum_n s_n(U(t)f, \varphi_n)\psi_n,$$

where the sets of vectors $\{\varphi_n\}$ and $\{\psi_n\}$ are orthonormal, and $s_n = s_n(G)$ are the singular numbers of the operator $G$. From this we obtain the equality

$$\|GU(t)f\|^2 = \sum_n s_n^2(U(t)f, \varphi_n)^2.$$

Integrating it over all $t \in \mathbb{R}$ and considering the estimate (2.5.3), we find that

$$\int_{-\infty}^{\infty} \|GU(t)f\|^2 \, dt \leq 2\pi^2 \sum_n s_n^2 \|P\varphi_n\|^2. \tag{2}$$

The interchange of integration on $t$ and summation on $n$ performed here can be justified by appeal to Fubini's theorem. To prove (1) it remains to note that the sum on the right-hand side of (2) does not exceed $\|G\|^2$. □

We shall now present a time-dependent proof of Theorem 2.3. Below, we systematically use the notation of (2.1.2). For a bounded operator $B$ $[H, B] = HB - BH$ denotes its commutator with the selfadjoint operator $H$. In a precise sense the commutator $[H, B]$ is defined in terms of the corresponding sesquilinear form on $\mathcal{D} \times \mathcal{D}$ where $\mathcal{D} = \mathcal{D}(H)$ and is, generally speaking, only a (bounded) mapping of $\mathcal{D}$ into $\mathcal{D}^*$.

**Proof of Theorem 2.3.** We consider, for example, the case $t \rightarrow +\infty$. To prove the existence of the WO $W_2(H, H_0; \mathcal{A})$ it is sufficient to show that for $f \in \mathcal{A}_0(\mathcal{A})$ the quantity

$$\|W(t) - W(s)f\|^2 = (f, W^*(t)(W(t) - W(s))f)$$

$$- (f, W^*(s)(W(t) - W(s))f)$$

tends to zero as $t \rightarrow +\infty$, $s \rightarrow -\infty$. We shall establish that each of the two terms on the right vanish in this limit. To be specific, we consider the first of them. From equality (4.5.1) it is evident that for $V \in \mathcal{S}_1$

$$W(t) - W(s) \in \mathcal{S}_1 \subset \mathcal{S}_\infty.$$

Therefore, by Lemma 1.4.1 for $f \in \mathcal{A}_0(\mathcal{A})$

$$s-lim_{\rho \rightarrow \infty}(W(t) - W(s))U_{\rho}(\rho)f = 0,$$  

and hence

$$(f, W^*(t)(W(t) - W(s))f) = \lim_{\rho \rightarrow +\infty} (f, W^*(t)(W(t) - W(s))f)$$

$$- U_{\rho}(\rho)W^*(t)(W(t) - W(s))U_{\rho}(\rho)f).$$
Thus, it suffices to show that each of the two expressions
\[
(f, (W^*(t)W(t) - U_0^*(\rho)W^*(t)W(t)U_0(\rho)))f,
\]
\[
(f, (W^*(t)W(s) - U_0^*(\rho)W^*(t)W(s)U_0(\rho)))f
\]
(4)
tends to zero as \( t \to \infty, s \to \infty \), uniformly with respect to \( \rho \geq 0 \).

It is possible to consider only the second scalar product, since the first can be obtained from it in the special case \( s = t \). We note that for \( f \in \mathcal{D}(H_0) \)
\[
\frac{\partial}{\partial \tau}(W^*(t)W(s)U_0(\tau)f, U_0(\tau)f) = i([H_0, W^*(t)W(s)]U_0(\tau)f, U_0(\tau)f). \tag{5}
\]
and by the definition of the operator \( W(\cdot) \)
\[
[H_0, W^*(t)W(s)] = U_0^*(\tau)(H_0^*U(t-s)J - J^*U(t-s)J)U_0(s) = U_0^*(\tau)(J^*U(t-s)V - V^*U(t-s)J)U_0(s). \tag{6}
\]
In a precise sense (6) must, of course, be understood as equality of the corresponding sesquilinear forms on \( \mathcal{D}(H_0) \times \mathcal{D}(H_0) \). Combining relations (5) and (6), we find that for \( f \in \mathcal{D}(H_0) \) the expression (4) is equal to
\[
-i \int_0^\infty \langle [H_0, W^*(t)W(s)]U_0(\tau)f, U_0(\tau)f \rangle d\tau = \int_0^\infty \langle (J^*U(t-s)V - V^*U(t-s)J)U_0(\tau+s)f, U_0(\tau+t)f \rangle d\tau. \tag{7}
\]
By the boundedness of all operators this representation extends to arbitrary elements \( f \in \mathcal{D}_0 \). The last integral again consists of two terms of the same type. It is possible to consider, for example, the first of them. The proof thus reduces to verifying that for \( f \in \mathcal{D}_0 \)
\[
\lim_{t \to \infty, s \to \infty} \int_0^\infty \langle (J^*U(t-s)V - V^*U(t-s)J)U_0(\tau+s)f, U_0(\tau+t)f \rangle d\tau = 0 \tag{7}
\]
uniformly with respect to \( \rho \geq 0 \).

Setting \( V = G^*G_0 \), where \( G_0 \in \mathcal{S}_2, G \in \mathcal{S}_2 \), we estimate the integral in (7) by means of the Schwarz inequality by
\[
\int_0^\infty \|G_0U_0(\tau+s)f\| \|GU(t-s)JU_0(\tau+t)f\| d\tau \leq \left( \int_0^\infty \|G_0U_0(\tau)f\|^2 d\tau \right)^{1/2} \left( \int_0^\infty \|GU(t-s)JU_0(\tau)f\|^2 d\tau \right)^{1/2}. \tag{8}
\]
In the first integral on the right we have replaced \( \tau + s \) by \( \tau \) and in the second we have replaced \( \tau + t \) by \( \tau \). We emphasize that (8) does not depend on the parameter \( \rho \). We now take into account that \( f \in \mathcal{D}_0 \). By Lemma 1 the first factor on the right in (8) tends to zero as \( s \to -\infty \), while the second does not exceed the quantity
\[
(2\pi)^{1/2}r(f) \|GU(t-s)J\|_2 \leq (2\pi)^{1/2}r(f) \|G\|_2 \|J\|,
\]
which depends neither on \( t \) nor \( s \). Thus, the right-hand side of (8) tends to zero as \( s \to -\infty \) (uniformly with respect to \( t \)). From this we obtain the desired relation (7). \( \Box \)

**Remark 2.** Along the way we have proved the inequalities
\[
\|(W(t) - W(s))f\|^2 \leq 4(2\pi)^{1/2}r(f) \|G\|_2 \|J\| \times \left( \int_0^\infty \|G_0U_0(\tau)f\|^2 d\tau \right)^{1/2} + \left( \int_0^\infty \|G_0U_0(\tau)f\|^2 d\tau \right)^{1/2}. \tag{9}
\]

2. The proof of Theorem 2.3 presented in Part 1 has a somewhat tricky character. We therefore discuss its key features and, in particular, compare it with the proof of the existence of the WO \( W_\omega(H, H_0; J) \) in the theory of (Kato) smooth perturbations. We again emphasize that convergence (on a dense set of elements) of integrals of the form (1) ensures the existence only of the weak WO \( W_\omega(H, H_0; J) \). In the theory of smooth perturbations the integral (1) can be estimated (see definition (4.3.1)) uniformly with respect to \( f \) with \( \|f\| \leq 1 \). It is this that made it possible to establish the existence of the strong WO in Theorem 4.5.1. In the trace class theory the integral (1) converges only on the set \( \mathcal{A} \omega \) and there is no estimate uniform with respect to \( f \). In the time-dependent proof of Theorem 2.3 a decisive role is played by the uniformity of the estimate (1) with respect to another parameter— with respect to operators \( G \in \mathcal{S}_2 \) with \( \|G\|_2 \leq 1 \). In connection with this we note that the argument of Part 1 actually establishes the following test for the existence of the WO.

**Proposition 3.** Suppose \( V = G^*G_0 \), where the operators \( G_0 \) and \( G \) are bounded, and the elements \( f \) of some dense set in \( \mathcal{D}_0 \) satisfy the conditions
\[
\int_0^\infty \|G_0U_0(t)f\|^2 dt < \infty,
\]
\[
\int_0^\infty \|GU(t)f\|^2 dt < \infty \tag{10}
\]
where \( C = C(f) \) does not depend on \( \tau \in \mathbb{R} \). Then the WO \( W_\omega(H, H_0; J) \) exists.

Strictly speaking, in addition to conditions (10), in the proof of Theorem 2.3 the compactness of the operator \( V \) was also used, but only to verify relation (3). Actually, the latter follows from the first condition of (10). Namely, by (4.5.1)
\[
\|(W(t) - W(s))U_0(\rho)f\| \leq \|G\| \int_t^s \|G_0U_0(\tau + \rho)f\| d\tau \leq \|G\| \left( \int_t^s (t - \tau) \int_{s+\rho}^{s+\rho} \|G_0U_0(\tau)f\|^2 d\tau \right)^{1/2}.
\]
For $\rho \to \infty$ and fixed $t, \omega$ the right-hand side here tends to zero because of the convergence of the first integral in (10).

3. The estimate (9)$_{\pm}$ makes it possible to obtain some information about the properties of the WO $W_{\pm} = W_{\pm}(H, H_0; \omega)$. We shall now assume that $G_0 = |V|^{1/2}, G = |V|^{1/2} \text{sgn} V$ and set $\omega = \pm \infty$. From (9)$_{\pm}$ we then obtain the inequality

$$\| (W_{\pm} - W(t)) f \|^2 \leq 4(2\pi)^{1/2} r(f) \| V \|_{H^1}^{1/2} \| \omega \| \right| \int_{-\infty}^{\infty} |V|^{1/2} U_0(\tau)f^2 \, d\tau \right|^{1/2}, \quad f \in \mathcal{H}_0.$$  

(11)

As $t \to \pm \infty$ (11) gives an estimate of the convergence of $W(t)$ to the corresponding WO. We moreover distinguish the special cases of (11) corresponding to $t = 0$ and $t = \pm \infty$. Inequality (1) is then taken into account in the next assertion.

**Theorem 4.** Under the conditions of Theorem 2.3 for $f \in \mathcal{H}_0$ there are the estimates

$$\| (W_{\pm} - \mathcal{J}) f \|^2 \leq 4(2\pi)^{1/2} r(f) \| V \|_{H^1}^{1/2} \| \omega \| \left( \int_{-\infty}^{\infty} |V|^{1/2} U_0(\tau)f^2 \, d\tau \right)^{1/2},$$

and also

$$\| (W_{\pm} - W_{\omega}) f \|^2 \leq 4(2\pi)^{1/2} r(f) \| V \|_{H^1}^{1/2} \| \omega \| \left( \int_{-\infty}^{\infty} |V|^{1/2} U_0(\tau)f^2 \, d\tau \right)^{1/2},$$

(12)

(13)

We shall now establish an estimate for the difference of the scattering operator defined by relation (2.4.1) and the WO for the auxiliary triple $H_0, H_0, \mathcal{J}_3$. This difference was already considered in §2.8 in connection with the stationary representation of the scattering operator. By the (multiplication) Theorem 2.1.7 the WO $W_{\omega}(H_0, H_0; \mathcal{J}_3)$ exist and are equal to $W_{\pm}$. By the way, their existence follows also from Theorem 2.3, since $H_0, \mathcal{J}_3 - \mathcal{J}_3H_0 \in \mathcal{S}_1$ for $\mathcal{J}_3 - \mathcal{J}_3H_0 \in \mathcal{S}_1$. Applying inequality (13) and noting that $\| W_{\pm} \| \leq \| \omega \|$, we immediately obtain

**Corollary 5.** Under the conditions of Theorem 2.3 for $f \in \mathcal{H}_0$

$$\| (S(H, H_0; \mathcal{J}) - W_{\omega}(H_0, H_0; \mathcal{J}_3)) f \|^2 \leq 4(2\pi)^{1/2} r(f) \| V \|_{H^1}^{1/2} \| \omega \| \left( \int_{-\infty}^{\infty} |V|^{1/2} U_0(\tau)f^2 \, d\tau \right)^{1/2},$$

$$\leq 8\pi^2 r(f) \| V \|_{H^1} \| \omega \|^2.$$  

(14)

§3. Time-Dependent Proofs

It is clear, of course, that a similar estimate holds also for

$$\| (S^*(H, H_0; \mathcal{J}) - W_{\omega}(H_0, H_0; \mathcal{J}_3)) f \|^2.$$  

For $\mathcal{J} \neq I$ these estimates give inequalities for

$$\| (S(H, H_0) - I) f \| \quad \text{and} \quad \| (S^*(H, H_0) - I) f \|.$$  

From Theorem 4 it is easy to deduce strong continuity of the WO as $H$ varies in the topoology of the trace norm. Namely, we have

**Theorem 6.** Suppose $\mathcal{J}_0, \mathcal{J}_0' \to \mathcal{J}$ and the WO $W_{\omega}(H_0, H_0; \mathcal{J}_0)$ exists. Assume that for a family of selfadjoint operators $H(e)$ in $\mathcal{H}$ and $\mathcal{J}_1 : \mathcal{H} \to \mathcal{H}$

$$\lim_{t \to 0} \| H(e) \mathcal{J}_1 - \mathcal{J}_0 \mathcal{J}_1 \| = 0.$$  

Then for $\mathcal{J} = \mathcal{J}_1 \mathcal{J}_0$ the WO $W_{\omega}(H(e), H_0; \mathcal{J})$ exist, and

$$\text{s-lim}_{t \to 0} W_{\omega}(H(e), H_0; \mathcal{J}) = \mathcal{J}_1 W_{\omega}(H_1, H_0; \mathcal{J}_0).$$  

(15)

**Proof.** By Theorem 2.3 the WO $W_{\omega}(H(e), H_1; \mathcal{J}_1)$ exist. Therefore, according to Theorem 2.1.7, the WO

$$W_{\omega}(H(e), H_0; \mathcal{J}) = W_{\omega}(H(e), H_1; \mathcal{J}_1) W_{\omega}(H_1, H_0; \mathcal{J}_0)$$  

(16)

also exist. Thus, to prove (15) it suffices to show that

$$\text{s-lim}_{t \to 0} W_{\omega}(H(e), H_1; \mathcal{J}_1) = \mathcal{J}_1 P_{H_1}.$$  

(17)

We apply the second inequality of (12) to the collection $H_1, H(e), \mathcal{J}_1$. By condition (14) relation (17) is then satisfied on the set $\mathcal{S}_1$ dense in $\mathcal{S}_1^{(a)}$. To prove (17) it is further necessary to note that the norms of the operators on the left-hand side are bounded by $\| \mathcal{J} \|$. □

We note that under condition (14) the operators $H_1$ and $\mathcal{J}_1$ must commute, i.e.,

$$\mathcal{J}_1 \mathcal{J}_1H_1 = H_1 \mathcal{J}_1 \mathcal{J}_1.$$  

(18)

The most important special case of Theorem 6 is obtained for $H_0 = H_1$, $\mathcal{J}_0 = \mathcal{J}_1 = I$.

**Corollary 7.** If $\| H(e) - H_0 \|_1 \to 0$, then

$$\text{s-lim}_{t \to 0} W_{\omega}(H(e), H_0) = P_{H_0}.$$  

Under the conditions of Theorem 6 (even for $\mathcal{J}_0 = \mathcal{J}_1 = I$), generally speaking, there is no convergence in norm of the WO (in this regard see the paper of Putnam [133]). Of course, from Theorem 6 it follows immediately that under condition (14) the scattering operator $S(H(e), H_0; \mathcal{J})$ is weakly continuous. In fact, it is also continuous in the strong sense.
THEOREM 8. Suppose the conditions of Theorem 6 are satisfied, and the WO $W_{\pm}(H_1, H_0; \mathcal{J}_0)$ exist for both signs. Then there exists

$$
s_{0} \lim_{\varepsilon \to 0} S(H(\varepsilon), H_0; \mathcal{J}_0) = W_{\pm}^*(H_1, H_0; \mathcal{J}_0) \mathcal{J}_1 W_{\pm}(H_1, H_0; \mathcal{J}_0).
$$

PROOF. We start from definition (2.4.1) of the operator $S(H(\varepsilon), H_0; \mathcal{J}_0)$ in terms of the WO $W_{\pm}(H(\varepsilon), H_0; \mathcal{J}_0)$. By equality (16), to prove (19) it suffices to show that

$$
s_{0} \lim_{\varepsilon \to 0} W_{\pm}(H(\varepsilon), H_1; \mathcal{J}_0) W_{\pm}(H(\varepsilon), H_1; \mathcal{J}_1) = \mathcal{J}_1^* \mathcal{J}_1 P_1.
$$

According to Theorem 2.1.7 and equality (18)

$$
W_{\pm}(H(\varepsilon), H_1; \mathcal{J}_0) W_{\pm}(H(\varepsilon), H_1; \mathcal{J}_1) = W_{\pm}(H_1, H_1; \mathcal{J}_0 \mathcal{J}_1) = \mathcal{J}_1^* \mathcal{J}_1 P_1.
$$

Thus, (20) is equivalent to the relation

$$
s_{0} \lim_{\varepsilon \to 0} W_{\pm}^*(H(\varepsilon), H_1; \mathcal{J}_0)(W_{\pm}(H(\varepsilon), H_1; \mathcal{J}_1) - W_{\pm}(H(\varepsilon), H_1; \mathcal{J}_1)) = 0.
$$

Here the norm of the first factor is bounded uniformly with respect to $\varepsilon$, while the second factor tends strongly to zero by (17). $\square$

We again distinguish the case $H_0 = H_1$, $\mathcal{J}_0 = \mathcal{J}_1 = I$.

COROLLARY 9. If $\|H(\varepsilon) - H_0\| \to 0$, then

$$
s_{0} \lim_{\varepsilon \to 0} S(H(\varepsilon), H_0) = P_0.
$$

4. Finally, we present a time-dependent proof of Theorem 2.5 (the invariance principle). We denote by $\Omega$ an open set on which the function $\varphi$ is admissible. By Lemma 2.6.3 it suffices to verify relation (2.6.15). Let $\Omega_0$ be one of the component intervals of the set $\Omega$. According to the first estimate of (12), to prove Theorem 2.5 it is only necessary to demonstrate that for $f \in \mathcal{R}_0 = \mathcal{R}_{H_0}$

$$
\lim_{t \to \pm \infty} \frac{1}{2\pi} \int_0^\infty \|G_0 U_0(\pm \tau) \exp(-i t h_0) E_0(\Omega_0)\|^2 d\tau = 0,
$$

where

$$
h_0 = \varphi(H_0), \quad G_0 = |V|^{1/2}, \quad \nu = \text{sgn } \varphi'(\lambda), \quad \lambda \in \Omega_0.
$$

We shall now give a time-dependent justification of this relation. The next assertion, by the way, essentially reduces to Lemma 3.5.9.

LEMMA 10. For $G_0 \in \mathfrak{H}_2$ and $f \in \mathcal{R}_0$ the relation (21) holds.

PROOF. To be specific, we consider the case of the upper sign. By representing the operator $G_0$ in its Schmidt series, we write the integral (21) in the form

$$
\sum_n s_n^2 \int_0^\infty \|U_0(\tau) g_\tau, \varphi_n\|^2 d\tau,
$$

where

$$
g_\tau = \exp(-i t h_0) E_0(\Omega_0) f,
$$

$s_n = s_n(G_0)$ are the singular numbers of the operator $G_0$, and the vectors $g_n$ form an orthonormal basis in $\mathcal{H}_0$. The interchange of the summation on $n$ and integration on $\tau$ carried out here can be justified by appeal to Fubini's theorem. By the spectral theorem

$$
\int_0^\infty \|U_0(\tau) g_\tau, \varphi_n\|^2 d\tau = \int_0^\infty d\tau \left\| \int_{\Omega} \exp(-i t \lambda - i t \varphi(\lambda)) \frac{d(E_0(\lambda) f, \varphi_n)}{d\lambda} \right\|^2,
$$

where, according to inequality (2.5.3), the function $d(E_0(\lambda) f, \varphi_n)/d\lambda$ belongs to $L_2(\Omega_0)$. Therefore, by Lemma 2.6.4 each term of the series in (22) converges to zero as $t \to \pm \infty$. Further, according to inequality (2.5.3) the terms of this series are bounded by $2\pi s_n^2 x_n^2(g_\tau)$, where $x_\tau = r_0(g_\tau)$. Since $\{s_n^2\} \in l_1$, we have $r_0(g_\tau) \leq r_0(f)$, which gives a summable majorant for the series (22). From this it follows that the sum (22) tends to zero as $t \to \pm \infty$. $\square$

With the help of Theorem 2.2.1 in the justification of the IP it is possible to avoid the estimate (12) obtained by the method of Pearson. Indeed, from Lemma 10 (we immediately obtain (cf. Theorem 5.3.10) the IP for the weak WO. This shows that $\mathcal{R}_0(H_0, H_0; 2)$ exist and for them the equality (5.3.11) holds. Similarly, since under the conditions of Theorem 2.5 $H_0^3 \mathcal{J}_0 = \mathcal{J}_0^* H_0^3 \mathcal{J}_0$, it follows that

$$
\mathcal{R}_0(H_0, H_0; 3) \subset \mathcal{R}_0(H_0, H_0; 2),
$$

where both weak WO exist. According to Theorem 2.2.1 from the existence of the strong WO $W_0(H_0, H_0; 3)$ we obtain equality (2.2.3). Therefore, for the triple $h, h_0, H$ equality (3.2.13) is satisfied. Applying now Theorem 2.2.1 to this triple, we find that the WO $W_{\pm}(h, h_0, H)$ exist. Hence, the IP holds also for the strong WO. In conclusion we note that the same scheme of justifying the IP was applied in Part 4 of §5.3 where, however, the existence of the WO $W_0(H_0, H_0; 3)$ was verified by stationary means.

4. Local criteria for the existence of the WO

1. In view of Theorem 2.6, Theorem 2.1, in the case of a single space, and Theorem 2.3, in the case of a pair of spaces, completely solve the existence problem for WO in terms of the classes $\mathcal{H}_0$ from the viewpoint of abstract operator theory. Nevertheless, for applications these theorems are clearly insufficient. Indeed, the operator of multiplication by a function, a typical perturbation in the theory of differential operators, has continuous spectrum, and hence cannot be even compact. Therefore, Theorems 2.1 and 2.3 are clearly not applicable to such perturbations. The "drawback" of these theorems is that their conditions are formulated in terms of the perturbation $V$ itself ($V \in \mathcal{H}_0$) without reference to the properties of the operators $H_0$ and $H$. For applications, however, a decisive role is played by passage to various classes of "relatively trace class" perturbations.
This and the next sections are devoted to a description of the conditions, obtained in this way, for existence and completeness (or 3-completeness) of the WO $\mathcal{W}_{\pm}(H, H_0; \mathfrak{J})$. In this section we consider criteria based on the assumption that the perturbation is of trace class "locally." One of the most convenient technical means for generalization of the results of §2 consists in passage to a new identification. Thus, in obtaining general criteria for the existence and completeness of the WO even in the case $\mathfrak{J} = 1$, we need to resort to Theorem 2.3, which pertains to arbitrary 3. As an auxiliary means we also use the weak and local WO introduced in §2.2.

A very simple and sufficiently effective condition can be obtained if the perturbation is bordered by a spectral cut-off from only one side.

**Theorem 1.** Suppose for any bounded interval $\Lambda$

$$\left( \mathfrak{J}^3 - 3H_0 \right) E_0(\Lambda) \in \mathfrak{S}_1,$$  

Then the WO $\mathcal{W}_{\pm}(H, H_0; \mathfrak{J})$ exist.

**Proof.** For the identification $\mathfrak{J}' = 3E_0(\Lambda)$ the perturbation $\mathfrak{J}^3 - 3H_0$ is of trace class. Therefore, by Theorem 2.3 the WO

$$\mathcal{W}_{\pm}(H, H_0; \mathfrak{J}) = \mathcal{W}_{\pm}(H, H_0; \mathfrak{J}, \Lambda)$$

exist. It remains to use the fact that it suffices to verify the existence of the "global" WO $\mathcal{W}_{\pm}(H, H_0; \mathfrak{J})$ on a dense set of compactly supported elements $f = E_0(\Lambda)f$. □

The drawback of Theorem 1 is that condition (1) is not symmetric with respect to interchange of the roles of the operators $H_0$ and $H$. Therefore, Theorem 1 does not ensure completeness or 3-completeness of the WO $\mathcal{W}_{\pm}(H, H_0; \mathfrak{J})$.

2. A symmetric condition of trace class type can be obtained by bordering the perturbation by spectral cut-offs from both sides. The weakest natural assumption of trace class type is given by the condition

$$E(\Lambda)(\mathfrak{J}^3 - 3H_0)E_0(\Lambda) \in \mathfrak{S}_1,$$  

for any bounded interval $\Lambda$. We point out that the operator (2) is well defined and bounded for any $H_0, H, \mathfrak{J}$. The following result is almost obvious.

**Lemma 2.** Suppose condition (2) is satisfied for any bounded interval $\Lambda$. Then the weak WO $\mathcal{W}_{\pm}(H, H_0; \mathfrak{J})$ exist.

**Proof.** It suffices to verify the existence of the limit of $(W(t)f_0, f)$ as $t \to \pm \infty$ on some sets of elements $f_0$ and $f$ dense in $\mathcal{H}_{\mathfrak{J}}(a)$ and $\mathcal{H}_{\mathfrak{J}}^{(a)}$. By Lemma 3.1 these limits exist if $f_0 \in \mathcal{H}_{\mathfrak{J}}$, $f \in \mathcal{H}$ and $f_0, f$ have compact support. □

In place of Lemma 3.1 in the proof of this assertion it would be possible to use also a combination of Lemma 5.3.1 with Corollary 1.7. We further note that the validity of (2) for any single interval $\Lambda$ ensures the existence of the weak local WO $\mathcal{W}_{\pm}(H, H_0; \mathfrak{J})$.

Condition (2) alone, however, does not suffice for the proof of the existence of the strong WO $\mathcal{W}_{\pm}(H, H_0; \mathfrak{J})$. We shall therefore discuss necessary additional conditions. The following result is of preliminary character.

**Lemma 3.** The WO $\mathcal{W}_{\pm}(H, H_0; \mathfrak{J})$ exist if for any arbitrary bounded interval $\Lambda$ condition (2) is satisfied and

$$\lim_{t \to \pm \infty} E(\Lambda)(\mathfrak{J}^3 - 3H_0)E_0(\Lambda) = 0,$$

where $\Lambda_0$ is any strictly interior subinterval of $\Lambda$.

**Proof.** Introducing the auxiliary identification $\mathfrak{J}' = E(\Lambda)\mathfrak{J}E_0(\Lambda)$, with the help of Theorem 2.3 we establish the existence of the limits

$$\lim_{t \to \pm \infty} E(\Lambda)(\mathfrak{J}^3 - 3H_0)E_0(\Lambda) = 0.$$

Therefore, (cf. Theorem 4.5.6) the strong WO $\mathcal{W}_{\pm}(H, H_0; \mathfrak{J})$ exists, provided the projection $E(\Lambda)$ on the left can be "removed." This is possible under condition (3). To prove the existence of $\mathcal{W}_{\pm}(H, H_0; \mathfrak{J})$ it is further necessary to use the fact that $\Lambda$ is arbitrary. □

We shall now present concrete criteria for the existence of the WO. A number of them are symmetric relative to the operators $H_0$ and $H$. Such criteria ensure the existence also of the "inverse" WO $\mathcal{W}_{\pm}(H_0, H; \mathfrak{J})$. Thus, in view of Theorem 3.2.4, both WO $\mathcal{W}_{\pm}(H, H_0; \mathfrak{J})$ and $\mathcal{W}_{\pm}(H_0, H; \mathfrak{J})$ are 3-complete (respectively, $\mathfrak{J}'$-complete).

We shall indicate effective conditions for relation (3) to be satisfied. We note, first of all, that by Lemma 1.4.1 it is ensured by the inclusion

$$E(\Lambda)(\mathfrak{J}^3 - 3H_0)E_0(\Lambda) \in \mathfrak{S}_\infty.$$  

In practice (5) can conveniently be verified with the help of the next elementary assertion.

**Lemma 4.** Suppose one of the following two conditions holds.

(1) For any $\varphi \in C_0^\infty(\mathbb{R})$

$$\varphi(H)\mathfrak{J} - 3\varphi(H_0) \in \mathfrak{S}_\infty.$$  

(2) For some (and then for all) $z$

$$R(z)\mathfrak{J} - 3R_0(z) = \mathfrak{S}_\infty, \quad z \in \rho(H_0) \cap \rho(H).$$

Then for an arbitrary bounded interval $\Lambda$ and any subinterval $\Lambda_0$ strictly contained in it the inclusion (5) holds.

**Proof.** We consider some function $\varphi \in C_0^\infty(\mathbb{R})$ equal to 1 for $\lambda \in \Lambda_0$ and equal to 0 for $\lambda \in \Lambda'$. Under condition (6) the operator

$$E(\Lambda')(\mathfrak{J}^3 - 3\varphi(H_0) - \varphi(H)\mathfrak{J})E_0(\Lambda_0)$$

is compact.
If (7) holds for some $\zeta = \zeta_1$, then by the identity
\[
R(z) - 3R_0(z) = (H - \zeta_1)R(z)[R(z_1) - 3R_0(z_1)](H_0 - \zeta_1)R_0(z)
\]
relation (7) is also satisfied for all $\zeta \in \rho(H_0) \cap \rho(H)$. To prove (5) we use the representation (4.5.6), where the contour $\Gamma$ enclosing $\Lambda_0$ does not intersect $\Lambda'$, and is traversed counterclockwise. The integrand in (4.5.6) is continuous and assumes compact values by (7). Therefore, the integral itself is a compact operator. □

We now combine Lemmas 3 and 4 and note that their conditions are symmetric with respect to the roles of the operators $H_0$ and $H$ in them.

**Theorem 5.** Suppose for any bounded interval $\Lambda \subset \mathbb{R}$ condition (2) is satisfied as well as one of the inclusions (6) or (7). Then there exist the WO $W_\pm(H, \Lambda)$ and the WO $W_{\pm}(H_0, H; \mathcal{S})$ and they are $\mathcal{S}$-complete (respectively $\mathcal{S}$-complete).

In fact, relations (6) and (7) are equivalent to one another. If (6) is satisfied, then, in particular, the operator $K_0 = \theta_0(H) \mathcal{S} - \theta_0(H_0)$ is compact, where $\theta_0(\lambda) = (\lambda - z)^{-1} \theta_0(\lambda)$ and $\theta_0(\lambda) = \theta(\lambda)$, $\theta(\lambda) = 1$ for $|\lambda| \leq 1$, $\theta(\lambda) = 0$ for $|\lambda| \geq 2$. Since $\theta_0(\lambda) \to (\lambda - z)^{-1}$ in $C(\mathbb{R})$, the operator (7) is approximated in norm by the compact operators $K_0$.

Conversely, suppose that (7) holds. Then (6) is satisfied for the function $\varphi_0(\lambda) = (\lambda - z)^{-1}$ and therefore also for the function $\varphi_0, \varphi_1(\lambda) = (\lambda - z)^{-1}(\lambda - \zeta)^{-1}$.

Thus, the operator-valued function
\[
\Phi_{\zeta, \zeta} = \varphi_0, \varphi_1(H) \mathcal{S} - \varphi_0, \varphi_1(H_0)
\]
assumes compact values and is analytic (separately) in the variables $\zeta$ and $\zeta$, $\Im \zeta < 0$. We use the representation
\[
f^{(0)}(z) = -2(2\pi i)^{-1} \int f(z')^{-1} f^{(0)}(z') \, dz'
\]
for the derivatives of an analytic function. Here $\Gamma$ is a counterclockwise oriented simple closed contour encompassing the point $z$. Applying (9) to $\Phi_{\zeta, \zeta}$ for fixed $\zeta$, we find that the operators $\Phi_{\zeta, \zeta}$ are compact. For fixed $z$ we now apply (9) in the variable $\zeta$. We then find that all the operators $\frac{d}{d\zeta} \Phi_{\zeta, \zeta}$ are also compact. This implies that the relation (7) holds for all functions $\varphi(\lambda) = (\lambda - i)^{-k} (\lambda + i)^{-l}$, where $k, l$ are nonnegative integers. We further note that according to one of the versions of the Stone-Weierstrass theorem (see [19], Volume 1) any continuous function tending to zero can be approximated in $C(\mathbb{R})$ by polynomials in $(\lambda - i)^{-k} (\lambda + i)^{-l}$. Therefore, (6) is satisfied for an arbitrary $\psi \in C_0^\infty(\mathbb{R})$.

3. Frequently in verifying the existence of WO condition (2) can conveniently be supplemented by an assumption regarding the domains of the operators $H_0$ and $H$ (or functions of them). In connection with this we have the useful

**Definition 6.** Suppose for some pair of locally bounded (Borel) functions $f_0$ and $f$ on $\mathbb{R}$ such that
\[
|f_0(\lambda)| \geq 1, \quad |f(\lambda)| \geq 1, \quad \lim_{|\lambda| \to \infty} |f_0(\lambda)| = \lim_{|\lambda| \to \infty} |f(\lambda)| = \infty
\]
the condition
\[
\mathcal{J}: \mathcal{D}(f_0(H_0)) \to \mathcal{D}(f(H))
\]
is satisfied. Then the operator $H$ is called subordinate (3-subordinate) to the operator $H_0$. In this case the operator $f(H) \mathcal{J} f^{-1}(H_0)$ is bounded.

Similarly, $H_0$ is subordinate (3'-subordinate) to the operator $H$ if $\mathcal{J}': \mathcal{D}(g(H)) \to \mathcal{D}(g_0(H_0))$ and
\[
g_0(H_0) \mathcal{J}' g^{-1}(H) \in \mathcal{B}
\]
for some pair of functions $g_0$, $g$ satisfying a condition of the form (10). Of course, the definition given of subordinacy of operators $H_0$, $H$ depends on the identification $\mathcal{J}$, but we shall not mention this each time. In applications for $\mathcal{J} = \mathcal{F}$, $\mathcal{J} = I$ the conditions $\mathcal{D}(H) = \mathcal{D}(H_0)$ or $\mathcal{D}(H)^{1/2} = \mathcal{D}(H_0)^{1/2}$ for $H \geq 0$, $H_0 \geq 0$ are often sufficient and guarantee mutual subordinacy of $H_0$ and $H$. Application of the concept of subordinacy in scattering theory is based on the following elementary observation.

**Lemma 7.** Suppose $X_0 = (-\tau, \tau)$, $X_0' = \mathbb{R} \setminus X_0$. If $H$ is subordinate to $H_0$, then for any bounded interval $\Lambda$
\[
\lim_{\tau \to \infty} \|E(X_0') \mathcal{J} E_0(\Lambda)\| = 0.
\]

**Proof.** On the right-hand side of the inequality
\[
\|E(X_0') \mathcal{J} E_0(\Lambda)\| \leq \|E(X_0') f^{-1}(H_0)\| \|f(H)\| \|f_0(H_0)\| \|f_0(H_0) E_0(\Lambda)\|
\]
the first factor tends to zero as $r \to \infty$, since $f(\lambda)^{-1} \to 0$ as $|\lambda| \to \infty$. The second factor is bounded by the subordinacy of $H$ to the operator $H_0$, while the third is bounded by the boundedness of $\Lambda$. □

It can actually be shown (see [99]) that (12) being satisfied for any bounded $\Lambda$ is equivalent to the subordinacy of $H$ to the operator $H_0$. Lemma 7 makes the verification of the next result elementary.

**Lemma 8.** Suppose the limits (4) exist for any bounded interval $\Lambda \subset \mathbb{R}$ and the operator $H$ is subordinate to $H_0$. Then the WO $W_{\pm}(H, H_0; 3)$ exist.

**Proof.** It suffices to establish the existence of $W_{+}(H, H_0; 3)$ on compactly supported elements $f = E_0(\Lambda) f$. Since by hypothesis for any $r$ the strong limits of the operators $E(X_0) U_{-}(r) U_0(T) E_0(\Lambda)$ exist as $t \to \pm \infty$, for this it is only necessary to use relation (12). □

Since under condition (2) the limits (4) exist, from Lemma 8 we obtain directly
6. SCATTERING FOR PERTURBATIONS OF TRACE CLASS TYPE

**Theorem 9.** Suppose condition (2) holds for any bounded interval $\Lambda \subset \mathbb{R}$ and the operator $H$ is subordinate to $H_0$. Then the WO $W_\pm(H, H_0; \mathfrak{J})$ exist.

Theorem 9 can also be derived from Lemma 3. Indeed, with the help of the representation (4.5.6) it is not hard to derive from condition (2) the compactness of the operator $E(\Lambda' \cap \mathfrak{X}) \mathcal{L}_0(A_0)$ for $\mathfrak{X} = (-r, r)$ and any $r > 0$ (this operator is also of trace class). Together with (12) this shows that under the conditions of Theorem 9 the inclusion (5) holds, and hence the assumptions of Lemma 3 are satisfied.

Convenient conditions for $\mathfrak{J}$-completeness of the WO can be obtained on the basis of Theorem 9.

**Corollary 10.** Suppose (2) holds and the operators $H$ and $H_0$ are mutually subordinate. Then the WO $W_\pm(H, H_0; \mathfrak{J})$ (and the WO $W_\pm(H_0, H; \mathfrak{J}^*)$) exist and they are $\mathfrak{J}$-complete (respectively, $\mathfrak{J}^*$-complete).

Sometimes more convenient is an asymmetric version of conditions of the same type.

**Corollary 11.** Suppose for any bounded interval $\Lambda$ (1) holds and the operator $H_0$ is subordinate to $H$. Then the WO $W_\pm(H, H_0; \mathfrak{J})$ (and the WO $W_\pm(H_0, H; \mathfrak{J}^*)$) exist and they are $\mathfrak{J}$-complete (respectively, $\mathfrak{J}^*$-complete).

**Proof.** According to Theorem 1, the existence of the WO $W_\pm(H, H_0; \mathfrak{J})$ is a consequence of condition (1) alone. The existence of $W_\pm(H_0, H; \mathfrak{J}^*)$ follows directly from Theorem 9 if the operators $H_0$ and $H$ change roles.

In applications, as a rule, only the spectral family $E_\Lambda(\mathfrak{J})$ of the unperturbed operator is effectively known. To prove (2) the spectral cut-off $E(\mathfrak{J})$ on the left can actually be used if it is further known that the operator $H_0$ is subordinate to the operator $H$.

**Corollary 12.** Suppose that the operators $H_0$ and $H$ are mutually subordinate, the identification $\mathfrak{J}$ is invertible, and $\mathfrak{J}^{-1} \in \mathfrak{B}$. Let

$$g_0^{-1}(H_0)\mathfrak{J}^{-1}V E_\Lambda(\mathfrak{J}) \in \mathfrak{S}_1,$$  \hspace{1cm} (13)

for any bounded interval $\Lambda$, where $g_0$ is the function appearing in condition (11). Then the WO $W_\pm(H, H_0; \mathfrak{J})$ (and the WO $W_\pm(H_0, H; \mathfrak{J}^*)$) exist and they are $\mathfrak{J}$-complete (respectively, $\mathfrak{J}^*$-complete).

**Proof.** Because of Corollary 10 it is only needed to verify condition (2). Let $A$ be the operator in (11), $T$ the operator in (13), and $\Lambda$ an arbitrary bound interval. Then on the right-hand side of the equality

$$E(\Lambda) V E_\Lambda(\mathfrak{J}) = (E(\Lambda) g(H)) A^* T$$

the first two factors are bounded operators, and the third is of trace class.

Corollary 12 shows that a preliminary spectral analysis of the operator $H$ is not needed to construct scattering theory. It is important that the trace class theory makes it possible to also manage without spectral analysis of the operator $H_0$. Thus, to apply Corollary 10 it is only needed to know the domains of the operators $H$ and $H_0$ or, more generally, of suitable functions of them. This consideration is illustrated in $\S$ 6.

Actually the results of this and the foregoing part are essentially equivalent to one another. This follows from the two following straightforward assertions.

**Proposition 13.** Compactness of the operator

$$(R(z) - 3R_0(z)) E_\Lambda(\mathfrak{J}) \in \rho(H_0) \cap \rho(H),$$

for any bounded interval $\Lambda$ is equivalent to

$$E(\Lambda)(HJ - 3H_0) E_\Lambda(\mathfrak{J}) \in \mathfrak{H}_\infty,$$  \hspace{1cm} (14)

and the subordinacy of $H$ to $H_0$.

**Proposition 14.** The inclusion (7) is equivalent to (14) and the mutual subordinacy of the operators $H$ and $H_0$.

A proof of these two assertions can be found in [99].

---

4. We shall now indicate effective conditions of trace class type which ensure that the conditions of Theorem 5.7.2 are satisfied. We now assume that $\mathfrak{H}_0 = \mathfrak{H}$, $\mathfrak{J} = I$. No a priori assumptions are made regarding the full Hamiltonian in Theorem 15. Under the conditions of it the operator $H$ can be defined by means of the construction of §1.10.

**Theorem 15.** Suppose for $\theta_0 + \theta = 1$, $\theta \in [0, 1/2]$, the conditions (1.9.6), (1.9.7) are satisfied, and at least one of the two operators $G_0([H_0] + I)^{-\theta_0}$ or $G([H_0] + I)^{-\theta}$ belongs to the class $\mathfrak{S}_p$ for some $p < \infty$.

Suppose also that for any bounded interval $\Lambda$

$$G_0 E_\Lambda(\mathfrak{J}) \in \mathfrak{S}_2, \quad GE_\Lambda(\mathfrak{J}) \in \mathfrak{S}_2,$$  \hspace{1cm} (15)

Then the WO $W_\pm(H, H_0)$ exist and are complete.

**Proof.** Let us verify that the conditions of Theorem 5.7.2 are satisfied for $I = 1$. For an arbitrary bounded interval $\Lambda$ and $\Lambda' = \mathbb{R} \setminus \Lambda$

$$G_0 R_0(z) G^* = G_0 E_\Lambda(\mathfrak{J}) R_0(z)(GE_\Lambda(\mathfrak{J}))^*$$

$$+ (G_0([H_0] + I)^{-\theta_0}(E_\Lambda(\mathfrak{J})([H_0] + I) R_0(z))([(H_0] + I)^{-\theta}].$$  \hspace{1cm} (16)

By Theorem 1.9 and Remark 1.12 under condition (15) the first term on the right has angular limit values in $\mathfrak{S}_2$ for a.e. $\lambda \in \mathbb{R}$. In the second term the operator

$$E_\Lambda(\mathfrak{J})([H_0] + I) R_0(z_0)/(I + (z - z_0) R_0(z)),$$

for $\operatorname{Re} z \in A$ depends holomorphically on $z$. The factors bordering it are bounded operators, and one of them belongs to the class $\mathfrak{S}_p$. Thus, the
operator (16) has angular limit values in the class $\mathcal{S}_q$ for a.e. $\lambda \in \Lambda$, where $q = \max\{2, p\}$. To verify the remaining assertions of Theorem 5.1 we apply again the decomposition $I = E_0(\Lambda) + E_0(\Lambda')$. In the terms with $E_0(\Lambda)$ condition (15) and Theorem 1.9 are used, and in the terms with $E_0(\Lambda')$ condition (1.9.6). This makes it possible to establish for a.e. $\lambda \in \mathbb{R}$ the limits in norm of the operators $G_0(\lambda, \epsilon) G_0^*$ and $G_0(\lambda + i \epsilon) G_0^*$. The existence on all elements $f_0$ of the strong limits (5.2.10) and (5.5.6) can be established similarly with the help of Corollary 1.11. □

5. The IP holds also under the local conditions for the existence of the WO. Let $\Lambda$ be an arbitrary bounded interval, and let $\mathcal{J}^3 = \mathcal{E}(\Lambda) \mathcal{E}(\Lambda)$. According to Theorem 2.5 under condition (2) for the identification $\mathcal{J}^3$ there exist the WO $W_\mathcal{J}(\varphi(H), \varphi(H_0); \mathcal{J}^3)$ and equalities of the form (2.6.11) hold. On the basis of Lemma 1.4.1 in them $\mathcal{J}^3$ can be replaced by the identification $\mathcal{J}^3 = \mathcal{E}(\Lambda)$. Condition (5) is hereby used. Since $\Lambda$ is arbitrary it is now possible to go over from $\mathcal{J}^3$ to the original identification $\mathcal{J}$.

The validity of the IP in the remaining assertions of Parts 1–3 is established completely similarly. Finally, under the conditions of Theorem 15 all the assumptions of Theorem 5.3.6 are satisfied. The IP therefore follows from Theorem 5.3.11.

§5. Further generalizations

1. In applications generalizations of the Kato-Rosenblum and Pearson theorems are often used in which for a suitable function $\varphi$ it is assumed that

$$\varphi(H) \mathcal{J} - \mathcal{J} \varphi(H_0) \in \mathcal{S}_1.$$  \hspace{1cm} (1)

This single condition makes it possible to replace in Theorem 4.5 the local trace class assumption (4.1) and the inclusion (4.6). As compared with the results of Part 3 of §4, now conditions involving subordination of the operators $H_0$ and $H$ do not arise explicitly. Condition (1) for $\varphi(\lambda) = (\lambda - z)^{-p}$ is most useful. Namely, we have

THEOREM 1. Suppose for some positive integer $p$ the following condition is satisfied:

$$T_p(z) := R^0(z) E - \mathcal{J} R_0^p(z) \in \mathcal{S}_1, \hspace{1cm} z \in \rho(H_0) \cap \rho(H).$$  \hspace{1cm} (2)

Then the WO $W_\mathcal{J}(H, H_0; 2)$ (and the WO $W_\mathcal{J}(H_0, H; \mathcal{J})$) exist, and they are $\mathcal{J}$-complete (respectively, $\mathcal{J}^3$-complete).

PROOF. We first verify that under condition (2) relation (1) holds also for $\varphi(\lambda) = (\lambda - z)^{-m}, \hspace{1cm} m > p$. Using equality (4.9) for $n = m - p$, $f(z) = R^0(z)$, and $f(z) = R_0^p(z)$, we obtain the integral representation

$$2\pi i (m - 1)! T_p(z) = -(m - p)! (p - 1)! \int_{\Gamma} (\zeta - z)^{-m-1} T_p(\zeta) d\zeta, \hspace{1cm} (3)$$

where the simple closed contour $\Gamma$ passes around the point $z$ in a counterclockwise direction. It follows from condition (2) that the operator (3) is of trace class.

In view of the symmetry of the conditions, to prove the theorem it suffices to establish the existence of the WO $W_\mathcal{J}(H, H_0; 3)$. We first consider the auxiliary identification

$$T_p(z) = R^0(z) \sum_{l=0}^{p-1} R^{l+1}(z) R_0^l(z),$$

where $z \in \rho(H_0) \cap \rho(H)$ is any fixed point. By condition (2) the operator

$$H \mathcal{J} - \mathcal{J} H_0 = - R^0 \mathcal{T}_p$$

is of trace class. Hence, the WO $W_\mathcal{J}(H, H_0; 3)$ exist by Theorem 2.3. Further, it follows from the compactness of the operators (3) that for $\mathcal{J}_2 = p R_0^p(z)$ the difference

$$\mathcal{J}_1 - \mathcal{J}_2 = \sum_{l=0}^{p-1} T_{p+l} R_0^{l-1}$$

is compact. Therefore, according to Proposition 2.1.9 the WO $W_\mathcal{J}(H, H_0; 3)$ also exist. To prove the existence of the desired WO $W_\mathcal{J}(H, H_0; 3)$ it remains to note that the range of the operator $R_0^p(z)$ is dense in $\mathcal{H}_0$. □

REMARK 2. To prove Theorem 1 it suffices to assume the validity of the inclusion (2) in a neighborhood of some point $z_1 \in \rho(H_0) \cap \rho(H)$. Moreover, this condition can also be replaced by assumption (2) for $z = z_1$ and all positive integers $p$ larger than some fixed number $p_1$. By the way, for $p = 1$ because of the identity (4.8), the validity of (2) for at least one point $z_1$ implies the validity of (2) for all $z \in \rho(H_0) \cap \rho(H)$.

It is clear that Theorem 1 for the case $p = 1$ can easily be derived from Theorem 4.5. For arbitrary $p$ this derivation requires the machinery of double operator integrals—see §8. We note that operators $H_0, H$ for which (2) is satisfied for $p = 1$ are sometimes called resolvent comparable.

2. A large number of practically convenient "trace class" criteria for the existence and $\mathcal{J}$-completeness of the WO are based on the invariance principle (IP). Namely, we have

THEOREM 2. Suppose a real function $\varphi$ is admissible (see Definition 2.6.2) for the pair of operators $H_0, H$ and the mapping defined by this function is one-to-one. Then under condition (1) the WO $W_\mathcal{J}(H, H_0; 2)$ (and the WO $W_\mathcal{J}(H_0, H; \mathcal{J}^3)$) exist, and they are $\mathcal{J}$-complete (respectively, $\mathcal{J}^3$-complete). Moreover, equality (2.6.11) holds.

PROOF. The function $\psi$ inverse to $\varphi$ is admissible for the pair $h_0 = \varphi(H_0), \ h = \varphi(H)$. Therefore, according to Theorem 2.5 for $h_3 - 3 h_0 \in \mathcal{S}_1$ there exist the WO for the pair $\psi(h_0) = H_0, \ \psi(h) = H$ (and the identification 3). Similarly, there exist the WO $W_\mathcal{J}(H_0, H; \mathcal{J})$. □

In concrete applications one of the following functions is often taken as $\varphi(\lambda)$: (1) $\varphi(\lambda) = \exp(-\lambda); \ (2) \varphi(\lambda) = (\lambda - a)^{-m}$, if $H_0$ and $H$ have a
common regular point \( a = \bar{a} \) and \( n \) is odd (but possibly negative); (3) \( \varphi(\lambda) = (\lambda - a)^{-n} \), where \( \alpha \neq 0 \) is any real number, if the operators \( H_0 \), \( H \) are semibounded, \( H_0 \geq cI \), \( H \geq cI \) and \( a < c \). The second and third versions are, of course, very close to Theorem 1.

Of course, the IP can be combined with more general conditions for the existence of WO than the basic condition \( V \in \mathfrak{G}_1 \). In this connection we present

**Remark 4.** Theorem 3 remains in force if the inclusion (1) is replaced by the more general condition

\[
(\varphi(H) - z)^{-p} - \mathcal{J}(\varphi(H_0) - z)^{-p} \in \mathfrak{G}_1, \quad \text{Im} \ z \neq 0.
\]

**Proof.** It is only necessary to demonstrate the validity of the IP under the conditions of Theorem 1. In the notation of the proof of it for the WO with the identification \( \mathcal{J} \), the IP is satisfied by Theorem 2.5. Passage to the original identification \( \mathcal{J} \) for functions of the operators \( H_0 \), \( H \) is realized in exactly the same way as for the operators themselves. \( \square \)

3. The IP can be used to carry over Theorem 2.3 to the case of unitary operators. We recall that for unitary operators the WO are defined by relation (2.2.12).

**Theorem 5.** Suppose \( U_0 \) and \( U \) are unitary operators in Hilbert spaces \( \mathcal{H}_0 \) and \( \mathcal{H} \), \( \mathcal{H}_0 \to \mathcal{H} \) is a bounded operator, and \( U^*U = I \in \mathfrak{G}_1 \). Then the WO \( W_{\pm}(U, U_0; t) \) (and \( W_{\pm}(U, U_0; J^*) \)) exist and are \( J \)-complete (respectively, \( J^\ast \)-complete).

**Proof.** By multiplying \( U_0 \) and \( U \) by a common factor equal to 1 in modulus, it can be arranged that 1 is not an eigenvalue for either of them. Then \( U_0 \) and \( U \) are the Cayley transforms (see equality (2.2.13)) of the selfadjoint operators \( H_0 = (I + U_0)(I - U_0)^{-1} \), \( H = I(U)I(U)^{-1} \). In view of equality (2.2.14), where \( \mathcal{H}_{H_0} = U_0 \), \( \mathcal{H}_H = U \), the pair \( H_0 \), \( H \) satisfies condition (2) (for \( p = 1 \)). On the basis of Remark 4 for any admissible function the WO \( W_{\pm}(\varphi(H), \varphi(H_0); J) \) exist.

Further, by the spectral theorem the operator \( U \) can be represented in the form \( U = \exp(IG) \) where

\[
U = \int_0^{2\pi} e^{it} \, dE(t), \quad G = \int_0^{2\pi} t \, dE(t).
\]

In a similar way, we set \( U_0 = \exp(iG_0) \). Then \( H = -\cotan(G_0/2) \), \( H_0 = -\cotan(G_0/2) \). The selfadjoint operators \( G \) and \( G_0 \) are functions of the operators \( H \) and \( H_0 \), and the corresponding function \( 2\arctan(-\lambda) \) is admissible. The WO \( W_{\pm}(G, G_0; J) \) thus exist. Passing in definition (2.1.1) of these WO to the limit as \( t \to \pm \infty \) only over integer \( t \), we find that the WO \( W_{\pm}(U, U_0; J) = W_{\pm}(G, G_0; J) \) exist. \( \square \)

**Remark 6.** Under the conditions of Theorem 5 the WO for the pair \( U_0 \), \( U \) and for their Cayley transforms \( H_0 \), \( H \) coincide, i.e.,

\[
W_{\pm}(U, U_0; J) = W_{\pm}(H, H_0; J).
\]

We emphasize that in spite of the formal analogy with Theorem 2.3, in terms of selfadjoint operators Theorem 5 corresponds to the case where the difference of the resolvents is of trace class.

We return to the consideration of selfadjoint operators. Conditions of existence of the WO formulated in terms of the unitary groups \( U_0(t), U(t) \), rather than in terms of their generators \( H_0 \), \( H \), are of particular interest.

**Theorem 7.** Suppose for all \( t \)

\[
U(t)J = JU_0(t) \in \mathfrak{G}_1.
\]

Then the WO \( W_{\pm}(H, H_0; J) \) (and the WO \( W_{\pm}(H_0, H; J^*) \)) exist and are \( J \)-complete (respectively, \( J^\ast \)-complete).

**Proof.** We verify, for example, the existence of the WO \( W_{\pm}(H, H_0; J) \).

We first consider the auxiliary identification \( \mathcal{J} = E(X)3E_0(X) \), where \( X = (-r, r) \). It follows from (4) that

\[
\sin(tH)J = J\sin(tH_0) \in \mathfrak{G}_1.
\]

For \( |t| < \pi(2\pi)^{-1} \) the mapping of the segment \( (-r, r) \) defined by the function \( \mu = \sin(tH) \) is one-to-one. Therefore, by the IP there exists the WO \( W_{\pm}(H, H_0; J) \). Since \( r \) is arbitrary, it follows from this that the weak WO \( W_{\pm}(H, H_0; J) \) exists.

To prove the existence of the strong WO \( W_{\pm}(H, H_0; J) \) we now apply Theorem 2.2.1. According to (4)

\[
J^*JU_0(t) = U_0(t)J^*J \in \mathfrak{G}_1,
\]

and hence, as already proved, there exists the weak WO \( W_{\pm}(H_0, H_0; J^*) \). It remains to establish equality (2.2.3). For this we consider the pair of unitary operators \( U_0(1), U(1) \). By Theorem 5 under condition (4) there exist the strong WO \( W_{\pm}(U(1), U_0(1); J) \) and \( W_{\pm}(U_0(1), U(1); J^*) \). From Theorem 2.2.1 (more precisely, from its analogue for unitary operators) it now follows that

\[
W_{\pm}(U(1), U_0(1); J)W_{\pm}(U(1), U_0(1); J) = W_{\pm}(U_0(1), U(1); J^*)W_{\pm}(U_0(1), U(1); J^*) \quad \text{(5)}
\]

Passing in the definition of the WO (2.2.1) to the limit on integer \( t \), we find that \( W_{\pm}(H, H_0; J) = W_{\pm}(U(1), U_0(1); J) \) and \( W_{\pm}(H_0, H_0; J^*) = W_{\pm}(U_0(1), U(1); J^*) \). Therefore, the desired equality (2.2.3) follows directly from (5). \( \square \)

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§5. FURTHER GENERALIZATIONS

213
4. In various assertions of §4 and Parts 1–3 of this section the WO \( W_\pm(H, H_0; \mathfrak{g}) \) may, generally speaking, not be isometric. Thus, the conditions for the existence of the WO and even \( \dot{3} \)-completeness may be separated from conditions for their isometricity. On the other hand, the conditions considered for the existence of the WO can easily be combined with conditions for isometricity (for example, of the form (2.1.9)) of §2.1. It is necessary to bear in mind here that, if both WO \( W_\pm(H, H_0; \mathfrak{g}) \) and \( W_\pm(H_0, H; \mathfrak{g}) \) exist and are isometric, then both these WO are complete (and not only \( \dot{3} \)-complete or \( \dot{3}' \)-complete). The next theorem gives a convenient combination of conditions immediately guaranteeing the existence, isometricity, and completeness of the WO.

**Theorem 8.** Suppose that the operator \( \mathfrak{g} : \mathfrak{g}_0 \to \mathfrak{g} \) has a bounded inverse and \( 3D(f, H_0) = 3D(f, H) \) for some pair of functions satisfying (4.10). Suppose for any bounded interval \( \Lambda \) conditions (4.2) and (2.1.9) are satisfied. Then the WO \( W_\pm(H, H_0; \mathfrak{g}) \) exist, are isometric on \( \mathfrak{g}_{0\mathfrak{g}} \), and are complete, i.e., \( R(W_\pm(H, H_0; \mathfrak{g})) = \mathfrak{g}_{0\mathfrak{g}} \). Moreover, there exist the WO \( W_\pm(H_0, H; \mathfrak{g}) \) and \( W_\pm(H_0, H; \mathfrak{g}) \); these WO are equal to one another, are isometric on \( \mathfrak{g}_{0\mathfrak{g}} \), and are complete.

**Proof.** In view of condition (4.2) and the subordinacy of \( H \) to the operator \( H_0 \), the existence of the WO \( W_\pm(H, H_0; \mathfrak{g}) \) follows from Theorem 4.9. These WO are isometric by (2.1.9). Further, according to Lemma 4.7, from the relation

\[
\mathfrak{g}^{-1}(f, H_0) = 3D(f, H_0)
\]

it follows that for \( X_\mathfrak{g} = (-r, r, X_\mathfrak{g}) \) and any bounded \( \Lambda \)

\[
\lim_{t \to \infty} \| E_0(X_\mathfrak{g})^{3-1} E(\Lambda) \| = 0.
\]

Therefore, by (2.1.9) there is the inclusion

\[
(3 - 3') E(\Lambda) = (3 - 3') E_0(X_\mathfrak{g})^{3-1} E(\Lambda)
\]

\[
+ (3 - 3') E_0(X_\mathfrak{g})^{3-1} E(\Lambda) \in \mathfrak{g}_0.
\]

Thus, together with the WO \( W_\pm(H_0, H; E_0(\Lambda)\mathfrak{g}) \), which exist under condition (4.2), the WO \( W_\pm(H_0, H; E_0(\Lambda)\mathfrak{g}) \) also exist. According to Lemma 4.8 from (6) it now follows that the WO \( W_\pm(H_0, H; \mathfrak{g}) \) exist. Again using (7), we find that there also exist the WO \( W_\pm(H_0, H; \mathfrak{g}) \), equal to \( W_\pm(H_0, H; \mathfrak{g}) \). Moreover, according to (7), \( (3 - 3') E(\Lambda) \in \mathfrak{g}_0 \), so that these WO are isometric on \( \mathfrak{g}_{0\mathfrak{g}} \). Finally, the completeness of the operators \( W_\pm(H, H_0; \mathfrak{g}) \) and \( W_\pm(H_0, H; \mathfrak{g}) \) follows, for example, from Proposition 2.3.11. \( \square \)

5. In the construction of scattering theory for a pair of selfadjoint operators we have always made some assumptions on their difference. Here we consider the more special case where trace class type condition is imposed on their product. This makes it possible to compare their ordinary sum with their direct sum.

**Theorem 9.** Suppose \( H_0 \) and \( H \) are bounded, selfadjoint operators in a Hilbert space \( \mathfrak{g} \). We define the identification \( \mathfrak{g} \oplus \mathfrak{g} \to \mathfrak{g} \) by the formula \( \mathfrak{g} \oplus \mathfrak{g} = \mathfrak{g} \oplus \mathfrak{g} \). Suppose that for any \( \varepsilon > 0 \) and \( X_\varepsilon = \mathbb{R}(\varepsilon, \varepsilon) \)

\[
H E_0(X_\varepsilon) \in \mathfrak{g}_1, \quad H_0 E_0(X_\varepsilon) \in \mathfrak{g}_1.
\]

Then the WO \( W_\pm(H_0 + H, H_0 + H; \mathfrak{g}) \) exist, are isometric, and complete.

A proof can be found in the paper of A. V. Suslov [78].

§6. An example. Perturbation by an integral operator of Fourier type

As an example of the application of trace class methods we now consider perturbation of the operator of multiplication by an integral operator of Fourier type. We recall that perturbation of the operator of multiplication by an integral operator with a smooth kernel decaying sufficiently fast at infinity was studied in §§4.1, 4.2. However, the kernel of the Fourier operator does not decay at infinity.

1. We denote by \( H_{00} \) the operator of multiplication by the function \( |x|^{2l}, l > 0 \), in the space \( \mathfrak{g} = L_2(\mathbb{R}^d) \). The operator \( H_{00} \) is selfadjoint on its domain \( \mathfrak{g}(H_{00}) \) consisting of functions \( f \) for which

\[
\int_{\mathbb{R}^d} (1 + |x|^{2l})^2 |f(x)|^2 \, dx < \infty.
\]

As the “unperturbed” operator \( H_0 \) we take an arbitrary selfadjoint operator with domain

\[
\mathfrak{g}(H_0) = \mathfrak{g}(H_{00}).
\]

We set

\[
(V_+ f)(x) = (2\pi)^{-d/2} \int_{\mathbb{R}^d} \cos(x, y) f(y) \, dy,
\]

\[
(V_- f)(x) = (2\pi)^{-d/2} \int_{\mathbb{R}^d} \sin(x, y) f(y) \, dy.
\]

If in the notation we do not distinguish the coordinate and momentum representations, then the operators \( V_\pm \) can be written in the form

\[
V_\pm = \Phi + \Phi^*, \quad 2iV_\pm = \Phi^* - \Phi.
\]

The operators \( V_\pm \) are bounded and symmetric. Therefore, for \( V = V_\pm \) the full Hamiltonian is selfadjoint on the domain

\[
\mathfrak{g}(H) = \mathfrak{g}(H_0).
\]

From (1), (4) it follows that

\[
(H_{00} - z) R_0(z) \in \mathfrak{g}, \quad (H_{00} - z) R(z) \in \mathfrak{g}, \quad \text{Im} \, z \neq 0.
\]
Similarly, the operators \( H_0 \) and \( H \) are bounded relative to \( H_0 \) and relative to one another.

We denote by \( \mathcal{H}_\pm \) the subspaces of even and odd functions \( f \in \mathcal{H} = L_2(\mathbb{R}^d); f(x) = \pm f(-x), \) and by \( P_\pm \) the orthogonal projection onto \( \mathcal{H}_\pm. \) The subspaces \( \mathcal{H}_\pm \) reduce both \( H_0 \) and the operators \( V_\pm \), and \( V_\pm f = 0 \) for \( f \in \mathcal{H}_\pm. \) Moreover, according to (3)

\[
y^2_\pm = P_\pm. \tag{6}
\]

The operators \( (2)_\pm \) are bounded but not compact. Moreover, these operators are not compact even relative to \( H_0 \), i.e., \( V_\pm R_0 \notin \mathcal{S}_\infty \). Indeed, by (1) and (6) the compactness of \( V_\pm R_0 \) would imply that \( R_0 P_\pm \in \mathcal{S}_\infty \). The last inclusion contradicts the fact that the operator \( H_0 \) has continuous spectrum on each of the subspaces \( \mathcal{H}_\pm \).

Although the operators \( V_\pm \) have no “improvable” properties, under perturbation by them the essential spectrum does not change. Namely, we have

**Theorem 1.** Suppose \( H_0 \) is an arbitrary selfadjoint operator for which condition (1) is satisfied, and \( V_\pm \) are defined by equalities (2). Then the essential spectra of the operators \( H_0 \) and \( H = H_0 + V \) coincide.

**Proof.** By the generalized Weyl theorem (see, for example, [4]) it suffices to show that

\[
R(z) - R_0(z) \in \mathcal{S}_\infty, \quad \text{Im} \, z \neq 0. \tag{7}
\]

According to the resolvent identity (1.9.5) and conditions (5), relation (7) is equivalent to the inclusion

\[
(H_0 + I)^{-1} V_\pm (H_0 + I)^{-1} \in \mathcal{S}_\infty. \tag{8}
\]

The operator (8) is an integral operator, and its kernel is given by the equality

\[
(2\pi)^{-d/2} (|x|^{2l} + 1)^{-1} \varphi_\pm((x, y))(|y|^{2l} + 1)^{-1}, \tag{9}
\]

where \( \varphi_+(s) = \cos s, \varphi_-(s) = \sin s. \) Hence, the operator in (8) is equal to the half-sum or half-difference of operators of the form (1.6.21). Therefore, the inclusion (8) for any \( l > 0 \) follows from Lemma 1.6.5. \( \square \)

For sufficiently large \( l \) it is possible to construct the WO for the pair \( H_0, H \) (corresponding to \( J = I \)).

**Theorem 2.** Suppose under the conditions of Theorem 1 the inequality \( 2l > d \) holds. Then the WO \( W_\pm(H, H_0) \) exist and are complete.

**Proof.** By Theorem 5.1 (for \( p = 1 \) and \( J = I \)) we need only to demonstrate that the operators (7) or (8) are of trace class. That for \( 2l > d \) an integral operator with kernel (9) is of trace class follows from Theorem 1.6.6. \( \square \)

**Corollary 3.** For \( 2l > d \) the absolutely continuous spectra of the operators \( H_0 \) and \( H \) coincide.
Theorem 5. Suppose $H_0$ is a semibounded, selfadjoint operator in $L^2(\mathbb{R}^d)$ for which $\mathcal{D}(|H_0|^{1/2}) = \mathcal{D}(H_0^{1/2})$, $V$ is the integral operator with kernel (10), and condition (11) holds for $a \in (-1, 0)$. Then the essential spectra of the operators $H_0$ and $H = H_0 + V$ coincide. If in addition $I > -a + d$, then the WO $W_\pm(H, H_0)$ exist and are complete.

3. In the proofs of Theorems 2, 4 and 5 we started from the criterion of Theorem 5.1 for the existence and completeness of the WO. It would have also been possible to use the local results of §4. In view of the conditions $\mathcal{D}(H) = \mathcal{D}(H_0)$ or $\mathcal{D}(|H|^{1/2}) = \mathcal{D}(|H_0|^{1/2})$, the operators $H_0$ and $H$ are mutually subordinate. Thus, by Corollary 4.10 it suffices to verify condition (4.2) (for $\tau = 1$). The problem, however, consists in the fact that the explicit form of the operators $H_0$ and $H$ is not known. Therefore, to verify (4.2) it is still necessary (cf. the derivation of Corollary 4.12) to replace the spectral projections $E_\pm(\cdot)$ by suitable functions of the operator $H_0$. Conditions of the form (1) and (4) are hereby required.

To prove (4.2) it suffices, for example, to show that for some $n$

$$R^n(z) V R^n_0(z) \in \mathcal{S}_1,$$

for $n > 1$ this condition is formally less restrictive than the assumption $R - R_0 \in \mathcal{S}_1$. However, replacement of $R^n$ and $R^n$ by $R^n_0$ is possible only under the conditions

$$\mathcal{D}(|H|^{n}) = \mathcal{D}(|H_0|^{n}) = \mathcal{D}(H_0^n).$$

These relations are satisfied only for $n \leq 1$ in Theorems 2 and 4, and only for $n \leq 1/2$ in Theorem 5. Thus, in Theorems 2 and 4 it is still necessary to consider the operator (8), and in Theorem 5 to consider the operator (12). For these reasons application of Theorem 5.1 for $p > 1$ gives nothing new for the pair $H_0, H$ in question.

For $H_0 = H_0$ the operator $H = H_0 + V$ reduces to the Hamiltonian of the Friedrichs-Faddeev model (see §§4.1, 4.2) in the space $L^2(\mathbb{R}^d)$, and the result holds for $d > 1$ and in $L^2(\mathbb{R})$ for $d = 1$. However, the kernel of the integral operator corresponding to (2.1) clearly does not satisfy the requirements of decay at infinity indicated at the end of §4.2. Thus, Theorem 2 does not follow from the results of Chapter 4 even in the special case $H_0 = H_0$. The proof of Theorem 2 given in Part 1 takes into account the oscillation of the kernels of the operators (2.1) at infinity. It is significant that the trace class technique requires only information on the domain of the unperturbed operator.

Under the conditions of Theorem 2, even the existence of the WO $W_\pm(H, H_0)$ for the pair $H_0 = H_0$, $H = H_0 + V$ cannot be deduced from Theorem 2.5.1 (Cook’s criterion).

$$\|V \exp(-iH_0t)f\| = \|f\|, \quad H_0 = H_0, \quad f \in \mathcal{H}_2.$$
6. SCATTERING FOR PERTURBATIONS OF TRACE CLASS TYPE

From this it follows that \( \mathcal{L} \subset \mathcal{L}_0 \). Changing the roles of the operators \( H_0 \) and \( H \), we find that \( \mathcal{L}_0 \subset \mathcal{L} \). □

The restrictions \( \tilde{H}_0 \) and \( \tilde{H} \) of the operators \( H_0 \) and \( H \) to the subspace \( \mathcal{L}_0 = \mathcal{L} \) have simple spectrum. Therefore (see Part 1 of §1.5), they can be realized as operators of multiplication by \( \lambda \) in the spaces \( L_2(\mathbb{R}; \rho_0) \) and \( L_2(\mathbb{R}; \rho) \), where the measures \( \rho_0 \) and \( \rho \) are given by the equalities

\[
\rho_0(\cdot) = (E_0(\cdot)v, v), \quad \rho(\cdot) = (E(\cdot)v, v). \tag{4}
\]

We introduce the functions

\[
\mathcal{R}_0(z) = (R_0(z)v, v), \quad \mathcal{R}(z) = (R(z)v, v), \tag{5}
\]

which by the spectral theorem are Cauchy-Stieltjes integrals with respect to the measures (4). From (2), (3) it follows that these functions are connected by the equality

\[
(1 + \gamma \mathcal{R}_0(z))(1 - \gamma \mathcal{R}(z)) = 1. \tag{6}
\]

2. On the basis of Theorem 2.1 for the pair \( H_0, H \) the WO \( W_\lambda(H, H_0) \) exist (and are complete). In terms of the function (2) it is not hard to obtain explicit expressions for these WO and the corresponding scattering matrix. Here we shall need

**Lemma 2.** For a.e. \( \lambda \in \mathbb{R} \) there exist the finite and nonzero limit values \( D(\lambda \pm i0) \) of the function (2).

**Proof.** According to Theorem 1.2.5, the functions (5) for \( z = \lambda \pm i\epsilon \) and \( \epsilon \to 0 \) have finite limits for a.e. \( \lambda \). The limits \( D(\lambda \pm i0) \) exist. The relation \( D(\lambda \pm i0) \neq 0 \) for a.e. \( \lambda \) now follows from the identity (6) extended to \( z = \lambda \pm i0 \). □

We further note that the relation \( D(\lambda \pm i0) \neq 0 \) for a.e. \( \lambda \) follows also from the uniqueness theorem for analytic functions with nonnegative imaginary part (see Remark 1.2.4). Moreover, since the function \( D(z) \) has angular limit values for a.e. \( \lambda \in \mathbb{R} \), it is possible to appeal to the Luzin-Privalov theorem (Theorem 1.2.1).

Now, substituting the expression (1) into the representation (4.2.18) for the sesquilinear form of the WO, we find that

\[
(W_\lambda(H, H_0)f_0, f) = (R_0f_0, f) - \int_{-\infty}^{\infty} w_\lambda(\lambda; f_0, f)d\lambda, \tag{7}
\]

where

\[
w_\lambda(\lambda; f_0, f) = \gamma D^{-1}(\lambda \pm i0) \frac{d(E_0(\lambda)f_0, v)}{d\lambda} (R_0(\lambda \pm i0)v, f). \tag{8}
\]

In a representation of the space \( \mathcal{L}_0^{(a)} = \mathcal{L}^{(a)} \) diagonal for \( H_0 \) the WO must (cf. §4.2), of course, act as a singular integral operator. Namely, suppose that \( \vartheta(\cdot) \) is the representative of the element \( v \) in the decomposition (2.4.2),

\[
\gamma D^{-1}(\mu \pm i0)\langle \cdot, \vartheta(\cdot) \rangle_{\mathcal{L}_0^{(a)}}(\lambda - \mu \pm i0)^{-1}.
\]

The same expression can be obtained by substituting equality (1) for \( T(z) \) into formula (4.2.1) for the WO.

Similarly, substituting (1) into the representation (2.8.11) for the scattering matrix, we find that

\[
S(\lambda) = I(\lambda) - 2\pi i \gamma D^{-1}(\lambda \pm i0)\langle \cdot, \vartheta(\cdot) \rangle_{\mathcal{L}_0^{(a)}}(\lambda - \mu \pm i0)^{-1}, \quad \text{a.e. } \lambda \in \delta_0. \tag{9}
\]

From this it is evident that the operator \( S(\lambda) - I(\lambda) \) acting in \( \mathfrak{h}(\lambda) \) is one-dimensional. The unitarity in \( \mathfrak{h}(\lambda) \) of the operator (9) is a consequence of the equality

\[
D(\lambda + i0) - D(\lambda - i0) = 2\pi i \gamma \langle \vartheta(\lambda) \rangle^2, \tag{10}
\]

which follows from relation (1.2.7) between the limit values of Cauchy-Stieltjes integral and the derivative of its measure. We formulate the result obtained.

**Theorem 3.** Let \( H = H_0 + \gamma(\cdot, v)v \). Then for the sesquilinear form of the WO \( W_\lambda(H, H_0) \) on arbitrary elements \( f_0 \in \mathcal{L} \) the representation (7), (8) holds. For a.e. \( \lambda \in \delta_0 \) the corresponding scattering matrix is given by relation (9).

3. With the help of relation (6) it is again possible to show that the absolutely continuous parts \( H_0^{(a)} \) and \( H^{(a)} \) of the operators \( H_0 \) and \( H \) are unitarily equivalent. More precisely, a proof of this fact is based on comparison of the measures (4). It is hereby convenient to introduce their Poisson integrals

\[
\mathcal{P}_0(\lambda, \epsilon) = \text{Im} \mathcal{R}_0(\lambda + i\epsilon), \quad \mathcal{P}(\lambda, \epsilon) = \text{Im} \mathcal{R}(\lambda + i\epsilon).
\]

These functions are connected by the equality

\[
\mathcal{P}(\lambda, \epsilon) = |D(\lambda + i\epsilon)|^{-2}\mathcal{P}_0(\lambda, \epsilon). \tag{11}
\]

To prove it, it is sufficient to take the imaginary part in the relation

\[
\mathcal{R}(z) = \mathcal{R}_0(z)(1 + \gamma \mathcal{R}_0(z))^{-1},
\]

which follows from (6). The next observation is completely elementary.

**Lemma 4.** Suppose the derivatives of the (nonnegative) measures \( m_0 \) and \( m \) satisfy the relation

\[
m'(\lambda) = p(\lambda)m_0(\lambda), \quad \text{a.e. } \lambda \tag{12}
\]
with a function \( p(\lambda) \) finite a.e. We denote by \( m_{0,a} \) and \( m_a \) the parts of these measures absolutely continuous with respect to Lebesgue measure. Then the measure \( m_a \) is absolutely continuous with respect to the measure \( m_{0,a} \).

**Proof.** Since for a.e. \( \lambda \) the derivative of a measure is equal to the derivative of its absolutely continuous part, it follows from (12) that \( m_a'(\lambda) = p(\lambda)m_{0,a}(\lambda) \). It remains to use the fact that \( m_{0,a}(X) \) and \( m_a(X) \) are the integrals over \( X \) of their derivatives.

It is now not hard to see that the absolutely continuous parts of the measures (4) are equivalent to one another.

**Lemma 5.** The measures \( \rho_{0,a} \) and \( \rho_a \) are equivalent.

**Proof.** In (11) we pass to the limit as \( \varepsilon \to 0 \). Taking Lemma 2 and relation (1.2.7) into account, we then find that for a.e. \( \lambda \) the derivatives of the measures (4) are connected by the equality

\[
\rho_a'(\lambda) = |D(\lambda + i0)|^2 \rho(\lambda).
\]

The absolute continuity of \( \rho_{0,a} \) with respect to \( \rho_a \) now follows from Lemma 4. To prove the absolute continuity of \( \rho \) with respect to \( \rho_0 \) it is further necessary to use the fact that \( D(\lambda \pm i0) \neq 0 \) for a.e. \( \lambda \).

We return to the consideration of the operators \( H_0 \) and \( H \). We shall now prove unitary equivalence of their absolutely continuous parts \( H_0^{(a)} \) and \( H^{(a)} \), bypassing the construction of the WO \( W_\kappa(H, H_0) \) and appealing only to Lemma 5.

**Theorem 6.** The operators \( H_0^{(a)} \) and \( H^{(a)} \) are unitarily equivalent.

**Proof.** By Lemma 1 it suffices to consider the restrictions \( \tilde{H}_0 \) and \( \tilde{H} \) of the operators \( H_0 \) and \( H \) to the subspace \( \mathcal{L}_0 = \mathcal{L} \). Their absolutely continuous parts \( \tilde{H}_0^{(a)} \) and \( \tilde{H}^{(a)} \) can be realized as the operators of multiplication by \( \lambda \) in the spaces \( L_1(\mathbb{R}; \rho_{0,a}) \) and \( L_1(\mathbb{R}; \rho_a) \) where \( \rho_0 \) and \( \rho \) are the measures (4). According to Lemma 5, the measures \( \rho_{0,a} \) and \( \rho_a \) are equivalent. On the basis of Lemma 1.3.10 the unitary equivalence of \( \tilde{H}_0^{(a)} \) and \( \tilde{H}^{(a)} \) follows from this.

4. In contrast to the absolutely continuous parts, the singular components \( \rho_{0,s} \) and \( \rho_s \) of the measures (4) are not only not equivalent but, on the contrary, are singular relative to one another.

**Lemma 7.** The measures \( \rho_{0,s} \) and \( \rho_s \) are concentrated on disjoint sets, i.e., there exist Borel sets \( Z_{0,s} \) and \( Z_s \) such that \( Z_{0,s} \cap Z_s = \emptyset \) and

\[
\rho_{0,s}(\mathbb{R} \setminus Z_{0,s}) = 0, \quad \rho_s(\mathbb{R} \setminus Z_s) = 0.
\]

**Proof.** The measure \( \rho_s \) is concentrated (see Part 3 of §1.1) on the set \( Z_s \) of those \( \lambda \) where the symmetric derivative \( d\rho/d\lambda = +\infty \). Similarly,

\[
|D(z)| = |1 + \gamma \mathcal{P}(z)| \geq \gamma \mathcal{P}(\lambda, \varepsilon), \quad z = \lambda + i\varepsilon.
\]

Combining this estimate with (11), we find that

\[
\mathcal{P}(\lambda, \varepsilon) \mathcal{P}(\lambda, \varepsilon) \geq \gamma^{-2}.
\]

However, at a point \( \lambda \in Z_{0,s} \cap Z_s \), the left-hand side of this equality would have to tend to +\infty.

For operators with a simple spectrum, it follows immediately from Lemma 7 that for one-dimensional perturbations the singular spectrum changes completely. Namely, we have

**Theorem 8.** Let \( H_0 \) be an arbitrary selfadjoint operator with simple spectrum, let \( \nu \) be any cyclic vector for \( H_0 \), and let \( H = H_0 + \gamma \nu \). Then for any real \( \gamma \neq 0 \) the singular parts of the spectral measures of the operators \( H_0 \) and \( H \) (the spectrum of \( H \) is also simple) are concentrated on disjoint Borel sets.

From this theorem it follows that for \( \gamma \neq \gamma' \) the singular parts (relative to Lebesgue measure) of the spectral measures of the operators \( H_0 \) and \( H' \) are mutually singular.

5. We now consider a one-dimensional perturbation within the framework of the Friedrichs-Faddeev model (see §4.1). We shall assume that \( H_0 \) is multiplication by the independent variable in \( \mathcal{A} = L_1(\sigma) \), \( \sigma = \{a, b\} \), the auxiliary space \( \eta = \mathcal{C} \), \( H_f = H_0 + \gamma(\eta, \nu) \), while \( \nu(\lambda) \) satisfies a Hölder condition with some exponent \( \alpha_0 > 0 \), and \( \nu(\lambda) = \nu(b) = 0 \). Thus, the perturbation has the form (4.1.1) with

\[
\nu(\lambda, \mu) = \gamma(\lambda)\nu(\lambda)\overline{\nu(\mu)}
\]

and conditions (4.1.2) and (4.1.3) hold for it. Therefore, in this formulation all the results of §§4.1, 4.2 are applicable. Some of them now admit a more concrete description. Moreover, consideration of a one-dimensional perturbation makes it possible to establish the necessity of the condition \( \alpha_0 > 1/2 \) for the absence of singular continuous spectrum.

Suppose, as in §4.1, the set \( \mathcal{A}^{(1)} \) consists of those points \( \lambda_0 \pm 0 \), where the homogeneous equation \( f = A(\lambda_0 \pm 0) f, \lambda_0 \in \sigma \), has a nontrivial solution. We shall present a description of this set independent of the considerations of §4.1. We first note that by Theorem 1.2.6 the function

\[
D(z) = 1 + \gamma \int_{\sigma} (\mu - z)^{-1} |\nu(\mu)|^2 d\mu
\]
is Hölder continuous with exponent \( \alpha_0 \) in the complex plane with a cut along \( \sigma \). Since now \( A(z) \) is the integral operator with kernel
\[
-\gamma v(\lambda)v(\mu)(\mu - z)^{-1},
\]
for a solution of the equation \( f = A(\lambda_0 \pm i0)f \) we must have \( f = cv \). Here the condition \( c \neq 0 \) is equivalent to \( D(\lambda_0 \pm i0) = 0 \). Since, according to Theorem 1.2.5,
\[
D(\lambda \pm i0) = 1 \pm \pi i|v(\lambda)|^2 + \gamma \ p.v. \int (\mu - \lambda)^{-1}|v(\mu)|^2 d\mu. \tag{13}
\]
It follows from this that also \( v(\lambda_0) = 0 \). Thus, the equations \( D(\lambda_0 \pm i0) = 0 \) have solutions simultaneously, and hence \( M^{(+) \sigma} = \sigma \sigma' \mathcal{N} \) if \( M^{(\pm \sigma)} \) are considered as subsets of \( \sigma \). We have thus established

**Lemma 9.** The set \( M \) of “singular” points \( \lambda_0 \in \sigma \) consists of those and only those points where \( D(\lambda_0 ; i0) = 0 \).

By (13) the equality \( v(\lambda_0) = 0 \) is a necessary condition for a point \( \lambda_0 \) to belong to the set \( M \). According to Theorem 4.1.6, the set \( M \) is closed and has measure zero, while, according to Theorem 4.2.1, on the complementary set \( \sigma \setminus M \) the spectrum of the operator \( H \) is absolutely continuous. In the present case these assertions can easily be verified directly. Thus, closedness of \( M \) follows from the continuity of the function (13), while the equality \( |M| = 0 \) follows from Lemma 2. Finally, to verify the last assertion it is necessary to establish the absolute continuity on \( \sigma \setminus M \) of the spectral measure \( \rho(\cdot) \) (see (4)) of the operator \( \tilde{H} \). Suppose a closed segment \( X \) is contained in \( \sigma \setminus M \). By the inversion formula (1.2.8) the measure (4) can be recovered by the equality
\[
\rho(X) = \pi^{-1} |X| \max_{\sigma \setminus M} \mathcal{P}(\lambda, \epsilon) d\lambda. \tag{14}
\]
According to (2) and (11),
\[
\mathcal{P}(\lambda, \epsilon) = |D(\lambda + i\epsilon)|^{-2} \text{Im} D(\lambda + i\epsilon),
\]
where for \( \lambda \in X \) the quantity \( |D(\lambda + i\epsilon)| \) is bounded away from zero. From this it follows that \( \mathcal{P}(\lambda, \epsilon) \) converges as \( \epsilon \to 0 \) uniformly with respect to \( \lambda \in X \) to the continuous function \( |D(\lambda + i\epsilon)|^{-2}|v(\lambda)|^2 \). It is thus possible to pass to the limit as \( \epsilon \to 0 \) under the integral sign in (14). This proves the absolute continuity of the measure (14) on the set \( \sigma \setminus M \).

By Theorem 4.1.1 and Lemma 4.1.4 for \( \alpha_0 > 1/2 \) the set \( M \) is finite, and, in particular, there is no singular continuous component in the spectrum of the operator \( H \). For smaller \( \alpha_0 \) it is possible to assert the absence of a singular continuous component if, for example, the function \( v(\lambda) \) has only a countable number of zeros. In the general case, however, the spectrum may have a rather complicated structure. This follows from the next theorem, established in [71].

**Theorem 10.** Suppose the set \( M \subset \sigma \) is closed, \( |M| = 0 \), and the component intervals \( I_n \) of the set \( \sigma \setminus M \) satisfy the condition
\[
\sum_{n} |I_n|^{-2\beta} < \infty, \quad 2\beta < 1.
\]

Let \( m \) be an arbitrary measure with support \( M \). Then there exists a function \( v(\lambda) \) satisfying a Hölder condition with exponent \( \alpha_0 = \beta(2(1 - \beta))^{-1} \) and the equalities \( v(a) = v(b) = 0 \), for which the singular part \( p_{\sigma} \) of the measure (4) is zero, and \( \rho(\cdot) \) coincides with \( m \).

We further note that even for \( \alpha_0 < 1/2 \) the set \( M \) cannot be too “thick.” Namely, as shown in [71], in this case for its \( \varepsilon \)-neighborhood \( M \), there is the estimate \( |M| \leq C\varepsilon^{2\alpha_0} \).

6. Finally, we shall consider the behavior of the scattering matrix \( S(\lambda) \) as \( \lambda \) approaches the eigenvalues of \( H \). We shall assume that \( H_0 \) is multiplication by \( \lambda \) in \( L_2(\sigma) \), while the function \( v(\lambda) \) satisfies a Hölder condition with exponent \( \alpha_0 > 1/2 \) and \( v(a) = v(b) = 0 \). By Lemma 4.1.4 and Theorem 4.1.1 the set \( M = \sigma \setminus M \cap \sigma \) is then finite, while by Lemma 9 the point \( \lambda_0 \in \sigma \) if and only if \( D(\lambda_0 \pm i0) = 0 \). The scattering matrix \( S(\lambda) \) is defined by relation (9), where \( \beta = 0 \), for all \( \lambda \in \sigma \setminus M \) and is continuous in \( \lambda \) on the component intervals of this set. We consider the behavior of \( S(\lambda) \) as \( \lambda \to \lambda_0 \in \sigma \).

In a neighborhood of such a \( \lambda_0 \) the function (13) has the asymptotics
\[
D(\lambda + i0) = \pi \gamma v_0(\lambda - \lambda_0) + o(\lambda - \lambda_0),
\]
where
\[
v_0 = \int (\mu - \lambda_0)^{-2}|v(\mu)|^2 d\mu > 0.
\]
By equality (9) from this it follows that as \( \lambda \to \lambda_0 \)
\[
S(\lambda) = 1 - 2\pi i v_0^{-1}(\lambda - \lambda_0)\left[|v(\lambda)|^2(1 + o(1))\right]. \tag{15}
\]
Since \( v(\lambda) = O(|\lambda - \lambda_0|^{\alpha_0}) \), the second term on the right tends to zero as \( \lambda \to \lambda_0 \). We have thus established

**Proposition 11.** Suppose \( \alpha_0 > 1/2 \) and \( D(\lambda_0 \pm i0) = 0 \). Then as \( \lambda \to \lambda_0 \) the relation (15) holds. Thus, \( S(\lambda) \to 1 \) as \( \lambda \to \lambda_0 \), where if \( \alpha_0 \) approaches \( \lambda_0 \) from the right (left) the number \( S(\lambda) \) tends to 1 from below (above). If, moreover, the function \( v \) is differentiable at the point \( \lambda_0 \), then
\[
S(\lambda) = 1 - 2\pi i v_0^{-1}|v'(\lambda)|^2(\lambda - \lambda_0) + o(\lambda - \lambda_0), \quad \lambda \to \lambda_0.
\]

§8. Double Stieltjes operator integrals

In spectral theory it is often necessary to investigate the difference of functions \( f(H) - f(H_0) \) of selfadjoint operators \( H_0 \) and \( H \) in dependence
on properties of the function $f$ and the known difference $H^3 - 3H_0$. The machinery of double Stieltjes operator integrals (DOI), developed in the papers of M. Sh. Birman and M. Z. Solomyak [45, 48], is very convenient for resolving such question. Here we restrict ourselves to formulations of the simplest results of this theory, and present an example of their application at the end of the section.

1. Suppose $E_0(\cdot), E(\cdot)$ are the spectral families of selfadjoint operators $H_0$ and $H$ acting in the Hilbert spaces $H_0$ and $H$, respectively. We consider the Hilbert space $\mathcal{S}_2 = \mathcal{S}_2(H_0, H)$ of Hilbert-Schmidt operators (see Part 2 of §1.7) and define the projection-valued function $\mathcal{E}$: $\mathcal{S}_2 \rightarrow \mathcal{S}_2$ by the equality

$$E(X \times Y)T = E(Y)TE_0(X), \quad T \in \mathcal{S}_2.$$  

The function $\mathcal{E}$, defined on measurable rectangles in $\mathbb{R}^2$, can be extended in the standard way to a spectral measure defined on all Borel sets in $\mathbb{R}^2$. The spectral theorem assigns to each essentially bounded (with respect to the measure $\mathcal{E}$) function $\varphi$ a bounded normal operator (the transformer)

$$A = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \varphi(\lambda, \mu) \, d\mathcal{E}(\lambda, \mu) : \mathcal{S}_2 \rightarrow \mathcal{S}_2; \quad (1)$$

hereby

$$\|A\| = (\mathcal{E})^{-\frac{1}{2}} \sup |\varphi(\lambda, \mu)|. \quad (2)$$

The value of the transformer $A$ on an operator $T \in \mathcal{S}_2$ is called, by definition, a double operator integral (DOI) and is denoted by

$$A = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \varphi(\lambda, \mu) \, dE(\mu)T \, dE_0(\lambda) =: AT. \quad (3)$$

We thus have

THEOREM 1. Suppose the quantity (2) is finite. Then for any $T \in \mathcal{S}_2$ the double operator integral (3) belongs to the class $\mathcal{S}_2$ and

$$\|AT\|_2 \leq (\mathcal{E})^{-\frac{1}{2}} \sup |\varphi(\lambda, \mu)| \|T\|_2.$$

By the spectral theorem the operator (transformer) adjoint to (1) (with respect to the scalar product in $\mathcal{S}_2$) is equal to

$$A^* = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \varphi(\lambda, \mu) \, d\mathcal{E}^*(\lambda, \mu). \quad (4)$$

Similarly, for two distinct transformers $A_j$ of the form (1) (with kernels $\varphi_j$) there is the equality

$$A_1 A_2 = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \varphi_1(\lambda, \mu) \varphi_2(\lambda, \mu) \, d\mathcal{E}(\lambda, \mu).$$

From this it follows that for any bounded functions $f_0$ and $f$

$$f(H)\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \varphi(\lambda, \mu) \, dE(\mu)T \, dE_0(\lambda) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f_0(\lambda) \varphi(\lambda, \mu) \, dE(\mu)T \, dE_0(\lambda). \quad (5)$$

In particular, if $f_0(H_0) = E_0(X_0)$ and $f(H) = E(X)$, the integrations in (5) are restricted to the sets $X_0$ (over $\lambda$) and $X$ (over $\mu$). This makes it possible to consider a DOI (3) with integration over arbitrary Borel sets $X_0$ and $X$.

2. Conditions for the boundedness of a transformer $A$ in other ideals $\mathcal{S}_p$ can be obtained by means of results ensuring that an integral operator belongs to the trace class and giving estimates for their $s$-numbers. We present here only the simplest assertion of this type.

THEOREM 2. Suppose the function $\varphi(\lambda, \mu)$ in (3) satisfies a Hölder condition with exponent $\alpha > 0$ in the variable $\lambda$ (or $\mu$) with a constant not depending on $\mu$ (on $\lambda$). Suppose, moreover, that the spectral measure $E_0$ (respectively, $E$) has compact support. Then the transformer $A$ acts boundedly from the space $\mathcal{S}_\gamma$ to $\mathcal{S}_p$, where $p > 2(1 + 2\alpha)^{-1}$ and $\gamma > 1$. In particular, for $\gamma > 1/2$ and $T \in \mathcal{S}_1$ the operator $A^* T$ is of trace class.

For $T \notin \mathcal{S}_2$ the DOI is defined on the basis of the duality of the classes $\mathcal{S}_p$. Namely, suppose, the transformer $A$,

$$A T = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \varphi(\lambda, \mu) \, dE(\mu)T \, dE_0(\lambda),$$

acts boundedly from $\mathcal{S}_1$ to $\mathcal{S}_1$. Then its adjoint transformer $A^*$ acts boundedly from $\mathcal{S}_1^* = \mathcal{S}_1$ to $\mathcal{S}_1^* = \mathcal{S}_1$. In accordance with (4), for arbitrary $T \in \mathcal{S}_1$ the DOI can now naturally be defined by the equality $A T = A^* T$. With this definition, conditions for the boundedness of $A$ in $\mathcal{S}_1$ automatically give conditions for boundedness of $A$ in $\mathcal{S}_p$.

3. We return, finally, to the study of properties of the difference $f(H)\mathcal{J} - 3f(H_0)$ mentioned at the beginning of the section. This difference can be (at least formally) expressed in terms of the operator

$$A = g(H)\mathcal{J} - 3g(H_0) \quad (6)$$

by means of the DOI

$$f(H)\mathcal{J} - 3f(H_0) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(\lambda) - f(\mu) \, dE(\mu)T \, dE_0(\lambda). \quad (7)$$

It is possible to give conditions regarding the functions $f$ and $g$ which guarantee the boundedness of the transformer on the right-hand side of (7) in various classes $\mathcal{S}_p$. Equality (7) itself hereby assumes a precise meaning.
6. SCATTERING FOR PERTURBATIONS OF TRACE CLASS TYPE

For a detailed exposition of these questions see the paper of M. Sh. Birman and M. Z. Solomyak [48].

Concrete conditions ensuring that the left-hand side of (7) be of trace class for $H^3 - 3H_0 \in \mathfrak{S}_1$, are discussed in §8.3. There, by the way, we get by without the machinery of DOI. With the help of this machinery the conditions on $f$ can be sharpened. Such a sharpening is presented in §8.5 for the case of unitary operators.

We further mention that in analogy to (5), (7) it is also possible to establish the equality

$$f(H) J_0 f(H_0) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{f_0(\mu)}{g(\mu) - g(\lambda)} dE(\mu) T dE_0(\lambda),$$

where $T$ is the operator (6).

4. As an example of the use of DOI we shall prove that Theorem 5.1 can be derived from the local results of §4. In view of the symmetry of condition (5.2) of Theorem 5.1 it suffices to establish the existence of the WO $W_{\pm} (H, H_0; \gamma)$. To this end by Lemma 4.3 it is necessary to verify relations (4.2) and (4.5).

According to (5), (7), there is the relation

$$E(\lambda)(H^3 - 3H_0) E_0(\lambda)$$

$$= \int \int_{(\mu - \lambda)^{-p} - (\lambda - z)^{-p} - 1} dE(\mu) T^p(z) dE_0(\lambda),$$

where $T^p(z)$ is the operator (5.2). The kernel on the right-hand side of (9) can be rewritten in the form

$$- (\lambda - z)^p (\mu - z)^{p-1} + (\lambda - z)^{p-2} (\mu - z) + \ldots + (\mu - z)^{p-1}.$$ (10)

We assume that $\Lambda = (-r, r)$ and that $|z|$ is sufficiently large as compared with $r$. Then for $|\lambda| \leq r$, $|\mu| \leq r$ the denominator in (10) does not vanish, so that the kernel (10) is a continuously differentiable function. Therefore, according to Theorem 2, the transformer defined by relation (9) takes the class $\mathfrak{S}_1$ into itself. That the operator (9) is of trace class is now a consequence of condition (5.2).

Similarly, according to (8)

$$E(\Lambda') J_0 E_0(\Lambda)$$

$$= \int \int_{(\mu - \lambda)^{-p} - (\lambda - z)^{-p} - 1} dE(\mu) T^p(z) dE_0(\lambda), \quad \Lambda' = \mathbb{R} \setminus \Lambda.$$ (11)

Since $\text{dist}(\Lambda', \Lambda_0) > 0$, for purely imaginary $z$, for example, the denominator in (11) does not vanish, so that the kernel in (11) is bounded. Thus, according to Theorem 1, the transformer (11) takes the class $\mathfrak{S}_2$ into itself which proves (4.5).

CHAPTER 7

Properties of the Scattering Matrix (SM)

In this chapter diverse facts are collected regarding the scattering matrix (SM). The results presented here group in some manner about its stationary representation. Sections 3 and 5 are not directly connect with the SM. In the first of them representations for the WO and scattering operator are studied. In the second auxiliary material on realization of trace class operators in the form of integral operators is presented. Sections 7–9 are devoted to the study of spectral properties of the SM.

§1. The multiplicity theorem for scattering operators and scattering matrices

Here we present facts regarding the scattering operator and matrix which follows from Theorem 2.1.7 on multiplicity of WO.

1. We first consider the scattering operator.

**Theorem 1.** Suppose the WO

$$W_{\pm}^{(0)} := W_{\pm}(H_1, H_0; \gamma_0), \quad W_{\pm}^{(1)} := W_{\pm}(H, H_1; \gamma_1)$$ (1)

exist for both signs. Then for $\gamma = \gamma_1, \gamma_0$ the scattering operator

$$S = S(H, H_0; \gamma)$$

exists. If the operator $W_{\pm}^{(0)}$ is isometric on $\mathcal{H}_0^{(\pm)}$ and $R(W_{\pm}^{(0)}) \subset R(W_{\pm}^{(0)})$, then

$$S(H, H_0; \gamma) = (W_{\pm}^{(0)})^* S(H_1, H_0; \gamma) W_{\pm}^{(0)} S(H, H_0; \gamma),$$ (2+)

if the operator $W_{\pm}^{(0)}$ is isometric on $\mathcal{H}_0^{(\pm)}$ and $R(W_{\pm}^{(0)}) \subset R(W_{\pm}^{(0)})$, then

$$S(H, H_0; \gamma) = S(H_1, H_0; \gamma) (W_{\pm}^{(0)})^* S(H_1, H_0; \gamma) W_{\pm}^{(0)}.$$ (2–)

**Proof.** By Theorem 2.1.7 the WO $W_{\pm} = W_{\pm}(H, H_0; \gamma) = W_{\pm}^{(1)} W_{\pm}^{(0)}$ exists. Therefore, the scattering operator $S$ is well defined by equality (2.4.1), and

$$S = W_{\pm}^* W_{\pm} = W_{\pm}^{(1)} W_{\pm}^{(0)} W_{\pm}^{(1)} W_{\pm}^{(0)} = W_{\pm}^{(0)*} S(H, H_1; \gamma_1) W_{\pm}^{(0)}.$$ (3)
To prove (2) \text{+} it is necessary to consider that, by hypothesis, the operator $W_+(0)^*W_+(0)^{-1}$ is the projection onto $R(W_+(0)^{-1}) \supset R(W_+(0)^{-1})$, and hence

$$W_-(0)^{-1} = W_-(0)^{-1}W_+(0)^{-1}W_-(0)^{-1} = W_+(0)^{-1}S(H_1, H_0; J_0).$$

Substituting this expression for $W_-(0)$ into the right-hand side of (3), we obtain the desired equality (2) \text{+}. Similarly, to prove (2) \text{−} it is necessary to use the equality

$$W_+(0)^{-1} = W_-(0)^{-1}W_+(0)^{-1}W_-(0)^{-1} = W_-(0)^{-1}S^*(H_1, H_0; J_0).$$

\[\square\]

2. We now write relations (2) \text{±} in terms of the corresponding SM. Suppose there is a given unitary mapping $\mathcal{F}_0$ of the space $\mathcal{H}_0^{(a)}$ onto the direct integral (2.4.2) which diagonalizes the operator $H_0$, while $S(H, H_0; J)$ and $S(H_1, H_0; J_0)$ act in this decomposition as multiplications by $S(\lambda)$ and $S_0(\lambda)$. Suppose also that the unitary mapping

$$\mathcal{F}_0 : \mathcal{H}_0^{(a)} \rightarrow \bigoplus_{\lambda} \mathcal{H}_0, \quad \mathcal{F}_0(\lambda) = \delta(\lambda), \quad \delta_\lambda = \delta(H_0),$$

diagonalizes the operator $H_1$, while the family $S(\lambda)$: $h_1(\lambda) \rightarrow h_1(\lambda)$ corresponds to $S(H, H_1; J_1)$ in the decomposition (4). According to Proposition 5.6.3 the operator $\mathcal{F}_0 W_+^{(0)} \mathcal{F}_0^* : S_0^{(a)} \rightarrow S_0^{(a)}$ acts as multiplication by the operator-valued function $w_+(0)(\lambda)$: $h_0(\lambda) \rightarrow h_+(\lambda)$. Therefore, in the direct integral (2.4.2) the operator

$$\mathcal{F}_0 W_+^{(0)}S(H, H_0; J_0)W_+^{(0)}\mathcal{F}_0^*$$

reduces to multiplication by

$$S^{(a)}_1(\lambda) = w_+^{(0)}(\lambda)S_1(\lambda)w_+^{(0)}(\lambda).$$

From Theorem 1 we now immediately obtain

**Corollary 2.** Under the conditions of validity of equality (2) \text{+},

$$S(\lambda) = S^{(a)}_1(\lambda)S_0(\lambda),$$

while under the conditions of validity of equality (2) \text{−},

$$S(\lambda) = S_0(\lambda)S^{(a)}_1(\lambda).$$

Relations (5) \text{±} are satisfied for a.e. $\lambda \in \mathcal{D}_0$. We note that for the unitary operator $W_+^{(0)} : \mathcal{F}_0^{(a)} \rightarrow \mathcal{F}_0^{(a)}$ the operators $S(\lambda)$ and $S_0(\lambda)$ are also unitarily equivalent for a.e. $\lambda \in \mathcal{D}_0$. In order to avoid superfluous stipulations we sometimes consider different SM on the entire spectral axis, extending them by the identity operators.

Suppose $\Lambda$ is some Borel set. In order that relations (5) \text{±} be satisfied on $\Lambda$ it suffices that the conditions of Theorem 1 be satisfied for the local WO corresponding to $\Lambda$.

3. We shall further obtain a stationary representation for the operator $S^{(a)}_1$, and write out formula realizations of the relations (5) \text{±}. We restrict attention to the most important case $\mathcal{F}_0 = \mathcal{F}_0 = \mathcal{F}_0$, $J_0 = J_1 = I$. We assume that for both pairs $H_0$, $H_1$ and $H_0$, $H_1 + V_1$ the WO exist and are complete, while for the SM there are representations of the form (2.8.11). The formal notation $\mathcal{L}_0^{(a)}(\lambda)$ and $\mathcal{L}_0^{(a)}(\lambda)$ introduced in Part 2 of §5.6 are now conveniently used. In terms of them

$$S_0(\lambda) = I - 2\pi i \mathcal{L}_0^{(a)}(V_0 - V_0R_0(\lambda + i0)V_0^*) \mathcal{L}_0^{(a)}(\lambda).$$

The mapping $\mathcal{F}_0 = \mathcal{F}_0 W_+^{(0)} : S^{(a)}_0 \rightarrow S^{(a)}_0$ is unitary and realizes a diagonalization of the operator $H_1$, while, according to (5.6.10),

$$(\mathcal{F}_0 f)(\lambda) = \mathcal{L}_0^{(a)}(f) = \mathcal{L}_0^{(a)}(I - R_0(\lambda + i0))f.$$

The operator $S^{(a)}_1(\lambda)$ can be considered as the SM obtained under this decomposition from the scattering operator $S(H, H_1)$. By the general formula (2.8.11) this implies that

$$S^{(a)}_1(\lambda) = I - 2\pi i \mathcal{L}_0^{(a)}(V_1 - V_1R(\lambda + i0)V_1^*) \mathcal{L}_0^{(a)}(\lambda).$$

Here we substitute the expression (7) for $\mathcal{L}_0^{(a)}(\lambda)$ and consider the resolvent identity. Setting $V = V_0 + V_1$, we then obtain the relations

$$S^{(a)}_1(\lambda) = I - 2\pi i \mathcal{L}_0^{(a)}(I - V R(\lambda + i0))V_1(I - R_0(\lambda + i0)V_0)\mathcal{L}_0^{(a)}(\lambda),$$

$$S^{(a)}_1(\lambda) = I - 2\pi i \mathcal{L}_0^{(a)}(I - V_0 R_0(\lambda + i0)V_1(I - R(\lambda + i0)V_0)\mathcal{L}_0^{(a)}(\lambda).$$

Thus, the SM $S(\lambda)$ satisfies the representation (5) \text{±}, where the operators on the right are defined by equalities (6) \text{±} (8) \text{±}. Various versions (for example, in the spirit of Chapter 5) of precise conditions are possible for the validity of these representations, but we shall not consider them.

Substituting into the left-hand side of (5) \text{±} the expression

$$S(\lambda) = I - 2\pi i \mathcal{L}_0^{(a)}(\lambda)(I - R(\lambda + i0))\mathcal{L}_0^{(a)}(\lambda),$$

we obtain an identity connecting the resolvents $R_1$, $R$ and the operators $\mathcal{L}_0$. This identity can also be verified directly without recourse to the time-dependent definition of the SM. We shall establish, for example, identity (5) \text{+}. In the notation we omit the dependence on $\lambda$, assuming the resolvents are evaluated at the point $\lambda + i0$. Since $2\pi i \mathcal{L}_0^{(a)} \mathcal{L}_0^{(a)} = R_0 - R_0^*$, the right-hand side of (5) \text{±} is equal to

$$S^{(a)}_1(\lambda)S_0(\lambda) = I - 2\pi i \mathcal{L}_0^{(a)}(I - V R)(V_1 - R_0(V_1 - V_0)R_0)V_0(R_0 - V_0R_0)V_0\mathcal{L}_0^{(a)}(\lambda)$$

In transforming the last term in square brackets we replace $\lambda + i0$ by $\lambda + it$. From the identity

$$I - R_1^*V_1(R_0 - R_0^*)(V_0 - V_0R_0)V_0 = 2\pi i (I - R_1^*V_1(R_0 - R_0^*)V_0)V_0$$

it follows that in the first and last terms part of the operators cancel. Therefore, the sum in square brackets of (10) is equal to

$$(I - V R)(V_1 - R_0(V_1 - V_0)R_0)V_0.$$

By the resolvent identity the last expression is equal to $V - V RV$. Thus, the right-hand sides of (10) and (9) coincide.
4. In scattering theory for relatively compact perturbations the operator $S(\lambda) - I$, as a rule, turns out to be compact. Suppose $|\cdot|_{\Theta}$ is a (quasi-)norm in some separable (quasi-)normed (see Part 4 of §1.6) ideal $\Theta$ (for example, in the ideal $C_p$, $0 < p < \infty$), of operators in the spaces $h(\lambda)$. We shall estimate the quasinorm in $\Theta$ of the difference of SM for two nearby (considering identifications) operators $H_1$ and $H$. According to equality (3), under the assumption of the existence of WO (1)

$$S(H, H_0; \mathcal{Z}) - S(H_1, H_0; \mathcal{Z}_1) = W^{(0)}_+ [S(H, H_1; \mathcal{Z}) - I]W^{(0)}_-,$$

$$W^{(0)}_+ = W^{(0)}_-(H, H_0; \mathcal{Z}_0).$$

In terms of the SM introduced in Part 2 and the operators $w^{(0)}_\pm(\lambda) : h_0(\lambda) \to h_1(\lambda)$ this equality can be written in the form

$$S(\lambda) - S_0(\lambda) = W^{(0)}_+(S_1(\lambda) - I)W^{(0)}_-(\lambda).$$

Since $|w^{(0)}_\pm(\lambda)|_0 \leq \|W^{(0)}_\pm\|_0 \leq \|\mathcal{Z}_0\|_0$, from this we obtain

**Proposition 3.** Suppose the WO (1) exist. Then

$$|S(\lambda; H, H_0; \mathcal{Z}) - S(\lambda; H_1, H_0; \mathcal{Z}_1)|_{\Theta} \leq \|\mathcal{Z}_0\|_0 \|S(\lambda; H, H_1; \mathcal{Z}_1) - I\|_{\Theta}.$$ (11)

The usual operator norm is not excluded here.

§2. The invariance principle for the SM.

The SM in the unitary case

In Part 1 the IP is applied to obtain new formula representations for the SM. Scattering theory for unitary operators is briefly discussed in Part 2.

1. We suppose that the real function $\varphi$ is admissible for a pair of selfadjoint operators $H_0, H$ in the sense of Definition 2.6.2, and that the mapping defined by this function is one-to-one. Then the function $\psi$ inverse to $\varphi$ is admissible for the operators $h_0 = \varphi(H_0)$, $h = \varphi(H)$. Assume that the decompositions into a direct integral for the operators $H_0^{(a)}$ and $h_0^{(a)}$ are coordinated (see Part 3 of §2.6) by the change of variable $\lambda = \varphi(\lambda)$. Any commuting with $H_0^{(a)}$ operator $A$, given by multiplication by $a(\lambda)$ in the first of these representations, acts as multiplication by $\varphi(a(\lambda))$ in the second representation. An integral operator $B$ with kernel $b(\mu, \nu)$ in the direct integral (2.4.2) has the kernel

$$\tilde{b}(\mu, \nu) = |\varphi'(\mu)\varphi'(\nu)|^{-1/2}b(\mu, \nu)$$ (1)

in the decomposition with respect to $h^{(a)}$.

Suppose the operators $T_\mathcal{Z}(z; \varphi)$ are constructed on the basis of the operators $h_0, h$ and identification $\mathcal{Z}$ in correspondence with equalities (2.8.4)$_\pm$, i.e.,

$$T_+(z; \varphi) = \mathcal{Z}^* - v - v^*(h - z)^{-1}v,$$

$$T_-(z; \varphi) = v^*\mathcal{Z} - v^*(h - z)^{-1}v,$$

where $v = h\mathcal{Z} - \mathcal{Z}h_0$. We denote by $t_\mathcal{Z}(\mu, \nu; z; \varphi)$ the kernel of the operator $T_\mathcal{Z}(z; \varphi)$ in the decomposition (2.4.2) and by $i_\mathcal{Z}(\mu, \nu; z; \varphi)$ its kernel in the decomposition with respect to $h^{(a)}$. In the last decomposition the operator $\mathcal{Z}_\mathcal{Z}(h_0, h_0; \mathcal{Z})$ acts as multiplication by an operator-valued function denoted by $\varphi_\mathcal{Z}(z; \varphi)$. Suppose that the collection $h_0, h, \mathcal{Z}$ satisfies the conditions of Theorem 5.5.3. Then, applying to this collection equality (2.8.9)$_\pm$, we find for the SM $s(\lambda) = S(\lambda; h, h_0; \mathcal{Z})$ the representation

$$s(\lambda) = \tilde{u}_\mathcal{Z}(\lambda) - 2\pi i\varphi'(\lambda) - t_\mathcal{Z}(\lambda, \lambda; \mathcal{Z}; \varphi),$$ (3)$_\pm$

By Theorem 5.3.11, the IP holds for the operators $h_0, h, \mathcal{Z}$ and the function $\varphi$. Applied to the SM this implies, according to (2.16.14), that $s(\lambda) = S(\lambda)$ for $\varphi'(\lambda) > 0$ and $s(\lambda) = S^*(\lambda)$ for $\varphi'(\lambda) < 0$. Moreover, $u_\mathcal{Z}(\lambda) = u_\mathcal{Z}(\lambda)$ for $\varphi'(\lambda) > 0$ and $u_\mathcal{Z}(\lambda) = u_\mathcal{Z}(\lambda)$ for $\varphi'(\lambda) < 0$. We further consider equalities (1), $u_\mathcal{Z}(\lambda) = u_\mathcal{Z}(\lambda)$, and $T_\mathcal{Z}(z) = T_\mathcal{Z}(z)$. Then relation (3)$_\pm$ gives a representation for the SM of the initial pair

$$S(\lambda) = u_\mathcal{Z}(\lambda) - 2\pi i\varphi'(\lambda) - t_\mathcal{Z}(\lambda, \lambda; \mathcal{Z}; \varphi),$$

$$\omega = \text{sgn} \varphi'(\lambda).$$ (4)$_\pm$

We have thus established

**Theorem 1.** Suppose a real function $\varphi$ is admissible for the operators $H_0$ and $H$, and the mapping defined by it is one-to-one. Suppose that the pair $\varphi(H_0), \varphi(H)$ satisfies the conditions of Theorem 5.5.3. Then for the SM $S(\lambda; H, H_0; \mathcal{Z})$ both representations (4)$_\pm$ hold, where $t_\mathcal{Z}(\cdot, \cdot; \mathcal{Z}; \varphi)$ is the kernel of the operator (2)$_\pm$.

2. We turn here our attention to scattering theory for a pair of unitary operators. We recall that the WO for unitary $U_0$ and $U$ were introduced by relation (2.2.12). In terms of them the scattering operator can be defined as previously by equality (2.4.1), while the SM $S(\mu) = S(\mu; U, U_0; \mathcal{Z}), \mu \in \delta_0 = \delta(U_0) \subset T$, is constructed on the basis of a decomposition of the space $\mathcal{H}_0^{(a)}$ into a direct integral which diagonalizes $U_0^{(a)}$ (see Part 1 of §1.11). We now obtain formula representations for the WO and the scattering operator and matrix. There are two approaches to this. The first of
them consists in repeating the arguments of §2.7, 2.8. Now, however, the integrals over \( \mathbb{R} \) are replaced by integrals over the unit circle \( \mathbb{T} \), and the Fourier transform is replaced by expansion in a Fourier series. The second possibility consists in applying the stationary expressions of §2.7, 2.8 for objects pertaining to the Cayley transforms \( H_0, H \) of the pair \( U_0, U \). By the IP this gives expressions of the corresponding objects for the pair \( U_0, U \) itself.

We shall briefly describe the first approach. If at least a weak limit in (2.2.12) exists, then for any \( f_0 \in \mathcal{H}^{(0)}_U, f \in \mathcal{H}_U^{(0)} \) the sesquilinear form of the corresponding WO \( \hat{W}_\pm = \hat{W}_\pm(U, U_0; \mathcal{J}) \) is equal to

\[
(\hat{W}_\pm f_0, f) = \lim_{r \to 0^+} \left( 1 - r^2 \right) \sum_{n=0}^{\infty} r^{2n}(3U_0^{(n+1)}f_0, U^{(n+1)}f).
\]

At the same time, for \(|c| < 1\)

\[
R(\zeta) = (U - \zeta)^{-1} = \sum_{n=0}^{\infty} \mu^{-1} \zeta^n = \sum_{n=0}^{\infty} U^{-n-1} r e^{in\theta}, \quad \zeta = re^{i\theta}.
\]

An analogous representation holds also for \( R_0(\zeta) = (U_0 - \zeta)^{-1} \). Therefore, by the Parseval equality

\[
(2\pi)^{-1} \int_0^{2\pi} (3R_0(re^{i\theta})f_0, R(re^{i\theta})f)d\theta = \sum_{n=0}^{\infty} r^{2n}(3U_0^{-n-1}f_0, U^{-n-1}f).
\]

Comparing this equality with (5) and setting \( e^{i\theta} = \mu, \quad r e^{i\theta} = \mu(1 - \eta) \), we find (for the upper sign)

\[
(\hat{W}_\pm f_0, f) = \lim_{\eta \to 0^+} \pi^{-1} \eta \int (3R_0(\mu(1 + \eta))f_0, R(\mu(1 + \eta))f)(i(\mu - 1))^{-1} d\mu.
\]

To prove (6) for the lower sign it is convenient to use the decompositions of \( R_0(\zeta) \) and \( R(\zeta) \) in Fourier series for \(|c| > 1\).

In the unitary case relation (6) plays the role of (2.7.3). The rest of the construction of scattering theory is also parallel to the selfadjoint case. Thus, by interchanging in (6) the integration on \( \mu \) and the limit on \( \eta \), we obtain the definition of the stationary WO \( \mathcal{H}_\pm = \mathcal{H}_\pm(U, U_0; \mathcal{J}) \). Under suitable assumptions regarding the perturbation \( V = U \mathcal{J} - 3U_0 \) it is possible to establish an equivalence analogous to (2.7.11). As in the selfadjoint case, it can be written in the form

\[
\mathcal{H}_+^* \mathcal{H}_+ = \mathcal{H}_\pm^{(0)}, \quad \mathcal{H}_-^{(0)} = \mathcal{H}_\pm(U_0, U_0; \mathcal{J}^* \mathcal{J}).
\]

Moreover, for \( X \subset \mathbb{T} \) we have the equality

\[
(E(X)\mathcal{H}_-^{(0)}, \mathcal{H}_+^{(0)}) = \int_X \lim_{\eta \to 0^+} \pi^{-1} \eta \eta(3 + VR_0(\mu(1 - \eta)^{-1}))f_0,
\]

\[
\mathcal{R}^*(\mu(1 - \eta))3R_0(\mu(1 - \eta))f_0)(i\mu)^{-1} d\mu,
\]

which corresponds to (2.8.2).

Distinguishing on the right side of (7) the form of the WO \( \mathcal{H}_+^{(0)} \) or \( \mathcal{H}_-^{(0)} \), we obtain two expressions for the SM. We now set

\[
T_\pm(\zeta) = \Gamma^* V - V^* \mathcal{R}(\zeta) V, \quad T_\pm(\zeta) = \Gamma^* \mathcal{J} - V^* \mathcal{R}(\zeta) V.
\]

We assume that the operators \( T_\pm(\zeta), \quad \omega = "\pm", \quad \zeta = \mu(1 + \eta) \), have limits \( T_\pm(\mu) \) as \( \eta \to 0^+ \). Moreover, we suppose that in the decomposition into a direct integral the limit operators are integral operators, and their kernels \( t_\omega(\nu, \nu'; \mu) \) are well defined on the triple diagonal \( \nu = \nu' = \mu \). We denote by \( u_\pm(\mu) \), \( \mu \in \mathcal{D}_0 \), the family of bounded operators corresponding in this decomposition to the WO \( \mathcal{H}_\pm^{(0)} \). Analogously to the derivation of (2.8.9),

for the SM \( S(\mu) = S(\mu; U, U_0; \mathcal{J}) \) we obtain the representations

\[
S(\mu) = u_+(\mu) - 2\pi \mathcal{J}u_+(\mu, \mu; \mu), \quad a.e. \mu \in \mathcal{D}_0,
\]

\[
S(\mu) = u_-(\mu) + 2\pi \mathcal{J}u_-(\mu, \mu; \mu), \quad a.e. \mu \in \mathcal{D}_0.
\]

In the case \( \mathcal{J} = I \) the SM \( S(\mu) \) can be written in terms of the kernel of the operator \( T(\zeta) = V - VR(\zeta) V \). Namely, considering equality (11.1.2) and the unitarity of the operators \( U_0 \) and \( U \), from relation (8) on (or (8) on it follows that

\[
S(\mu) = I(\mu) - 2\pi \mathcal{J}Z_0(\mu, \mu; \mu), \quad a.e. \mu \in \mathcal{D}_0.
\]

Of course, this representation can be rewritten in terms of the multiplicative perturbation \( M = UU_0^{-1} \).

A precise meaning can be assigned to relations (8) and (9) in analogy to the selfadjoint theory (see Chapter 5). Namely, it is necessary to require that \( V = G^* G_0 \), where the factors \( G_0 \) and \( G \) satisfy with respect to \( U_0 \) and \( U \) (in place of \( H_0 \) and \( H \) assumptions of same type as in Theorem 5.3.1). This makes it possible, in particular, to define by the equality

\[
Z_0(\mu; G_0) = (\mathcal{F}_0 G_0^*)(\mu), \quad a.e. \mu \in \mathcal{D}_0,
\]

mappings in terms of which the kernels of the operators \( T_\pm(\zeta) \) and \( T(\zeta) \) can be constructed. Here \( \mathcal{F}_0 \) realizes a unitary mapping of \( \mathcal{H}_0^{(0)} \) onto the corresponding direct integral, while \( G_0 \) is assumed to be weakly \( U_0 \)-smooth (see Definition 5.1.1). In these terms the realization of relation (9) has the form

\[
S(\mu) = I(\mu) - 2\pi \mathcal{J}Z_0(\mu; G)(1 - \mathcal{B}(\mu_+)^{-1})Z_0^*(\mu; G_0),
\]

where \( \mathcal{B}(\zeta) = G_0 R(\zeta) G^* \). In conclusion we note the equality

\[
w-lim_{\eta \to 0^+} G_0(\mu(1 - \eta)) - R_0(\mu(1 - \eta)^{-1}))G^* = 2\pi \mathcal{J}Z_0^*(\mu; G_0)Z_0(\mu; G),
\]
valid for weakly \( H_\sigma \)-smooth operators \( G_0 \) and \( G \). This equality follows directly from relations (1.2.13) (or (1.11.4)) and (1.11.6).

§3. Stationary representations for the WO and the scattering operator

The present section is of auxiliary character. It is needed for the study of the SM in §§4 and 6. Here the representations of §§2.7 and 2.8 for the WO and the scattering operator are concretized under the assumptions of the smooth and trace class methods. Simultaneously these methods are compared with the stationary scheme of Chapter 5.

1. We first consider smooth perturbations. The conditions of Chapter 4 are as a whole essentially more stringent than those of Chapter 5. Indeed, the definition (4.3.4) of Kato \( H \)-smoothness is similar to definition (5.1.2) of weak \( H \)-smoothness. However, the first of these is considerably more restrictive due to the uniformity of the estimate (4.3.4) with respect to \( \lambda \). Further, according to definition (4.3.2) for a Kato \( H \)-smooth operator \( G \) the vector-valued functions \( GR(\lambda \pm it)f \) belong to the Hardy classes \( \mathcal{H}_\infty^\circ(\Theta) \) for any \( f \in \mathcal{D} \). Therefore, as noted already in Part 2 of §4.3, they have strong limits as \( \varepsilon \to 0 \) for a.e. \( \lambda \in \mathbb{R} \).

From this, in particular, it follows that under the conditions of Chapter 4 the formula representations for the WO and scattering operator are valid. Thus, Definition 2.7.2 of the stationary WO is now good on all elements \( f_0 \in \mathcal{H}_0, \ f \in \mathcal{H} \). The next assertion is a direct consequence of Theorems 5.2.4, 5.2.8, and 5.5.1.

**Theorem 1.** Suppose the conditions of Theorem 4.5.1 are satisfied. \( f_0, g_0 \) are arbitrary elements in \( \mathcal{H}_0 \), \( f \) is an arbitrary element in \( \mathcal{H} \), and \( X \) is any Borel set on \( \mathbb{R} \). Then for the sesquilinear form \( \langle W(\lambda \pm it)f \rangle \) of the WO \( W_\pm = W_\pm(H, H_\sigma; \mathbb{Z}) \) the representation (2.7.5) holds, the representation (2.7.11) holds for the form \( \langle W_\pm f_0, g_0 \rangle \), and the representations (2.8.2) and (2.8.6) hold for the form \( \langle E(x)W_\pm f_0, g_0 \rangle \).

We point out that the results of Chapter 5 give expressions for the stationary WO \( W_\pm = W_\pm(H, H_\sigma; \mathbb{Z}) \) which under the conditions of Theorem 4.5.1 coincide with the time-dependent WO \( W_\pm \). As concerns the time-dependent WO, from the results of Chapter 5 we immediately obtain the existence of only the weak WO \( \tilde{W}_\pm(H, H_\sigma; \mathbb{Z}) \). Indeed, under the conditions of Theorem 4.5.1 the operators \( G_0 \) and \( G \) are, respectively, weakly \( H \)-smooth and \( H \)-smooth, and hence it is possible to appeal to Theorem 5.3.2. At the same time for a Kato \( H \)-smooth operator \( G \) condition (5.3.3), generally speaking, is not satisfied, and, moreover, the product \( GR(\lambda \pm it)G^* \) is not defined even for \( \text{Im} \ z \neq 0 \). Therefore, Theorem 4.5.1 does not follow from the results of §5.3 regarding the existence of the strong time-dependent WO. By the way, in the case \( \mathcal{H}_0 = \mathcal{H}, \mathcal{J} = I \) (and, more generally, under condition (2.1.9))
With the help of Lemma 5.2.1 from this we deduce that the representation (2.7.5) holds for \((\overline{W}_\pm f_0, f)\) if one of the elements \(f_0\) or \(f\) is compactly supported.

The problem is more involved with representations of the form (2.7.11)\(_\pm\). Below we assume that \(f_0 = E_0(\Lambda)f_0\). By (1) from equality (2.7.11)\(_\pm\) for \(W_{\pm}(\lambda^\alpha)\) it follows that

\[
(\overline{W}_\pm f_0, \overline{W}_\pm g_0) = \int_{-\infty}^{\infty} \lim_{r \to 0} \pi^{-1} e(i(\Lambda)\overline{2\overline{R}_0}(\lambda \pm i\varepsilon)f_0, 2\overline{R}_0(\lambda \pm i\varepsilon)g_0) d\lambda.
\]  

(2)

The spectral cut-off \(E_0(\Lambda)\) has been removed from \(g_0\) by means of Lemma 5.2.1 applied to the pair \(H_0, H_0\) and the identification \(\overline{\partial}E(\Lambda)\). Removal of \(E(\Lambda)\) in (2), however, requires additional assumptions. For example, this is possible if \(H\) is subordinate to the operator \(H_0\), which simultaneously guarantees the existence of the strong WO \(W_{\pm}(H, H_0; \mathcal{J})\). Indeed, it suffices to show that

\[
\int_{-\infty}^{\infty} \lim_{r \to 0} e(i(E(\Lambda)\overline{2\overline{R}_0}(\lambda \pm i\varepsilon)f_0, 2\overline{R}_0(\lambda \pm i\varepsilon)g_0)) d\lambda = o(1), \quad r \to \infty,
\]

where \(\Lambda' = \mathbb{R}\setminus \Lambda\), \(\Lambda = (-r, r)\). The last integral is bounded by

\[
\|E(\Lambda)\overline{2\overline{R}_0}(\lambda \pm i\varepsilon)f_0\| \|2\overline{R}_0(\lambda \pm i\varepsilon)g_0\| d\lambda.
\]  

(3)

According to the Schwarz inequality and relations (1.3.11), (1.4.11) the integral in (3) does not exceed \(\|P_{0\infty}\| \|P_{0\infty}\|\). Therefore, the entire expression (3) tends to zero as \(r \to \infty\) by Lemma 6.4.7.

In representations for the form \((E(\Lambda)\overline{W}_{\pm} f_0, \overline{W}_0 g_0)\) it is necessary to assume that on the right-hand sides of (2.8.2) and (2.8.6) the role of \(\mathcal{J}\) is played by \(\mathcal{J}'_\mathcal{J}\), while the role of \(\mathcal{V}\) is played by the operator \(E(\Lambda)\overline{V}E_0(\Lambda)\). In particular, \(\mathcal{J}'_{\mathcal{J}}\) in (2.8.6) must be replaced by \(W_{\pm}(H_0, H_0; \mathcal{J}'_{\mathcal{J}})\). We summarize our considerations.

Theorem 2.

1. Suppose \(V \in \mathcal{O}_1\). Then on all elements \(f_0, g_0 \in \mathcal{X}_\mathcal{O}\), \(f \in \mathcal{X}\) for the forms \((W_{\pm} f_0, f)\), \((W_{\pm} f_0, W_0 g_0)\), and \((E(\Lambda)\overline{W}_{\pm} f_0, W_0 g_0)\) the representations (2.7.5) and (2.7.11)\(_\pm\) hold (2.8.2), (2.8.6) hold.

2. Suppose condition (6.4.1) is satisfied. Then the representations (2.7.5) and (2.7.11)\(_\pm\) for the forms \((W_{\pm} f_0, f)\) and \((W_{\pm} f_0, W_0 g_0)\) are preserved if one of the elements of the pairs \(f_0, f\) and \(g_0, g_0\) is compactly supported. The representations (2.8.2), (2.8.6) for \((E(\Lambda)\overline{W}_{\pm} f_0, W_0 g_0)\) are valid in the case where both elements \(f_0, g_0\) are compactly supported.

3. Suppose condition (6.4.2) is satisfied. Then the representation (2.7.5) for \((\overline{W}_{\pm} f_0, f)\) holds if one of the elements \(f_0, f\) is compactly supported. The representation (2.7.11)\(_\pm\) for \((\overline{W}_{\pm} f_0, W_0 g_0)\) holds in the case where one of the elements \(f_0\) or \(g_0\) is compactly supported and the operator \(H\) is subordinate to \(H_0\). The representations (2.8.2), (2.8.6) for \((E(\Lambda)\overline{W}_{\pm} f_0, W_0 g_0)\) are preserved if \(f_0 = E_0(\Lambda)f_0\) and \(g_0 = E_0(\Lambda)g_0\) for a bounded \(\Lambda\), while in the right-hand sides of (2.8.2), (2.8.6) the operators \(\mathcal{J}\) and \(\mathcal{V}\) are replaced, respectively, by \(E(\Lambda)\overline{E}E_0(\Lambda)\) and \(E(\Lambda)\overline{V}E_0(\Lambda)\).

§4. The SM for smooth perturbations

1. In the theory of (Kato) smooth perturbations without additional assumptions it is hardly possible to count on the validity of the stationary representations for the SM. The reason for this is that (cf. Part 1 of §3) under the conditions of Theorem 4.5.1 relation (5.3.3) may be violated. However, the remaining conditions of Theorem 5.5.3 are satisfied. Thus, relations (5.2.10)\(_\pm\) hold for an \(H_0\)-smooth operator \(G_0\). Therefore, from Theorem 5.5.3 we immediately obtain

Proposition 1. Suppose the operator \(G_0\) is \(H_0\)-smooth, the operator \(G\) is \(|H|^{1/2}\)-bounded, and condition (5.3.3) holds. Then there exist the strong WO \(W_{\pm}(H, H_0; \mathcal{J})\), while for the corresponding SM for a.e. \(\Lambda \in \mathcal{O}_0\) the representations (2.8.9)\(_\pm\) or (5.5.3)\(_\pm\) hold.

Similarly, under the validity of the conditions of Proposition 1 on some interval \(\Lambda\) the representations for the SM defined in terms of the local WO will also hold for a.e. \(\Lambda \in \mathcal{O}_0 \cap \Lambda\).

2. More substantial information can be obtained in the case \(\mathcal{X}_0 = \mathcal{X}\), \(\mathcal{J} = I\), which we consider below. Suppose again that \(\Lambda\) is some interval, the operator \(H_0\) on \(\Lambda\) has absolutely continuous spectrum of constant multiplicity \(k\), and the mapping \(\mathcal{S}\) of the space \(E_0^{(\alpha)}(\Lambda)\mathcal{X}\) onto \(L_2(\Lambda; h)\), \(\dim h = k\), is fixed. Then the SM \(S(\lambda) = S(\lambda; \mathcal{H}, H_0); h \to h\) is defined uniquely, and not only up to unitary equivalence. We suppose that the perturbation is represented in the form \(V = G\mathcal{V}G\), where \(\mathcal{V} \in \mathcal{S}(\mathcal{O})\), and the operator \(G: \mathcal{X} \to \mathcal{O}\) is \(|H|^{1/2}\)-bounded and satisfies the condition \(\mathcal{O}\). According to (5.4.2), this means precisely that in the operator norm the mapping \(Z_0(\lambda; G): \mathcal{X} \to h\) depends on \(\lambda \in \Lambda\) in a Hölder continuous fashion (with some exponent \(\alpha > 0\)). According to 4.4.7 from this it follows that the operator-valued function \(B_0^{(\alpha)}(z) = GR_0(z)G^*\) is continuous in norm in the strip \(\Re z \in \Lambda\) up to the cut along \(\Lambda\).

As in §4.6, below we distinguish cases of small (Theorem 2) and relatively compact (Theorem 3) perturbations. Under the conditions of these assertions the existence and completeness of the local WO \(W_{\pm}(H_0, H_0; \Lambda)\) were established in Theorems 4.6.1 and 4.6.4, respectively. Results on the WO follow also from Theorems 5.7.1 and 5.8.1 were, in addition, a stationary representation for the SM was obtained.
7. PROPERTIES OF THE SCATTERING MATRIX

THEOREM 2. Suppose the operator $G$ is strongly $H_0$-smooth on $\Lambda$ and for $B_0(z) = -\mathcal{V} B^{(0)}(z)$, the following condition is satisfied:
\[
\sup_{\lambda \in \Lambda} \|B_0(\lambda + i0)\| < 1. \tag{1}
\]
Then for the SM $S(\lambda) = S(\lambda; H, H_0)$ for all $\lambda \in \Lambda$ the representation (5.7.6) holds, $S(\lambda)$ is continuous in $\lambda \in \Lambda$, and there is the expansion in a series
\[
S(\lambda) = I - 2\pi i \sum_{n=0}^{\infty} Z_0(\lambda; G) B_0^n(\lambda + i0) \mathcal{V} Z_0^*(\lambda; G), \tag{2}
\]
which converges in the operator norm.

Proof. Under condition (1) the operator (5.7.5) exists and is continuous in $\lambda \in \Lambda$. Therefore, the representation (5.7.6) follows from Theorems 5.7.1 and 5.8.1. According to (1), from (5.7.6) we obtain both the continuity of $S(\lambda)$ and the possibility of the expansion (2).

By introducing the coupling constant $\gamma$, i.e., replacing $V$ by $\gamma V$, we find that for sufficiently small $\gamma$ condition (1) is always satisfied. In this case (2) becomes a power series in the parameter $\gamma$. It is the series of perturbation theory for the SM. In the physics literature this series is called the Born series.

3. For compact operators $B_0(\lambda \pm i0)$ the set $\mathcal{N}$, on which condition (5.7.5) is violated, consists of points $\lambda$ for which equation (4.7.1) has a nontrivial solution. By Theorem 4.7.2 the set $\mathcal{N}$ is closed and has measure zero.

THEOREM 3. Suppose the operator $G$ is strongly $H_0$-smooth on $\Lambda$ and $B^{(0)}(z) \in \Theta_{\infty}$ for $1m z \neq 0$. Then for the SM $S(\lambda) = S(\lambda; H, H_0)$ for all $\lambda \in \Lambda \backslash \mathcal{N}$ the representation (5.7.6) holds, $S(\lambda)$ is continuous with respect to $\lambda \in \Lambda \backslash \mathcal{N}$, and
\[
S(\lambda) - I \in \Theta_{\infty}. \tag{3}
\]

Proof. The representation (5.7.6) for the SM and its continuity with respect to $\lambda \in \Lambda \backslash \mathcal{N}$, as before, are consequences of Theorem 5.8.1. We further note that by equality (5.7.7) $Z_0(\lambda; G) \in \Theta_{\infty}$. Therefore, relation (3) follows directly from (5.7.6).

We emphasize that under the conditions of this theorem the SM is continuous on the component intervals of the open set $\Lambda \backslash \mathcal{N}$. However, $S(\lambda)$ may not have a limit as the end points of these intervals are approached. Under the conditions of Theorems 2 and 3 the continuity of $S(\lambda)$ in $\lambda$ can be understood in a qualified sense— as Hölder continuity with exponent $\alpha > 0$.

With the help of the representation (5.7.6) it is possible to obtain an expansion of $S(\lambda)$ in the scale $\Theta_p$, $p > 0$. 

§5. Trace class integral operators

Here we shall discuss the realization of trace class operators in the form of integral operators in the decomposition of a Hilbert space into a direct integral. The facts presented here are needed in the next section in the study of the SM for trace class perturbations.

1. Let $H$ be an arbitrary selfadjoint operator, and suppose that (1.5.6) is the decomposition of its absolutely continuous subspace $H^{(a)}$ into a direct integral. For a trace class operator $A$ in $H$ we construct the kernel of the operator $P A P$. Just as any Hilbert-Schmidt operator, the trace class operator $P A P$ is an integral operator (see Part 5 of §1.6), and for its kernel the quantity (1.6.10) is finite. Now, however, it is important to ascribe to the kernel $a(\mu, \nu)$ values on the direct product $A \times A$, where $A$ is some set of full measure in $\delta$.

We start from the results of §5.4. An operator $A \in \Theta_1$ can be represented in the form (5.4.1) where $G : H \to \Theta$ is a Hilbert-Schmidt operator, while the operator $A : \Theta \to \Theta$ is bounded. According to Theorem 6.1.5 any Hilbert-Schmidt operator is weakly $H$-smooth. Thus, Definition 5.4.2 gives the required interpretation of the kernel of the operator $P A P$. We recall that the operator $Z(\lambda; G)$ in equality (5.4.6) is given by relation (5.4.2).

Here we consider some additional properties of the kernel $a(\mu, \nu)$ valid for $A \in \Theta_1$. We note first of all that, according to Theorem 6.1.5, the operator $A(\lambda; G)$, equal to the limit (5.1.1) or (5.1.4), belongs to the class $\Theta_1$. Therefore by equality (5.4.4) for $G \in \Theta_2$, $Z(\lambda; G) \in \Theta_2$, a.e. $\lambda \in \delta$. 

\[
Z(\lambda; G) \in \Theta_2, \quad \text{a.e. } \lambda \in \delta. \tag{1}
\]
Further, according to Corollary 6.1.7, for \( G \in \Theta_2 \), any \( f \in \mathcal{F} \), and a.e. \( \lambda \in \mathbb{R} \) there exist the strong derivative \( dG\mathcal{E}(\lambda)f/d\lambda \) and the strong limit of \( G\delta(\lambda, \varepsilon)f \) as \( \varepsilon \to 0 \). This makes it possible with the help of Lemma 5.4.7 to recover the sesquilinear form of the kernel \( a(\mu, \nu) \). We have thus established

**Proposition 1.** For any trace class operator \( A \) in \( \mathcal{F} \) the kernel \( a(\mu, \nu) \) of the operator \( \mathcal{P} \) is well defined by equality (5.4.6) on a square \( \Lambda \times \Lambda \) of full measure in \( \tilde{\sigma} \times \tilde{\sigma} \). For \( \mu, \nu \in \Lambda \) the operator \( a(\mu, \nu): h(\nu) \to h(\mu) \) belongs to the trace class. Moreover, for any \( f, g \in \mathcal{F} \) equality (5.4.13) holds, where the limit \( G \) on the right can be understood as a double limit.

For an operator \( G \in \Theta_2 \) a somewhat more graphic form can be given to definition (5.4.2) of the mapping \( Z(\lambda; G) : \Theta \to h(\lambda) \). For an arbitrary orthonormal basis \( q_n \) in \( \Theta \) we consider the expansion

\[
G = \sum_n \gamma_n(\cdot, r_n)q_n, \tag{2}
\]

where (see Part 5 of §1.6)

\[
\gamma_n = \|G^*q_n\|, \quad \sum_n \gamma_n^2 < \infty, \quad r_n = \gamma_n^{-1}G^*q_n. \tag{3}
\]

Let \( \tilde{r}_n = \mathcal{T}r_n \). We denote by \( \Lambda \) the set of points \( \lambda \) on which all the functions \( \tilde{r}_n(\lambda) \) are defined and

\[
r^2(\lambda) := \sum_n \gamma_n^2\|\tilde{r}_n(\lambda)\|_{h(\lambda)}^2 < \infty. \tag{4}
\]

This set has full measure in \( \tilde{\sigma} \), since by (3) the function (4) is integrable over \( \tilde{\sigma} \). Substituting now (2) into (5.4.2), we obtain the representation

\[
Z(\lambda; G) = \sum_n \gamma_n(\cdot, r_n)\tilde{r}_n(\lambda), \quad \lambda \in \Lambda. \tag{5}
\]

This operator is bounded, since by the orthonormality of \( q_n \)

\[
\|Z(\lambda; G)\|_{\mathcal{F}} \leq \sum_n \|\psi, q_n\|_2 \gamma_n^2\|\tilde{r}_n(\lambda)\|_{h(\lambda)}^2 < \|\psi\|^2r^2(\lambda)
\]

and the right-hand side is finite by condition (4).

Proceeding from (5), it is easy to give a direct proof of relation (1). Namely, by definition (5) \( Z(\lambda; G)q_n = \gamma_n\tilde{r}_n(\lambda) \). Computing the Hilbert-Schmidt norm of the operator \( Z(\lambda; G) \) by means of an equality of the form (1.6.12), we find that for \( \lambda \in \Lambda \)

\[
\|Z(\lambda; G)\|_{h(\lambda)}^2 = \sum_n \|Z(\lambda; G)q_n\|^2 = r^2(\lambda) < \infty.
\]

The operator \( Z(\lambda; G) \) can be constructed even simpler in the case where \( \Theta = \mathcal{F} = H_2 \) and \( G \in \Theta_2 \) is realized as an integral operator whose kernel \( g(\lambda, \mu) \) satisfies condition (1.6.16). We denote by \( \Lambda \) the set of those \( \lambda \in \Lambda \) where

\[
\int_\Lambda |g(\lambda, \mu)|^2 d\mu < \infty. \tag{6}
\]

Since the function (6) is integrable on \( \lambda \), the set \( \partial \Lambda \) has measure zero. Now in accordance with (5.4.2)

\[
Z(\lambda; G)f = \int_\Lambda g^*(\mu, \lambda)f(\mu) d\mu. \tag{7}
\]

By the Schwarz inequality under condition (6) this integral converges for any \( f \in \mathcal{F} \). Moreover, the Hilbert-Schmidt norm of the operator \( Z(\lambda; G) \) is equal to the square root of the integral (6) and is hence finite for \( \lambda \in \Lambda \).

2. Together with Definition 5.4.2 there also exist other means making it possible to ascribe a kernel to an operator of the class \( \Theta_1 \) on a measurable square of full measure. One of the most natural of them is by approximating a trace class operator by finite-dimensional operators. To the one-dimensional operator \( A = (\cdot, \mu) \) there corresponds the kernel \( a(\mu, \nu) = (\cdot, \tilde{u}(\nu))\tilde{v}(\mu) \) defined on the square \( \Lambda \times \Lambda \), where \( \Lambda \) is the set on which the functions \( \tilde{u} \) and \( \tilde{v} \) are defined. The kernel of a finite-dimensional operator can be constructed similarly. In Part 5 of §1.6 the correspondence between operators and kernels was extended to the Hilbert-Schmidt class. However, the series (1.6.17) converged only in the metric (1.6.16), and therefore the sum of it was defined on a set of full measure in \( \tilde{\sigma} \times \tilde{\sigma} \) not having, generally speaking, the structure of a direct product. We shall now see that for operators of \( \Theta_1 \) that procedure ascribes to the kernel values on a measurable square of full measure.

Let

\[
A = \sum_n \alpha_n(\cdot, u_n)v_n, \tag{8}
\]

where

\[
\alpha_n \geq 0, \quad \|u_n\| = \|v_n\| = 1, \quad \sum_n \alpha_n < \infty. \tag{9}
\]

Orthonormality of \( v_n \) (or of \( u_n \)) is not assumed here. The series (8) converges in the norm of \( \Theta_1 \), so that its sum is the trace class operator. For any \( A \in \Theta_1 \) a representation of the form (8) is always possible—the canonical decomposition (1.6.3) gives one such example. We consider a set \( \Lambda \) on which

\[
\sum_n \alpha_n\|\tilde{u}_n(\lambda)\|^2_h < \infty, \quad \sum_n \alpha_n\|\tilde{v}_n(\lambda)\|^2_h < \infty. \tag{10}
\]

Under condition (9) the set \( \partial \Lambda \) has measure zero. On the square \( \Lambda \times \Lambda \) the kernel \( a(\mu, \nu) \) can be defined by an equality of the form (1.6.17), i.e.,

\[
a(\mu, \nu) = \sum_n \alpha_n(\cdot, \tilde{u}_n(\nu))\tilde{v}_n(\mu). \tag{11}
\]

By the estimates

\[
|a(\mu, \nu)| \leq \sum_n \alpha_n\|\tilde{u}_n(\nu)\|\|\tilde{v}_n(\mu)\| \leq \left( \sum_n \alpha_n\|\tilde{u}_n(\nu)\|^2 \sum_n \alpha_n\|\tilde{v}_n(\mu)\|^2 \right)^{1/2}. \tag{12}
\]
and (10) the series (11) converges on $\Lambda \times \Lambda$ in the trace norm. Formula (11) is also valid in the case where $u_n$ and $v_n$ are determined by the expansion into the series (8) of the operator $PAP$ (rather than $A$).

For operators $A \in \Theta_1$ definitions (11) and (5.4.6) coincide. Indeed, for an arbitrary orthonormal basis $\{z_n\}$ we write the operator (8) in the form $A = G^*G_0$ where

$$G_0 = \sum_n \gamma_n(\cdot, u_n)z_n, \quad G = \sum_n \gamma_n(\cdot, v_n)z_n, \quad \gamma_n = \alpha_n^{1/2}$$

are the Hilbert-Schmidt operators. According to Definition 5.4.2, the kernel of $PAP$ can be found by the formula

$$a(\mu, \nu) = Z(\mu; G)Z^*(\nu; G_0).$$

Substituting here the expressions (5) for the operators $Z(\cdot; G_0), Z(\cdot; G)$, we again arrive at the representation (11) for the kernel $a(\mu, \nu)$.

In the case $\Theta = \mathcal{H} = \mathcal{B}(\mathcal{H})$ still another expression for $a(\mu, \nu)$ can be obtained on realizing the Hilbert-Schmidt operators $G_0$ and $G$ as integral operators with square-integrable kernels $g_0(\mu, \nu)$ and $g(\mu, \nu)$. Suppose $A = G^*G_0$ and $\Lambda$ consists of points $\lambda$ where the integral (6) and the analogous integral of $|g_0(\mu, \lambda)|^2$ are finite. Substituting into (13) the expression (7) for $Z(\mu; G)$ and an analogous expression for $Z(\nu; G_0)$, we find that

$$a(\mu, \nu) = \int_\delta g^*(\lambda, \mu)g_0(\lambda, \nu) d\lambda.$$  

(14)

This integral also defines the kernel $a(\mu, \nu)$ of the operator $A$ on the set $\Lambda \times \Lambda$.

We have shown that for operators $A \in \Theta_1$ the representations (11) and (14) on a measurable square $\Lambda \times \Lambda$ of full measure can be obtained from the general Definition 5.4.2. Thus in all the versions of construction the sesquilinear kernel $a(\mu, \nu)$ can be recovered on $\Lambda \times \Lambda$ by relations (1.5.13) or (5.4.13). From this it follows that for $A \in \Theta_1$ the kernel $a(\mu, \nu)$ of the operator $PAP$ is well defined on the square $\Lambda \times \Lambda$ (while for $A \in \Theta_2$ such a set of full measure in $\delta \times \delta$ may not have the structure of a direct product). In particular, for a.e. $\lambda \in \delta$ the diagonal values $a(\lambda, \lambda)$ do not depend on the manner of constructing the kernel.

3. We note two elementary assertions regarding kernels of operators in $\Theta_1$. The first of them generalizes the well-known properties of integral operators in $L_2$ spaces.

**Proposition 2.** Suppose $A \in \Theta_1$ and $a(\mu, \nu)$ is the kernel of $PAP$ in the decomposition (1.5.6). Then

$$\int_\delta |a(\lambda, \lambda)|_1 d\lambda \leq \|PAP\|_1,$$

$$\text{Tr} PAP = \int_\delta \text{Tr}_{\Theta(\delta)} a(\lambda, \lambda) d\lambda.$$  

(15)

(16)

**Proof.** Consider the canonical decomposition of the operator $PAP = \sum_n a_n(\cdot, u_n)v_n, \quad u_n \in \mathcal{H}(\delta), \quad v_n = \mathcal{H}(\delta)$, and define the kernel $a(\mu, \nu)$ by equality (11). We use the estimate (12) for it. Setting in (12) $\mu = \nu = \lambda$, integrating over $\lambda \in \delta$, and applying the Schwarz inequality, we find that

$$\int_\delta |a(\lambda, \lambda)|_1 d\lambda \leq \sum_n \alpha_n.$$

Since $\alpha_n$ are the singular numbers of the operator $A$, the right-hand side is equal to $\|PAP\|_1$.

Further, according to (11),

$$\text{Tr}_{\Theta(\delta)} a(\lambda, \lambda) = \sum_n a_n(\theta_n(\lambda), \theta_n(\lambda)).$$

Integrating this relation over $\lambda \in \delta$, we find that the right-hand side is equal to $\sum_n a_n(v_n, v_n)$. The left-hand side of (16) is also equal to this sum.

With the help of inequality (15) it is possible to give still another method of computing the diagonal values of the kernel. This method does not require preliminary construction of the kernel on the measurable square of full measure. We consider some sequence of finite-dimensional operators $A_n$ converging to $A$ in $\Theta_1$. For $A_n$ the diagonal values $a_n(\lambda, \lambda)$ are well defined, and by (15)

$$\int_\delta |a_n(\lambda, \lambda) - a_n(\lambda, \lambda)|_1 d\lambda \leq \|A_n - A_m\|_1 \to 0$$

as $n, m \to \infty$. From this it follows that in the metric of the integral on the left the sequence $a_n(\lambda, \lambda)$ has a limit which is taken as $a(\lambda, \lambda)$. Inequality (15) is also satisfied for this quantity.

The next assertion is one of the versions of the Parseval equality.

**Proposition 3.** Suppose $\mathcal{B}_H$ is the set of elements $f \in \mathcal{H}(\delta)$ with a finite quantity (2.5.2). Then for any $A \in \Theta_1$ and $f, g \in \mathcal{B}_H$

$$\iint_\delta \langle AU(t)f, U(t)g \rangle dt = 2\pi \int_\delta (a(\lambda, \lambda)f(\lambda), \check{g}(\lambda)) d\lambda.$$  

(17)

**Proof.** We start from some expansion of the form (8). We write the left-hand side of (17) in the form

$$\sum_n a_n \int_{-\infty}^{\infty} \langle U(t)f, u_n \rangle(v_n, U(t)g) dt.$$  

(18)

By inequality (2.5.3) the interchange of integration on $t$ and summation on $n$ performed here is justified by appeal to Fubini's theorem. The factors $(U(t)f)u_n$ and $(U(t)g, v_n)$ are the Fourier transforms of the functions $(2\pi)^{1/2}(f(\lambda), \check{u}_n(\lambda))$ and $(2\pi)^{1/2}(g(\lambda), \check{v}_n(\lambda))$ extended by zero off $\delta$. Therefore, according to the Parseval equality, the expression (18) is equal to

$$2\pi \sum_n a_n \int_{-\infty}^{\infty} (f(\lambda), \check{u}_n(\lambda))(v_n(\lambda), \check{g}(\lambda)) d\lambda.$$  

(19)

For $f, g \in \mathcal{B}_H$ the functions $f$ and $g$ are bounded, so that there is absolute convergence in (19). Finally, we interchange summation on $n$ and
integration on \( \lambda \). By (11) the sum on \( n \) is equal to \( (a(\lambda, \lambda), \tilde{f}(\lambda), \tilde{g}(\lambda)) \). Thus, the expression (19) coincides with the right-hand side of (17). \( \square \)

\[ \text{§6. The SM for Trace Class Perturbations} \]

In Part 1 it is shown that the usual stationary representations of §2.8 for the SM hold under trace class perturbations. In Part 2 properties of the SM are considered which are specific for such perturbations. Finally, continuous dependence of the SM on the perturbation is established in Part 3.

1. We now apply the considerations already used in Part 2 of §3 in the derivation of representations of the WO and scattering operator. The information obtained in §5 regarding the representation of trace class operators as integral operators is also needed for the construction of the SM.

We start from Theorem 5.5.3. Suppose the operator \( T_a(z) \) is defined by formula (2.8.4). As in §3, we first suppose that \( V = H \tilde{J} + \mathbb{H} \mathbb{H}_0 \in \mathcal{E}_1 \).

Then by Theorem 6.1.9 and Corollary 6.1.11 for any factorization \( V = G^*G \), the conditions of Theorem 5.5.3 are satisfied. Then the kernels \( t_{\pm} (\mu, \nu; \lambda + i0) \) (in the decomposition into a direct integral (2.4.2)) of the operators \( P_0 T_{\pm} (\lambda + i0) P_0 \) are well defined by an equality of the form (5.4.6), while both representations (2.8.9) hold for \( S(\lambda) \). We recall that in a precise sense these representations are realized in the form (5.5.3).

These representations are little changed in the general case, when only condition (6.4.1) is satisfied. Namely for any bounded interval \( \Lambda \) the operator \( E_0(\Lambda) T_a(\lambda + 0) E_0(\Lambda) \) has a limit as \( \epsilon \to 0 \) for a.e. \( \lambda \in \mathbb{R} \), and the kernel of the limit operator is well defined for a.e. \( \mu \in \delta_0 \) and a.e. \( \nu \in \delta_0 \). Passing to the new identification \( J_a = 3E_0(\Lambda) \), we see that the representations (2.8.9) are preserved if by \( t_{\pm}(\lambda, \lambda; \lambda + 0) \) we understand the values at the point \( \mu = \nu = \lambda \) of the kernel of the operator \( E_0(\Lambda) T_a(\lambda + 0) E_0(\Lambda) \), where \( \lambda \in \Lambda \). The operator-valued function \( u_{\pm}(\lambda) \) in (2.8.9) now corresponds to the decomposition (2.4.2) into the WO \( \tilde{W}_{\pm}(H_0, H_\mathbb{H}; \mathcal{S}) \) or to the weak WO \( \tilde{W}_{\pm}(H_0, H_\mathbb{H}; \mathcal{S}) \). Of course, the representations for \( S(\lambda) \) do not depend on the choice of \( \Lambda \) if \( \lambda \in \mathcal{S} \).

Suppose now that only local trace class condition (6.4.2) is satisfied. Then only the weak WO \( \tilde{W}_{\pm} = \tilde{W}_{\pm}(H_0, H_\mathbb{H}; \mathcal{S}) \) exist, and \( S(\lambda) \) can be constructed on the basis of the weak scattering operator \( \tilde{W}^{-1}_{\pm} \tilde{W}_{\pm} \). We set

\[ J_a = E(\lambda) 3E_0(\Lambda), \quad V = H J_a - \mathcal{S}_a H_0, \]

\[ \tilde{V}_{\pm}(\Lambda) = W_{\pm}(H_0, H_\mathbb{H}; J_a V_a) \]

and in accordance with (2.8.4), we set

\[ T_a(z; \lambda) = J_a V_a - V_a^* R(z) V_a. \]

The operator \( T_a(z; \lambda) \) is constructed in a similar way. Suppose in the decomposition (2.4.2) the operator \( T_a(\lambda + 0; \Lambda) \) is given by the kernel \( t_{\pm}(\mu, \nu; \lambda + i0; \Lambda) \), while the operator \( \tilde{V}_{\pm}(\Lambda) \) acts as multiplication by \( u_{\pm}(\lambda; \Lambda) \). Then by Theorem 5.5.3 there is the representation

\[ S(\lambda) = u_{\pm}(\lambda; \Lambda) - 2\pi i u_{\pm}(\lambda; \lambda; \lambda + i0; \Lambda), \quad \text{a.e.} \ \lambda \in \Lambda. \]

On the right-hand side of (1) both terms, generally speaking, depend on the choice of the interval \( \Lambda \ni \lambda \). We now additionally suppose that the strong WO \( W_{\pm}(H_0, H_\mathbb{H}; \mathcal{S}) \) exists and hence also the weak WO \( \tilde{W}_{\pm}(H_0, H_\mathbb{H}; \mathcal{S}) \). Then

\[ \begin{align*}
& \quad \lim_{t \to \infty} E(X) J U_0(t) E_0(X) P_0 = 0. \\
& \quad \lim_{t \to \infty} E(X) J U_0(t) E_0(X) E_0(X) P_0 = 0.
\end{align*} \]

From this it follows that

\[ W_{\pm}(H_0, H_\mathbb{H}; \mathcal{S}) = \tilde{W}_{\pm}(H_0, H_\mathbb{H}; \mathcal{S}) \]

and consequently \( u_{\pm}(\lambda; \Lambda) = u_{\pm}(\lambda) \langle \chi \rangle \) where \( u_{\pm}(\lambda) \) corresponds to the WO \( \tilde{W}_{\pm}(H_0, H_\mathbb{H}; \mathcal{S}) \). Thus, if the strong WO \( W_{\pm}(H_0, H_\mathbb{H}; \mathcal{S}) \) exists the corresponding SM has the representation (1) with \( u_{\pm}(\lambda; \Lambda) \) in place of \( u_{\pm}(\lambda; \Lambda) \). In this case, of course, the term \( t_{\pm}(\lambda, \lambda; \lambda + i0; \Lambda) \) does not depend on \( \Lambda \). It is not, however, possible to drop \( \Lambda \) here since \( T_a(z) \) and even \( E_{\pm}(\Lambda) T_a(z) E_{\pm}(\Lambda) \) may not have an operator meaning.

Finally, under the conditions of Theorem 6.5.3 new representations for the SM \( S(\lambda) \) can be obtained by means of Theorem 2.1 (the IP). We recall that the operators \( T_a(z; \varphi) \) are defined by equalities (2.2) while \( t_{\pm}(\mu, \nu; z; \varphi) \) are their kernels in the decomposition (2.4.2).

We collect the results obtained.

**Theorem 1.**

1. For \( V \in \mathcal{E}_1 \) both representations (2.8.9) hold for \( S(\lambda) \).

2. Under condition (6.4.1) and for any bounded interval \( \Lambda \) the representations (2.8.9) are preserved for a.e. \( \lambda \in \Lambda \) if \( t_{\pm}(\mu, \nu; \lambda + i0) \) is the kernel of the operator \( E_{\pm}(\Lambda) T_a(\lambda + i0) E_{\pm}(\Lambda) \).

3. Under condition (6.4.2) both representations (1) hold. If, moreover, there exists the strong WO \( W_{\pm}(H_0, H_\mathbb{H}; \mathcal{S}) \), then in (1) \( u_{\pm}(\lambda; \Lambda) \) can be replaced by the operator-valued function \( u_{\pm}(\lambda) \) corresponding to the WO \( \tilde{W}_{\pm}(H_0, H_\mathbb{H}; \mathcal{S}) \).

4. Under the conditions of Theorem 6.5.3 the representations (2.4) hold for \( S(\lambda) \).

**Corollary 2.** Under condition (6.4.1) there are the relations

\[ S(\lambda) - u_{\pm}(\lambda; \Lambda) \in \mathcal{E}_1, \quad \text{a.e.} \ \lambda \in \delta_0. \]

Relation (2) remains in force also under assumption (6.4.2) if it is additionally known that the strong WO \( W_{\pm}(H_0, H_\mathbb{H}; \mathcal{S}) \) exists. If, moreover, \( W_{\pm}(H_0, H_\mathbb{H}; \mathcal{S}) \) is isometric, then \( u_{\pm}(\lambda) = I(\lambda) \) in (2).
7. Properties of the Scattering Matrix

Proof. Under condition (6.4.1) $VE_{0}(\Lambda) = G^{T}_{\Lambda}G_{0,\Lambda}$, where both factors belong to the class $\mathcal{S}_{1}$. Therefore, by Theorem 6.1.9 the operator $E_{0}(\Lambda)T_{\Lambda}(\lambda + it)E_{0}(\Lambda) \in \mathcal{S}_{1}$. The inclusion (2) now follows from the representation (2.8.9) and Proposition 5.1. The second assertion can be established in an altogether similar way with the help of the representation (1).

Corollary 3. Suppose for any bounded interval $\Lambda$ the operator $VE_{0}(\Lambda)$ has finite rank $r$. Then for a.e. $\lambda \in \Lambda$, the rank of $S(\lambda) - u_{\pm}(\lambda)$ does not exceed $r$. This happens, in particular, if the operator $V$ itself is finite-dimensional.

Proof. By (5.5) the rank of the operator $Z(\lambda; G)$ for a.e. $\lambda$ does not exceed $r$ if the rank of $G$ is equal to $r$. It remains to use the representation (5.5.3).

In the case $\mathcal{K}_{0} = \mathcal{K}$, $J = I$ stationary representations for the SM hold also under the conditions of Theorem 6.4.15. Namely, we have

Theorem 4. Under the conditions of Theorem 6.4.15 for a.e. $\lambda \in \mathcal{K}$ the representation (5.7.3) holds for $S(\lambda)$ and $S(\Lambda) - I(\lambda) \in \mathcal{S}_{1}$.

Proof. In the proof of Theorem 6.4.15 it was shown that under its conditions the conditions of Theorem 5.7.1 are also satisfied. Thus, the representation (5.7.3) holds where, according to (5.1), the operators

$$Z_{0}(\lambda; G_{0}) = Z_{0}(\lambda; G_{0}E_{0}(\Lambda)), \quad Z_{0}(\lambda; G) = Z_{0}(\lambda; GE_{0}(\Lambda)), \quad \lambda \in \Lambda,$$

belong to the Hilbert-Schmidt class.

2. We shall now discuss a somewhat more special representation of the SM adapted just to the theory of trace class perturbations. Suppose the WO $W_{\pm} = W_{\pm}(H, \Lambda)$ exist. Because of the intertwining property of the WO for the sesquilinear form of the scattering operator there is the representation

$$\langle [S - W_{+}^{*}W_{+}]f, g \rangle = \langle (W_{-} - W_{+})f, W_{+}g \rangle = -i \int_{-\infty}^{\infty} \langle VU_{0}(t)f, U(t)W_{+}g \rangle dt$$

and hence

$$S(\lambda) = u_{\pm}(\lambda) - 2\pi i\omega_{\pm}(\lambda, \lambda).$$

In an entirely similar manner, by considering the operator $S - W_{+}^{*}W_{-} = (W_{+} - W_{-})^{*}W_{-}$, it can be shown that

$$S(\lambda) = u_{\pm}(\lambda) - 2\pi i\omega_{\pm}(\lambda, \lambda).$$

We recall that $u_{\pm}(\lambda)$ is the operator-valued function corresponding to the operator $W_{\pm}^{*}W_{\pm} = W_{\pm}(H_{0}, H_{0}; J)$. The representations (4) extend easily to the case where only local trace class conditions are satisfied. This generalization is realized in complete analogy to Part 1. We formulate only the final result.

Theorem 5.

1. For $V \in \mathcal{S}_{1}$ both representations (4) hold for $S(\lambda)$.

2. Under condition (6.4.1), for any bounded interval $\Lambda$ these representations are preserved for a.e. $\lambda \in \Lambda$ if $\omega_{\pm}(\mu, \nu)$ is the kernel of the operator $W_{\pm}^{*}VE_{0}(\Lambda)$.

3. Under condition (6.4.2) we denote by $\omega_{\pm}(\mu, \nu; \Lambda)$ the kernel of the operator $W_{\pm}^{*}E(\Lambda)VE_{0}(\Lambda) \in \mathcal{S}_{1}$. Then both representations

$$S(\lambda) = u_{\pm}(\lambda; \lambda) - 2\pi i\omega_{\pm}(\lambda, \lambda), \quad \lambda \in \Lambda,$$

hold. If, moreover, the strong WO $W_{\pm}(H, H_{0}; J)$ exists, then in (5) $u_{\pm}(\lambda)$ can be replaced by $u_{\pm}(\lambda)$.

The inclusions (2) again follow from Theorem 5. Moreover, the representations (4) can be used for an effective estimate of the trace norm of the operator (2).

Theorem 6. Suppose condition (6.4.2) holds and the strong WO $W_{\pm}(H, H_{0}; J)$ exists. Then for any bounded interval $\Lambda$

$$\int_{\Lambda} |S(\lambda) - u_{\pm}(\lambda)| d\lambda \leq 2\pi \|E^{(a)}(\Lambda)V E_{0}(\Lambda)\|_{1}.$$

For $V \in \mathcal{S}_{1}$ it may here be assumed that $\Lambda = \mathcal{K}$.

Proof. We apply the estimate (5.15) to the trace class operator $W_{\pm}^{*}E(\Lambda)VE_{0}(\Lambda)$ and consider the representation (5), where $u_{\pm}(\lambda; \lambda) = u_{\pm}(\lambda)$. Then the left-hand side of (6) is bounded by $2\pi \|W_{\pm}^{*}E(\Lambda)VE_{0}(\Lambda)\|_{1}$, which does not exceed the right-hand side of (6).

We have formulated the estimate (6) under the conditions of Part 3) of Theorem 5. It is thus also valid under the conditions of Parts 1) and 2) of it. The estimate (6) can, as usual, be combined with the IP. Namely, under the conditions of Theorem 6.5.3 we have the inequality

$$\int_{\mathcal{K}} |S(\lambda) - u_{\pm}(\lambda)| |\varphi(\lambda)| d\lambda \leq 2\pi \|\varphi(H)J - \varphi(H_{0})\|_{1}.$$

Comparing the estimates (6) and (1.11), we immediately obtain
7. PROPERTIES OF THE SCATTERING MATRIX

Proposition 7. Suppose both WO $W_\pm(H_1, H_0; \mathcal{I}_0)$ exist and $(H\mathcal{I}_1 - \mathcal{I}_1 H_1)E_H(\Lambda) \in \mathcal{G}_1$ for any bounded interval $\Lambda \subset \mathbb{R}$. Then for $\mathcal{I} = \mathcal{I}_1 \mathcal{I}_0$

$$\int_\Lambda |S(\lambda; H, H_0; \mathcal{I}) - S(\lambda; H_1, H_0; \mathcal{I}_0)|^2 d\lambda \leq 2\pi \|\mathcal{I}\|_2^2 \|\Lambda\|_2 \|H\mathcal{I}_1 - \mathcal{I}_1 H_1)E_H(\Lambda)\|_1. \quad (8)$$

We note that in (8) the operator $(H\mathcal{I}_1 - \mathcal{I}_1 H_1)E_H(\Lambda)$ can be replaced by $E_H(\Lambda)(H\mathcal{I}_1 - \mathcal{I}_1 H_1)E_H(\Lambda)$ if the existence of the WO $W_\pm(H, H_1; \mathcal{I}_1)$ is additionally required. Moreover, according to (7), it is possible to estimate an integral of the form (8) by $\|\phi(H\mathcal{I}_1 - \mathcal{I}_1 \phi(H_1))\|_1$.

3. We shall now discuss the continuity of the SM in dependence on the perturbation. Suppose the family of operators $H(\gamma)$ satisfies the condition

$$\lim_{\gamma \to 0} \|H(\gamma)\mathcal{I}_1 - \mathcal{I}_1 H_1)E_H(\Lambda)\|_1 = 0,$$

where $\Lambda$ is any bounded interval. From (8) it then follows that

$$S(\lambda; H(\gamma), H_0; \mathcal{I})$$

is continuous as $\gamma \to 0$ in the metric of the integral on the left-hand side. By the familiar theorem of F. Riesz (see, for example, the book [15]) from this it follows that along some sequence $\gamma_n \to 0$ for a.e. $\lambda \in \mathcal{G}_0$

$$\lim_{\gamma_n \to 0} |S(\lambda; H(\gamma_n), H_0; \mathcal{I}) - S(\lambda; H_1, H_0; \mathcal{I}_0)|_1 = 0. \quad (9)$$

In the special case where the family $H(\gamma)$ depends linearly on the parameter $\gamma$, a relation of the form (9) is valid for all $\gamma \to 0$. The proof of this assertion requires an entirely different technique. In this regard see Part 4 of the next section.

§7. THE STRUCTURE OF THE STATIONARY REPRESENTATION OF THE SM

In this section we consider arbitrary operators which in their structure copy the stationary representation of the SM for the case $\mathcal{K}_0 = \mathcal{K}$, $\mathcal{I} = \mathcal{I}$. This makes it possible to study many properties of the SM in a maximally general situation.

1. Let $\mathfrak{h}$ and $\mathfrak{G}$ be abstract Hilbert spaces, and suppose the operator $S$ in $\mathfrak{h}$ has the form

$$S = I - 2\pi i \mathcal{V} (I + \mathcal{B} \mathcal{V})^{-1} \mathcal{V}^*. \quad (1)$$

It is assumed that

$$\mathcal{V} = \mathcal{V}^* \in \mathcal{B}(\mathfrak{G}), \quad (2)$$

while the operators $\mathcal{I}: \mathfrak{G} \to \mathfrak{h}$ and $\mathcal{B}: \mathfrak{G} \to \mathfrak{G}$ are bounded and satisfy the relation

$$2\pi i \mathcal{I}^* \mathcal{I} = \mathcal{B} - \mathcal{B}^*. \quad (3)$$

Of course, equality (3) requires that the operator $\mathcal{B}$ have a nonnegative imaginary part.

It is clear that the representation (5.7.9), obtained in Theorem 5.7.1'A for the SM $S(\lambda; H, H_0)$, has the form (1). Here the operators $Z_0(\lambda; G)$ and $B_0(\lambda + i \delta)$ play the roles of $\mathcal{Z}$ and $\mathcal{B}$, while relation (5.7.7) plays the role of (3). We recall that in Theorem 5.7.1'A the existence of an inverse operator on the right-hand side of (5.7.9) was assumed for a.e. $\lambda$.

We suppose that either the inverse operator in (1) exists in the usual sense or that under the additional condition $\mathcal{B} \in \mathcal{S}_\infty$, this inverse is understood in the generalized sense indicated in Part 3 of §1.6.

According to (1), the subspaces $\mathcal{R}(\mathcal{Z})$ and $\mathcal{N}(\mathcal{Z}^*)$ are invariant relative to the operator $S = S(\mathcal{Z}, \mathcal{B}, \mathcal{V})$, and on $\mathcal{N}(\mathcal{Z}^*)$ the operator $S$ reduces to the identity. Thus, in studying the operator (1), it suffices to consider its restriction $S'$ to the subspace $\mathcal{R}(\mathcal{Z})$. Up to unitary equivalence these operators do not depend on the choice of the operator $\mathcal{Z}$ satisfying relation (3). Namely, we have

**Lemma 1.** Let

$$\mathcal{Z}' = ((2\pi i)^{-1}(\mathcal{B} - \mathcal{B}^*))^{1/2}, \quad \mathcal{Z}' \geq 0. \quad (4)$$

Then the operators $S'(\mathcal{Z}, \mathcal{B}, \mathcal{V})$ and $S'(\mathcal{Z}', \mathcal{B}, \mathcal{V})$ are unitarily equivalent to one another.

**Proof.** We start from the polar representation (see Part 1 of §1.6) of the operator $\mathcal{Z} = F\mathcal{Z}$, where $F$ is a unitary mapping of $\mathcal{R}(\mathcal{Z})$ onto $\mathcal{R}(\mathcal{Z})$. Comparing (3) and (4), we find that $\mathcal{Z} = |\mathcal{Z}|$ and therefore by (1)

$$S'(\mathcal{Z}, \mathcal{B}, \mathcal{V})F = FS'(\mathcal{Z}', \mathcal{B}, \mathcal{V}). \quad \square$$

We proceed a further investigation of the operator (1) with an observation of technical character. The need for it drops out if the inverse operator in (1) exists in the usual sense.

**Lemma 2.** Under conditions (2) and (3) there are the relations

$$\mathcal{N}(I + \mathcal{B}\mathcal{V}) \cap \mathcal{N}(I + \mathcal{B}^*\mathcal{V}) \subset \mathcal{N}(\mathcal{Z})$$

and hence

$$\mathcal{R}(\mathcal{Z}^*) \subset \mathcal{R}(I + \mathcal{B}\mathcal{V}) \cap \mathcal{R}(I + \mathcal{B}^*\mathcal{V}). \quad (5)$$

**Proof.** Suppose, for example, that $f + \mathcal{V}\mathcal{B}f = 0$. We form the scalar product of this equation with the element $\mathcal{B}f$ and take its imaginary part. From the equality $\text{Im}(f, \mathcal{B}f) = 0$ obtained and (3) it follows that $\mathcal{Z}f = 0. \quad \square$

**Corollary 3.** The operator (1) can be written in the form

$$S = I - 2\pi i \mathcal{I}(I + \mathcal{V}\mathcal{B})^{-1}\mathcal{V}^*. \quad (6)$$
PROOF. Let \( P_1 \) and \( P_2 \) be the orthogonal projections onto \( R(I + \mathcal{B} \mathcal{Y}) \), and \( R(I + \mathcal{B}^* \mathcal{Y}) \), respectively. According to (5), \( P_j \mathcal{Y} = \mathcal{Y}^* \), \( j = 1, 2 \). Therefore, to prove (6) it suffices to show that

\[
(I + \mathcal{Y} \mathcal{B})^{-1} \mathcal{Y} P_1 = P_2 \mathcal{Y} (I + \mathcal{B} \mathcal{Y})^{-1}.
\]

To this end it is only necessary to multiply the equality \( \mathcal{Y} (I + \mathcal{B} \mathcal{Y}) = (I + \mathcal{Y} \mathcal{B}) \mathcal{Y} \) by \( (I + \mathcal{B} \mathcal{Y})^{-1} \) on the right, by \( (I + \mathcal{Y} \mathcal{B})^{-1} \) on the left, and consider relations (1.6.7).

2. We have always derived the unitarity of the SM from the isometricity and completeness of the corresponding WO. However, unitarity of the SM can be established also on the basis of its stationary representation.

Theorem 4. Suppose conditions (2) and (3) are satisfied. Then the operator (1) is unitary.

PROOF. We shall verify, for example, that \( S^* S = I \). Passing to the adjoints in (1), we find that

\[
(2\pi i)^{-1} (S^* S - I) = \mathcal{Y} [(I + \mathcal{Y} \mathcal{B})^{-1} \mathcal{Y} - \mathcal{Y} (I + \mathcal{B} \mathcal{Y})^{-1} - 2\pi i (I + \mathcal{Y} \mathcal{B})^{-1} \mathcal{Y} \mathcal{Y} (I + \mathcal{B} \mathcal{Y})^{-1}] \mathcal{Y}^*.
\]

By (1.6.7) and (5)

\[
\mathcal{Y}^* = (I + \mathcal{B} \mathcal{Y})(I + \mathcal{B} \mathcal{Y})^{-1} \mathcal{Y}^*.
\]

Substituting this expression into the right-hand side of (7), we rewrite it in the form

\[
\mathcal{Y} (I + \mathcal{Y} \mathcal{B})^{-1} [(I + \mathcal{Y} \mathcal{B})^{-1} \mathcal{Y} - \mathcal{Y} (I + \mathcal{B} \mathcal{Y})^{-1} - 2\pi i (I + \mathcal{Y} \mathcal{B})^{-1} \mathcal{Y} \mathcal{Y} (I + \mathcal{B} \mathcal{Y})^{-1}] \mathcal{Y}^*.
\]

According to (3), the operator in square brackets is equal to zero.

The proof of the equality \( SS^* = 1 \) can be obtained in an entirely similar way if in place of (1) we use the expression (6) for the operator \( S \) and in place of (8) the equality

\[
\mathcal{Y}^* = (I + \mathcal{B} \mathcal{Y})(I + \mathcal{B} \mathcal{Y})^{-1} \mathcal{Y}^*.
\]

which also follows from (1.6.7) and (5).

By Theorem 4 the spectrum of the operator (1) lies on the unit circle \( T \). We shall show that for small operators \( \mathcal{B} \) the operator \( S \) differs little from the identity, and hence its spectrum is contained in a small neighborhood of the point \( 1 \in T \). With a view to applications to the SM we denote the operator norms in the space \( \mathcal{H} \) by \( \| \cdot \| \) and in other spaces (or a pair of spaces) by \( \| \cdot \| \).

PROPOSITION 5. Suppose \( \| \mathcal{Y} \| \| \mathcal{B} \| = b < 1/2 \). Then the spectrum of the operator (1) lies on the arc \( \{ e^{-i\theta(b)}, e^{i\theta(b)} \} \) where \( 0 < \theta(b) < \pi \) and \( \theta(b) \to 0 \) as \( b \to 0 \).

PROOF. According to (3)

\[
\pi \| \mathcal{Y} \|^2 \leq b \| \mathcal{Y} \|^2.
\]

Therefore, it follows from definition (1) that

\[
|S - I| \leq 2\pi \| \mathcal{Y} \|^2 \| \mathcal{Y} \| (I - \| \mathcal{Y} \| \| \mathcal{B} \| )^{-1} \leq 2b (1 - b)^{-1}.
\]

For \( b < 1/2 \) the right-hand side here is less than 2. Thus, the spectrum of \( S \) lies on the arc \( \{ e^{-i\theta(b)}, e^{i\theta(b)} \} \) where \( \sin(\theta(b)/2) = b (1 - b)^{-1} \), so that \( \theta(b) \to 0 \) as \( b \to 0 \).

Additional information regarding the spectrum of the operator \( S \) can be obtained for compact operators \( \mathcal{B} \).

PROPOSITION 6. Suppose \( \mathcal{B} \in \mathcal{S}_\infty \). Then the spectrum of the operator (1) consists of eigenvalues, which may accumulate only at the point 1. Eigenvalues different from 1 have finite multiplicity.

PROOF. From (3) it follows that \( \mathcal{Y} \in \mathcal{S}_\infty \), and hence by (1) \( S - I \in \mathcal{S}_\infty \). It remains to use Corollary 1.11.2.

3. As an application of the results obtained here we consider the SM under the conditions of Chapter 5. We assume that \( \mathcal{K}_0 = \mathcal{H} \), \( I = \mathcal{I} = I \), and that the perturbation \( \mathcal{V} \) can be represented in the form \( \mathcal{V} = G^* \mathcal{G} \), where the operator \( \mathcal{G} : \mathcal{H} \to \mathcal{B} \) is \( \{ H_1 \} \)-bounded, and \( \mathcal{Y} = \mathcal{Y}^* \in \mathcal{B}(\mathcal{S}) \).

The following agreement is used systematically in discussing the SM. By the definition of \( \| \mathcal{Y} \| \) the SM is defined on some set of points \( \lambda \) of full measure. Without concrete assumptions this set remains undefined. Under conditions ensuring the validity, for example, of the stationary representation (5.7.6), it is natural to assume that it consists of points \( \lambda \) for which all the operators on the right in (5.7.6) are well defined. Moreover, according to Lemma 1, as \( Z_0(\lambda; \mathcal{G}) \) in (5.7.6) it is possible to take any operator satisfying the identity (5.7.7). With this agreement the SM is defined for all \( \lambda \) at which the limits \( B^{(0)}(\lambda \pm i0) \) and the inverse operator (5.7.5) exist. Under the conditions of Theorem 5.7.1 the set \( \mathcal{M} \) of such \( \lambda \) has full measure.

The next assertion follows directly from Theorems 5.7.1', 5.8.1, and Proposition 6.

Theorem 7. Suppose on a Borel set \( \Lambda \) the conditions of Theorem 5.7.1' are satisfied and \( B^{(0)}(z) = GR_0(z)G \in \mathcal{S}_\infty \), \( \text{Im} z \neq 0 \). Then for a.e. \( \lambda \in \Lambda \) the SM \( S(\lambda) = S(\lambda; H, H_0) \) is unitary, \( S(\lambda) - I \in \mathcal{S}_\infty \), and the spectrum of \( S(\lambda) \) consists of eigenvalues which may accumulate only at the point 1. Eigenvalues different from 1 have finite multiplicity.
4. We shall now study the continuity of the SM $S(\lambda; H(\gamma), H_0)$ with respect to the parameter $\gamma$ for the family of Hamiltonians $H(\gamma) = H_0 + \gamma V$. In order that a condition of the form (5.7.5) be satisfied for all $\gamma$, it is simplest to require the existence of the angular limits for the operator-valued function $B^{(0)}(z)$. In comparing the SM for different $\gamma$ it must be born in mind that they are defined (see Part 3) on different sets $\mathcal{M}(\gamma)$ of full measure. In the next assertion the SM are considered for small values of $\gamma$. It is hereby used that for some set of full measure $\mathcal{M}_0$ and every $\lambda \in \mathcal{M}$ there exists $\gamma_0 = \gamma_0(\lambda)$ such that $\lambda \in \mathcal{M}(\gamma)$ for all $\gamma \in (0, \gamma_0)$.

**Theorem 8.** Suppose $H(\gamma) = H_0 + \gamma V$ and the operator-valued function $B^{(0)}(z) = GR_0(z)G^*$ satisfies on the set $\Lambda$ the conditions of Theorem 5.7.2'. Then for a.e. $\lambda \in \Lambda$

$$\lim_{\gamma \to 0} S(\lambda; H(\gamma), H_0) - I_{\Lambda} = 0$$

and, hence, the spectrum of the SM $S(\lambda; H(\gamma), H_0)$ lies on the arcs $[e^{-i\theta(\gamma)}, e^{i\theta(\gamma)}]$, where $\theta(\gamma) \to 0$ as $\gamma \to 0$. If, moreover, $\Im B^{(0)}(\lambda + i0) \in \Theta$, for some $r < \infty$, then

$$\lim_{\gamma \to 0} |S(\lambda; H(\gamma), H_0) - I_{\Lambda}| = 0.$$  

**Proof.** In view of the local version of Theorem 5.7.2' (see §5.8)

$$S(\lambda; H(\gamma), H_0) = I - 2\pi i\gamma Z_0(\lambda; G)(I - \gamma B^{(0)}(\lambda + i0))^{-1} Z_0^*(\lambda; G).$$  

We denote by $\mathcal{M}$ the set of those $\lambda$ where the limits $B^{(0)}(\lambda + i0)$ exist. On this set relation (9) follows directly from Proposition 5 applied to the operator (11).

Under the condition $\Im B^{(0)}(\lambda + i0) \in \Theta$, it follows from (5.7.7) that $Z_0(\lambda; G) \in \Theta_2$. Depending on $\lambda \in \mathcal{M}$, we now choose the number $\gamma$ so small that $\gamma \|B^{(0)}(\lambda + i0)\| \leq 1/2$. Then (11) gives the estimate

$$|S(\lambda; H(\gamma), H_0) - I_{\Lambda}| \leq 4\pi \gamma \|Z_0(\lambda; G)\|_{\Theta_2},$$

which ensures relation (10). □

With the help of the multiplication theorem for the SM this assertion can be extended to a more general situation.

**Corollary 10.** Suppose the WO $W_\pm(H, H_0, \mathfrak{J}, \Lambda)$ exist and the conditions of Theorem 5.7.2' are satisfied for the pair $H_1 = H + \gamma V$ on $\Lambda$. Then for a.e. $\lambda \in \Lambda$

$$\lim_{\gamma \to 0} S(\lambda; H(\gamma), H_0, \mathfrak{J}) - S(\lambda; H_1, H_0, \mathfrak{J}) = 0.$$  

If, moreover, $\Im GR_{H_1}(\lambda + i0)G^* \in \Theta$, then

$$\lim_{\gamma \to 0} |S(\lambda; H(\gamma), H_0, \mathfrak{J}) - S(\lambda; H_1, H_0, \mathfrak{J})| = 0.$$  

**Proof.** We shall clarify, for example, (13). According to Proposition 1.3, the quantity under the limit sign in (13) is bounded by

$$|3|^2 |S(\lambda; H(\gamma), H_1) - I|.$$

It remains to use relation (10). □

In particular, by (12), (13) under the conditions of Theorem 9 for any $\gamma_i$

$$\lim_{\gamma \to \gamma_i} |S(\lambda; H(\gamma), H_1) - S(\lambda; H(\gamma), H_0)| = 0,$$

where the value $r = \infty$ is not excluded, when $\|\cdot\|_{\Theta_\infty}$ becomes the ordinary norm.

We shall apply the estimates obtained to trace class theory. We note that condition (5.5.8), adopted for simplicity of the formulations, is of little consequence. Therefore, the next assertion follows directly from Corollary 10.

**Corollary 11.** Suppose there exist the WO $W_\pm(H, H_0, \mathfrak{J})$, while for the pair $H_1 = H_1 + \gamma V$, where $V = G^*G_1$, the conditions of Theorem 6.4.15 are satisfied. Then relation (13) for $r = 1$ is valid for a.e. $\lambda \in \mathbb{R}$.

**Proof.** It is sufficient to note that, as shown in the proof of Theorem 6.4.15, under its conditions the assumptions of Theorem 5.7.2 are satisfied for $l = 1$. In addition, relations of the form (5.1) should be taken into account. □

The facts about the SM expounded in Parts 3 and 4 can be considered as a generalization of the results of §4 to the more abstract situation. The method used here is essentially the same as in §4.

In analogy to §4, under the conditions of this part it is possible also to construct an expansion of $S(\lambda; H(\gamma), H_0)$ in a power series in $\gamma$, but we shall not consider this. Other results presented here were, on the other hand, not discussed in §4, although they are also true under "smooth" assumptions. Thus, relation (12) is preserved if the WO $W_\pm(H_1, H_0, \mathfrak{J})$ exist and the operator-valued function $GR_{H_1}(z)G^*$ is continuous for $Re z \in \Lambda$ up to the cut along $\Lambda$.

In conclusion we show that under the conditions of §4 the SM $S(\lambda)$ can be chosen continuous in $\Lambda$ even in the case when the strong $H_0$-smoothness of the operator $G$ is replaced by the weaker condition of continuity of the operator-valued function $GR_0(z)G^*$. It is here necessary to assume that the SM is defined by equality (5.7.6), where $Z_0(\lambda; G)$ is any operator satisfying relation (5.7.7). We set (cf. (4))

$$Z_0(\lambda; G) = ((2\pi i)^{-1}(B^{(0)}(\lambda + i0) - B^{(0)}(\lambda - i0)))^{1/2}.$$  

By the inequality

$$\|A^{1/2} - A^{-1/2}\| \leq C\|A_2 - A_1\|^{1/2}, \quad A_j \geq 0,$$

the operators $Z_0(\lambda; G)$, and hence also (5.7.6), are continuous in $\lambda$. 

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§7. STRUCTURE OF THE STATIONARY REPRESENTATION

255
§8. The spectrum of the SM for perturbations of definite sign

Here we establish two related assertions regarding the spectrum of the SM for relatively compact perturbations of definite sign. The first of them is that for positive (negative) perturbations the eigenvalues of the SM may accumulate at the point 1 \( \mu \in T \) only from below (from above). The second asserts that by introduction of an additional positive (negative) perturbation the spectrum of the SM rotates in a clockwise (counterclockwise) direction.

1. We first consider an arbitrary operator of the form (7.1). As previously, conditions (7.2) and (7.3) are assumed. In studying the spectrum of the operator (7.1) it is convenient to introduce its imaginary part

\[ \mathcal{F} = (2i)^{-1}(S - S^*) . \]

This selfadjoint operator has the representation

\[ \mathcal{F} = -2\pi \Re (I + \mathcal{B}^*)^{-1} \mathcal{B}^* . \]

The next assertion follows directly from the spectral theorem and is satisfied for any unitary operator.

**Lemma 1.** If a point \( \mu = e^{i\varphi} \in \sigma(S) \), then \( \Im \mu = \sin \varphi \in \sigma(\mathcal{F}) \). Conversely, if \( \lambda = \sin \varphi \in \sigma(\mathcal{F}) \subset [-1, 1] \), then at least one of the points \( e^{i\varphi} \) or \( e^{i(\pi - \varphi)} \) belongs to \( \sigma(S) \).

We now assume that the operator \( \mathcal{B} \) has definite sign. We first note that the case \( \mathcal{B} \geq 0 \) reduces to \( \mathcal{B} = 1 \). For this it is only necessary to go over to the new operators \( \mathcal{F} = \mathcal{B}^{1/2} \mathcal{B}^{1/2} \) and \( \mathcal{B} = \mathcal{B}^{1/2} \mathcal{B}^{1/2} \). Then

\[ S = I - 2\pi \mathcal{F} (I + \mathcal{B}^*)^{-1} \mathcal{B}^* \]

and for \( \mathcal{F} \) and \( \mathcal{B} \) an equality of the form (7.3) is preserved. Similarly, the case \( \mathcal{B} \leq 0 \) reduces to \( \mathcal{B} = -1 \).

The next two assertions sharpen Propositions 7.5 and 7.6 in the case of definite sign.

**Theorem 2.** Suppose \( \| \mathcal{B} \| \| \mathcal{B}^* \| < 1 \) and \( \mathcal{B} \geq 0 \) or \( \mathcal{B} \leq 0 \). Then the spectrum of \( S \) lies on the closed lower (upper) semicircle.

**Proof.** Suppose, for example, that \( \mathcal{B} \geq 0 \), so that it may be assumed that \( \mathcal{B} = 1 \). Setting \( g = (I + \mathcal{B}^*)^{-1} \mathcal{B}^* f \), we find by (1) that

\[ -(2\pi)^{-1} (\mathcal{F} f, f) = \Re (I + \mathcal{B}^{-1} \mathcal{B}^* f, \mathcal{B}^* f) = \Re (I + \mathcal{B} g, g) \geq (1 - \| \mathcal{B} \|) \| g \|^2 \geq 0 . \]

Therefore, \( \mathcal{F} \geq 0 \), and hence the desired assertion follows directly from Lemma 1. \( \Box \)

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\[ \text{§8. The spectrum of the SM} \]

**Theorem 3.** Let \( \mathcal{B} \in G_{\infty} \) and \( \mathcal{B} \geq 0 \) (or \( \mathcal{B} \leq 0 \)). Then the eigenvalues of the operator \( S \) can accumulate at the point 1 only from below (above).

**Proof.** Since 1 is the only accumulation point of the spectrum of the operator \( S \), it suffices by Lemma 1 to show that the eigenvalues of the operator \( \mathcal{F} \) can accumulate at the point 0 only from the left (right). Suppose again that \( \mathcal{B} \geq 0 \), so that it may be assumed that \( \mathcal{B} = 1 \). We represent the compact operator \( \mathcal{B} \) in the form of a sum \( \mathcal{B} = K + \mathcal{B}_1 \), where \( K \) is finite-dimensional and \( \| \mathcal{B}_1 \| < 1 \), and we set

\[ \mathcal{F} = -2\pi \Re (I + \mathcal{B}_1)^{-1} \mathcal{B}^* . \]

According to (1.6.8)

\[ (I + \mathcal{B}_1)^{-1} - (I + \mathcal{B}_1)^{-1} = -(I + \mathcal{B}_1)^{-1} K (I + \mathcal{B}_1)^{-1} - P (I + \mathcal{B}_1)^{-1} , \]

where \( P \) is the orthogonal projection onto the kernel of the operator \( I + \mathcal{B}_1 \). Since the operators \( K \) and \( P \) are finite-dimensional, from this it follows that the difference

\[ \mathcal{F} - \mathcal{F}_1 = -2\pi \Re [(I + \mathcal{B}_1)^{-1} - (I + \mathcal{B}_1)^{-1} \mathcal{B}^* . \]

is finite-dimensional. At the same time, in analogy to Theorem 2 we see that

\[ -(2\pi)^{-1} (\mathcal{F}_1 f, f) = ((I + \Re \mathcal{B}_1) g, g) \geq 0 , \quad g = (I + \mathcal{B}_1)^{-1} \mathcal{B}^* f . \]

Thus, \( \mathcal{F} \) is the sum of a negative and a finite-dimensional operator. By the spectral theorem the multiplicity of the positive spectrum of the operator \( \mathcal{F} \) is finite. \( \Box \)

2. The direction of rotation of the spectrum we also first consider in an abstract setting. Let \( S_{\gamma} \) be some family of unitary operators in a Hilbert space which depends on a parameter \( \gamma \geq 0 \). We suppose that

\[ S_0 - I \in G_{\infty} \]

and

\[ \lim_{\gamma \to 0} \| S_0 - S_0 \| = 0 . \]

According to (2), the spectrum of the operator \( S_0 \) consists of eigenvalues \( \mu_n \) which can accumulate only at the point 1 \( \mu \in T \). By Proposition 1.1.1.4 under condition (3) the spectrum of the operator \( S_{\gamma} \) is contained in the union of the arcs \( \mu_n e^{-i\theta_n(\gamma)} \), \( \mu_n e^{i\theta_n(\gamma)} \) where \( \theta(\gamma) \to 0 \) as \( \gamma \to 0 \). Moreover, on each of these arcs there is at least one point of the spectrum of the operator \( S_{\gamma} \).

Since the spectrum of the operator \( S_0 \) has an accumulation point, the concept of rotation of the spectrum needs a precise definition. We say that the spectrum of the family \( S_{\gamma} \) rotates at the point \( \gamma = 0 \) in a clockwise (counterclockwise) direction if for each \( \mu_n \) there exist small numbers \( \theta_n > 0 \) and \( \gamma_n > 0 \) such that for \( \gamma \in (0, \gamma_n) \) the operators \( S_{\gamma} \) have no spectrum on the arc \( (\mu_n, e^{i\theta_n}) \) (respectively, on the arc \( (\mu_n, e^{-i\theta_n}) \)). This definition
does not require compactness of the operators $S_{\gamma} - I$ for $\gamma > 0$. However, the idea of rotation of the spectrum becomes more graphic if the spectrum of $S_{\gamma}$ for $\gamma > 0$ also consists of eigenvalues.

We now suppose that
\begin{equation}
\bar{S}_{\gamma} := S_{\gamma}S_{0}^{-1} = I - 2\pi i \gamma \mathcal{L}_{1} \mathcal{V}(I + \gamma \mathcal{B}_{1} \mathcal{V})^{-1} \mathcal{L}_{1}^{*},
\end{equation}
where the operators $\mathcal{V} = \mathcal{V}^{*}$, $\mathcal{L}_{1}$, and $\mathcal{B}_{1}$ are bounded, and
\begin{equation}
2\pi i \mathcal{L}_{1}^{*} \mathcal{L}_{1} = \mathcal{B}_{1} - \mathcal{B}_{1}^{*}.
\end{equation}
Relation (4) copies the stationary representation for the SM corresponding to a small perturbation (the parameter $\gamma$ has the meaning of a coupling constant). Thus, we suppose that the multiplicative perturbation $\bar{S}_{\gamma}$ of $S_{0}$ has the form (7.1); here (5) plays the role of (7.3). According to Theorem 7.4, the operator (4) is unitary. From Proposition 7.5 it follows immediately that condition (3) is now automatically satisfied. The direction of rotation of the spectrum is determined in the next assertion.

**Theorem 4.** Suppose $S_{0}$ is unitary and condition (2) is satisfied. If in (4) $\mathcal{V} \geq 0$ (or $\mathcal{V} \leq 0$), then the spectrum of the family $S_{\gamma}$ rotates at the point $\gamma = 0$ in a clockwise (respectively, counterclockwise) direction.

**Proof.** Suppose, for example, that $\mathcal{V} \geq 0$. According to Proposition 7.5 and Theorem 2 the spectrum of the operator (4) lies on the arc $[e^{-i\theta(\gamma)}, 1]$ where $\theta(\gamma) > 0$ and $\theta(\gamma) \to 0$ as $\gamma \to 0$. It remains to use Corollary 1.11.6 for $U_{0} = S_{0}$, $M = \bar{S}_{\gamma}$. $\square$

3. In analogy to individual eigenvalues, their product—the determinant of the operator $S_{\gamma}$—can also be considered. For finiteness of the determinant it is, of course, necessary to assume that the operator $S_{\gamma} - I$ is of trace class.

**Theorem 5.** Suppose $S_{\gamma} = \bar{S}_{\gamma}S_{0}$, where $S_{0}$ is unitary $S_{0} - I \in \mathcal{E}_{1}$ and the operator $\bar{S}_{\gamma}$ is given by equality (4) with $\mathcal{L}_{1} \in \mathcal{E}_{2}$. Then for $\mathcal{V} \geq 0$ (or $\mathcal{V} \leq 0$) the number $\text{Det}S_{\gamma}$ rotates at the point $\gamma = 0$ in a clockwise (counterclockwise) direction.

**Proof.** On the basis of (4) $\bar{S}_{\gamma} - I \in \mathcal{E}_{1}$, and hence we have the well defined
\begin{equation}
\text{Det}S_{\gamma} = \text{Det}\bar{S}_{\gamma} \cdot \text{Det}S_{0}.
\end{equation}
Moreover, according to (4),
\begin{equation}
|\bar{S}_{\gamma} - I| \leq 2\pi \gamma \|\mathcal{L}_{1}\|\|\mathcal{V}\|\|I - \gamma \mathcal{B}_{1} \mathcal{V}\|^{-1}.
\end{equation}
Since the determinant is continuous in the trace norm, from this it follows that $\text{Det}\bar{S}_{\gamma} \to 1$ and hence $\text{Det}S_{\gamma} \to \text{Det}S_{0}$ as $\gamma \to 0$.

Let, for example, $\mathcal{V} \geq 0$. We write the eigenvalues of the operator $\bar{S}_{\gamma}$ in the form $\mu_{n}(\gamma) = e^{-i\phi_{n}(\gamma)}$. According to Theorem 4 it may be assumed that $\phi_{n}(\gamma) \in [0, \pi]$. Under this condition
\begin{equation}
\phi(\gamma) := \sum_{n=1}^{\infty} \phi_{n}(\gamma) \leq 2^{-1} \pi \sum_{n=1}^{\infty} |\mu_{n}(\gamma) - 1| = 2^{-1} \pi |\bar{S}_{\gamma} - I|.
\end{equation}
We have here used the fact that $|\mu_{n}(\gamma) - 1|$ are the $s$-numbers of the operator $\bar{S}_{\gamma} - I$. Inequality (7) now shows that $\phi(\gamma) \to 0$ as $\gamma \to 0$. Since $\phi(\gamma) \geq 0$, this follows that for sufficiently small $\gamma > 0$ the number $\text{Det}S_{\gamma} = \exp(-i\phi(\gamma))$ lies on the lower semicircle. By (6) this implies that $\text{Det}S_{\gamma}$ rotates in a clockwise direction. $\square$

4. We return to the consideration of the SM. We shall first discuss the SM under conditions of the general stationary scheme. We assume that the assumptions of Part 3 of §7 are satisfied. For the perturbation $V = G^{*}G$ we accept by definition that $V \geq 0$ (or $V \leq 0$) if $\mathcal{V} \geq 0$ (respectively, $\mathcal{V} \leq 0$). The next assertion supplements Theorem 7.7. With Theorems 5.7.4 and 5.8.1 taken into account, it follows immediately from Theorem 3.

**Theorem 6.** Suppose on a Borel set $\Lambda$ the conditions of Theorem 5.7.1' are satisfied and $B^{(0)}(z) = G\mathbb{R}_{0}(z)G \in \mathcal{E}_{\infty}$ for $\text{Im} z \neq 0$. Then for $\mathcal{V} \geq 0$ (or $\mathcal{V} \leq 0$) on the upper (respectively, lower) semicircle the spectrum of the SM $S(\lambda; H_{1}, H_{0})$ consists of a finite number of eigenvalues.

In other words, for positive (negative) perturbations $V$ the spectrum of the SM can accumulate at $1$ only from below (from above). We now present smooth and trace class versions of this theorem.

**Theorem 7.** Suppose the operator-valued function $B^{(0)}(z) \in \mathcal{E}_{\infty}$ for $\text{Im} z \neq 0$ and is continuous in norm up to the cut along the interval $\Lambda$. Then for $\lambda \in \Lambda$ for the SM $S(\lambda; H_{1}, H_{0})$ all the conclusions of Theorem 6 hold.

**Theorem 8.** Suppose $G(|H_{1}| + I)^{-1/2} \in \mathcal{E}_{p}$ for some $p < \infty$ and $GE_{p}(\Lambda) \in \mathcal{E}_{\infty}$ for any bounded interval $\Lambda$. Then for a.e. $\lambda \in \mathbb{R}$ for the SM $S(\lambda; H_{1}, H_{0})$ all the conclusions of Theorem 6 hold.

The difference between these theorems is that in the first of them all $\lambda \in \Lambda$ are allowed, while the second is valid for a.e. $\lambda \in \mathbb{R}$.

5. We now consider the rotation of the spectrum of the SM $S(\lambda) = S(\lambda; H(\gamma), H_{0}, 2)$ for the family of selfadjoint operators $H(\gamma) = H_{1} + \gamma V$ depending on a parameter $\gamma \geq 0$. We assume the spectral parameter $\lambda$ to be fixed.

**Theorem 9.** Suppose for some Borel set $\Lambda$ the WO $W_{\gamma}(H_{1}, H_{0}; \lambda, \Lambda)$ exist, are isometric on $E_{\gamma}(\Lambda) \mathcal{H}_{\gamma}$, are complete, and $S(\lambda; H_{1}, H_{0}, 2) - I \in \mathcal{E}_{\infty}$ for a.e. $\lambda \in \Lambda$. We suppose that $V = G^{*}G$ and the operator-valued
7. PROPERTIES OF THE SCATTERING MATRIX

function \( B^{(1)}(z) = G(H_1 - z)^{-1} G^* \) satisfies on \( \Lambda \) the conditions of Theorem 5.7.2' (for some \( l \) and \( p \)). Suppose on a dense set of elements \( f \) there exist the strong limits of the vector-valued functions \( G(H_1 - \lambda \pm i\epsilon)^{-1} f \). Then for the family of Hamiltonians \( H(\gamma) = H_1 + \gamma V \) the spectrum of the SM \( S(\lambda; H(\gamma), H_0; J) \) for a.e. \( \lambda \in \Lambda \) rotates at the point \( \gamma = 0 \) in a clockwise (respectively, counterclockwise) direction if \( \gamma' \geq 0 \) (or \( \gamma' \leq 0 \)). If, moreover, \( S(\lambda; H_1, H_0; J) - I \in \Theta_1 \) and \( dG E_{S(\lambda)} G^* d\lambda \in \Theta_1 \) for a.e. \( \lambda \in \Lambda \), then the same conclusion holds regarding the function \( \text{Det} S(\lambda; H(\gamma), H_0; J) \).

**Proof.** By Theorems 5.7.2' and 5.8.1 the WO \( W_\pi(H(\gamma), H_0) \) exist for all \( \gamma \geq 0 \) and are isometric and complete. Thus, according to Corollary 1.2

\[
S(\lambda; H(\gamma), H_0; J) = S_\gamma(\lambda)S(\lambda; H_1, H_0; J),
\]

where the operator \( S_\gamma(\lambda) \) is unitarily equivalent to the SM \( S(\lambda; H(\gamma), H_1) \), and hence a representation of the form (4) holds for it. Thus, the assertion regarding the direction of rotation of the spectrum follows directly from Theorem 4 applied to the operators \( S_\gamma = S(\lambda; H(\gamma), H_1; J), S_0 = S(\lambda; H_1, H_0; J) \) and \( S_\gamma = S^{(4)}(\lambda) \). Similarly, the assertion regarding the direction of rotation of \( \text{Det} S_\gamma \) follows from Theorem 5. \( \Box \)

We note that equation (8) follows from (55). To prove Theorem 9 it would have been possible to use also equality (15), but either of them suffices for our purposes.

In exact analogy to Part 4, Theorem 9 can be made concrete in the framework of the smooth and trace class methods. In particular, the assertion regarding the direction of rotation of the spectrum of \( S(\lambda; H_0 + \gamma V, H_0) \) is clearly satisfied for all \( \gamma \geq 0 \) if the pair \( H_0, V \) satisfies the conditions of Theorems 7 or 8. Under the conditions of Theorem 8 the assertion regarding the direction of rotation of the determinant also holds.

§9. The scattering cross section. Upper bounds

1. In this section we shall establish conditions for the validity of the inclusion

\[
S(\lambda) - I \in \Theta_p, \quad p > 0,
\]

for the SM \( S(\lambda) = S(\lambda; H, H_0) \) and obtain effective estimates of the corresponding (quasi-) norms \( |S(\lambda) - I|_p \). Such estimates are of interest primarily because in concrete problems \( |S(\lambda) - I|_2^2 \) is the basic quantity observable in scattering experiments. It is called the scattering cross section. Actually, in the definition of the scattering cross section a factor is introduced which depends on the energy \( \lambda \) and the dimension of the problem (see Part 4 of the Introduction for a precise definition of the Schrödinger operator).

For the scattering cross section integrated over some interval \( \Lambda \), an effective estimate can easily be obtained within the framework of trace class assumptions. Namely, from Theorem 6.6 and the obvious inequality

\[
|T|^2 \leq |T| |T|, \quad T \in \Theta_1,
\]

we immediately obtain

**Theorem 1.** Under the conditions of Theorem 6.6

\[
\int_\Lambda |S(\lambda; H, H_0; J) - u_\lambda(\lambda)|^2 d\lambda \leq 4\pi\|E(\lambda)\| \|E_0(\lambda)\|,
\]

(2)

In particular, for \( \Theta_0 = \mathcal{K}, \ J = I \) an estimate of \( |S(\lambda) - I|_2^2 \) follows from (2). In this section we give an estimate for the scattering cross section for fixed \( \lambda \). For this estimate the condition \( \Theta_0 = \mathcal{K}, \ J = I \) is essential. On the other hand, the quantity \( |S(\lambda) - I|_p \) is considered for all \( p > 0 \).

2. We shall find it convenient to first of all consider an arbitrary operator (7.1) satisfying conditions (7.2), (7.3). It is moreover assumed that either the inverse operator in (7.1) exists and is bounded or \( \mathcal{K} \in \Theta_\infty \). In the latter case the inverse operator is understood in the generalized sense indicated in Part 3 of §1.6. We first establish conditions for the operator \( S - I \) to belong to the class \( \Theta_p \).

**Lemma 2.** The inclusion \( S - I \in \Theta_p \) holds if \( \mathcal{K} \in \Theta_{2p} \). Moreover,

\[
s_p(S - I) \leq Cn^{-\alpha}, \quad \alpha > 0,
\]

if \( s_p(\mathcal{K}) = O(n^{m/2}) \).

**Proof.** By inequalities (1.6.9), (1.6.11) it follows from (7.1) that

\[
|S - I|_p \leq 2\pi\|\mathcal{K}\| \|\mathcal{K}\|_2 \|\mathcal{K}^{-1}\| \|I + \mathcal{K}\mathcal{K}^{-1}\|.
\]

Similarly, the estimate (3) can be obtained by means of inequalities (1.6.5) and (1.6.6). \( \Box \)

**Corollary 3.** The assertion of Lemma 1 hold if, respectively, \( \mathcal{K} \in \Theta_p \) or \( s_p(\mathcal{K}) = O(n^{m/2}) \).

**Proof.** It suffices to consider relation (7.3). \( \Box \)

The estimate (4) for the norm \( |S - I|_p \) is not very effective since, generally speaking, the norm of the inverse operator on the right-hand side of (4) cannot be controlled. Only for the small perturbations, where \( \|\mathcal{K}\| \|\mathcal{K}\| = b < 1 \), (4) ensures the concrete estimate

\[
|S - I|_p \leq 2\pi(1 - b)^{-1}\|\mathcal{K}\| \|\mathcal{K}\|_2 \leq 2(1 - b)^{-1}\|\mathcal{K}\| \|\mathcal{K}\|_p.
\]

(5)

It turns out that an estimate of the form (5) is preserved in the general case. Its derivation uses Proposition 1.6.2, which, for sufficiently large \( n \), gives an effective estimate for the singular numbers \( S_n \) of the inverse operator.
Theorem 4. Suppose conditions (7.2), (7.3) are satisfied and $\mathcal{B} \in \Theta_p$. Then for the operator (7.1) there is the estimate
\[
|S - I|_p \leq \kappa_p \|\mathcal{B}\| \|\mathcal{B}\|_p,
\]
where
\[
\kappa_p = 2^p \min_{\beta \in \mathbb{R}} \{\beta^p + 2(1 - \beta)^{-p}\}.
\]
Proof. For any $\beta \in (0, 1)$ we choose a number $N \geq 2$ such that
\[
S_N(\mathcal{B}^*) \leq \beta \leq S_{N-1}(\mathcal{B}^*) \quad \text{and} \quad n \geq N \text{ respectively. According to (8),}
\]
\[
\beta^p(N - 1) \leq s^p_{N-1}(\mathcal{B}^*)(N - 1) \leq \|\mathcal{B}\|_p^p.
\]
Since $|A| \leq 2$ by Theorem 7.4, it follows that
\[
\Sigma_2 = \sum_{n=1}^{N-1} s^p_n(A) \leq 2^p(N - 1) \leq (2\beta^{-1}) \|\mathcal{B}^*\|_p^p.
\]
To estimate the second sum we apply inequalities (1.6.5) and (1.6.6) to the product of the right in (7.1). Then for any natural number $k$
\[
s_{N+2k-2}(A) \leq 2\pi\|\mathcal{B}\|s_{N}(1 + \mathcal{B}^*)^{-1} s_k(\mathcal{B}^*) s_k(\mathcal{B}^*).
\]
According to Proposition 1.6.2, from condition (8) it follows that
\[
s_{N}(1 + \mathcal{B}^*)^{-1} \leq (1 - \beta)^{-1},
\]
and hence
\[
\Sigma_2 \leq 2 \sum_{k=1}^{\infty} s^p_{N+2k-2}(A)
\]
\[
\leq 2(2\pi)^p (1 - \beta)^{-p} \|\mathcal{B}\|_p^p \sum_{k=1}^{\infty} s_k(\mathcal{B}^*).\]

The sum on the right is $\|\mathcal{B}^*\|_p^p$ and hence by (3) does not exceed $\pi^{-p} \|\mathcal{B}\|_p^p$. Combining inequalities (9) and (10), we establish the estimate (6). The expression (7) for $\kappa_p$ can be established by minimizing the estimate constant with respect to $\beta$.  \[\square\]

3. We now apply Theorem 4 to estimate the scattering cross section. By Theorems 5.7.1’ and 5.8.1 under the conditions of the next assertion the WO $W_{\pm}(H, H_0; \lambda)$ exist, and the SM $S(\lambda) = S(\lambda; H, H_0)$ has the representation (5.7.9). We recall that $B^{(0)}(z) = GR_0(z)G^*$. Theorem 5. Suppose on a Borel set $\Lambda$ the conditions of Theorem 5.7.1’ are satisfied and $B^{(0)}(\lambda + i0) \in \Theta_p$ (for $\lambda \in \Lambda$). Then the inclusion (11) holds, and
\[
|S(\lambda; H, H_0) - I|_p \leq \kappa_p \pi^{-p} \|B^{(0)}(\lambda + i0)\|_p,
\]
where the constant $\kappa_p$ is given by equality (7).

As in §8, the conditions of this theorem can be replaced by more concrete assumptions of smooth or trace class types. We present the corresponding formulations.

Theorem 6. Suppose $B^{(0)}(z) \in \Theta_\infty$ for $\text{Im} z \neq 0$, and the operator-valued function $B^{(0)}(z)$ is continuous in norm up to the cut along the interval $\Lambda$. Suppose also that $B^{(0)}(\lambda + i0) \in \Theta_p$ (for $\lambda \in \Lambda$). Then the estimate (11) holds for all $\lambda \in \Lambda_0$, where the set $\Lambda_0 \subset \Lambda$ is open and $|\Lambda\setminus\Lambda_0| = 0$.

We recall that under the conditions of Theorem 6 the set $\Lambda\setminus\Lambda_0$ consists of points $\lambda$ at which condition (5.7.5) is violated. Therefore, for $\lambda \in \Lambda\setminus\Lambda_0$ the representation (5.7.6) loses meaning and the SM $S(\lambda)$ is not defined at all. Nevertheless, inequality (11) (with an estimation constant not depending on $\lambda$) is preserved as the end points of the component intervals of the open set $\Lambda_0$ are approached.

Theorem 7. Suppose $GE_0(\Lambda) \in \Theta_2$ for any bounded interval and $G((H_0 + I)^{-1/2}) \in \Theta_r$ for some $r < \infty$. Then for a.e. $\lambda \in \mathbb{R}$ the estimate (11) holds, where $\rho \geq 2$ and $\rho \geq 2^{-1}r$.

Proof. It is only needed to show that the operator $B^{(0)}(z)$ has angular limit values in $\Theta_p$ as $z$ tends to $\mathbb{R}$. It is clear that for a bounded interval $\Lambda$ $GR_0(z)G^* = G_\Lambda R_0(z)G_\Lambda^* + G_1 E_0(\Lambda')((H_0 + I)R_0(z)G_1^*$, where $G_\Lambda = GE_0(\Lambda)$, $G_1 = G((H_0 + I)^{-1/2}$, $\Lambda' = \mathbb{R}\setminus\Lambda$. According to Theorem 6.1.9 and Remark 6.1.12 the first term (on the right) has angular limits in $\Theta_\infty$ for a.e. $\lambda \in \Lambda$. The second term in (12) depends holomorphically on $z$ for $\text{Re} z \in \Lambda$ in the class $\Theta_\infty$.

4. The estimate (11), just as the “averaged” estimate (2), is linear in the coupling constant. This implies that for the family of Hamiltonians $H(\gamma) = H_0 + \gamma V$
\[
|S(\lambda; H(\gamma), H_0) - I|_p \leq \kappa_p \pi^{-p} \|B^{(0)}(\lambda + i0)\|_p,
\]
where, as previously, $B^{(0)}(\zeta) = GR_0(z)G^*$. As in §6, this estimate can sometimes be improved by applying the IP. In the next assertion we consider the semibounded case and take into account the nonnegativity of the perturbation. The Hamiltonian $H = H_0 + V$ is hereby defined according to Theorem
1.10.9 by means of its quadratic form defined on $\mathcal{D}(H_0^{1/2})$. We present only
the formulation in terms of the smooth theory.

**Theorem 8.** Suppose $H_0 \geq cI$, $c > 0$, the perturbation $V \geq 0$, and the
operator $V^{1/2}$ is $H_0^{1/2}$-bounded. We set

$$L^2 = H_0^{-1/2} V H_0^{1/2} (I + H_0^{-1/2} V H_0^{-1/2})^{-1}, \quad 0 \leq L \leq I.$$ 

Suppose that the operator-valued function $K_0(z) = LR_0(z)L \in \mathcal{S}_\infty$ for $|z| \neq 0$ and
is continuous in norm up to the cut along the interval $\lambda$. Let also
$K_0(\lambda + i0) \in \mathcal{S}_p$, $p > 0$, for $\lambda \in \Lambda$. Then there exist the complete WO
$W_k(H; H_0; \Lambda)$, and for the corresponding SM on an open set $\Lambda_0$ of full
measure in $\Lambda$ there is the estimate

$$|S(\lambda) - I|_p \leq \kappa_p \|K_0(\lambda + i0)\|_p, \quad \lambda \in \Lambda_0, \quad \text{(13)}$$

where the constant $\kappa_p$ is defined by equality (7).

**Proof.** We consider the auxiliary pair of operators $h_0 = H_0^{-1}$, $h = H^{-1}$.
It is clear that the corresponding perturbation

$$v = h - h_0 = -H_0^{-1} V (H_0 + V)^{-1}$$

$$= -H_0^{-1} V H_0^{-1/2} (I + H_0^{-1/2} V H_0^{-1/2})^{-1} H_0^{-1/2} = -H_0^{-1} L^2 H_0^{-1/2}$$

admits the factorization $v = g^* \omega g$ where $g = LH_0^{-1/2}$ and $\omega = -I$.
Moreover, the resolvent $r_0(\zeta)$ of $h_0$ at the point $\zeta = z^{-1}$ equals $r_0(\zeta) = -zH_0R_0(z)$, and hence

$$b_0(\zeta) := g r_0(\zeta) g^* = -zLR_0(z)L = -zK_0(z). \quad \text{(14)}$$

Thus, all the conditions of Theorem 6 are satisfied for the pair $h_0, h$ on the
set $Y = \{ \mu \in \mathbb{R} : \mu^{-1} \in \Lambda \}$. It shows that the WO $W_k(h, h_0; \Lambda)$ exist, and
for the corresponding SM

$$|S(\mu; h, h_0) - I|_p \leq \kappa_p \|b_0(\mu + i0)\|_p.$$ 

We further note that for the pair $h_0, h$ and $\varphi(\mu) = \mu^{-1}$ the IP holds (see Part
3 of §4.5), and hence, according to (2.6.14), $S(\mu; h, h_0) = S^*(\lambda; H, H_0)$,
$\lambda = \mu^{-1}$. The estimate (13) now follows directly from equality (14). □

We emphasize that, independently of the magnitude of $V$, the "effective perturbation" $L^2$ is
bounded in norm by one. For the three-dimensional Schrödinger operator with a nonnegative, compactly supported potential it is possible to derive from Theorem 8 (see [137]) an estimate for the scattering cross section which is uniform with respect to the coupling constant $\gamma$.

We note that all results of this section can easily be extended [76, 77] to
arbitrary symmetrically quasinormed ideals.

**Chapter 8**

The Spectral Shift Function (SSF)
and the Trace Formula

The basic objects of scattering theory (the wave operators and the scattering
operator and matrix) pertain to perturbation theory on the continuous
spectrum. The concept of the spectral shift function (SSF) goes beyond the
framework of scattering theory itself.

The SSF arises in the theory of trace class perturbations in connection
with integral representation for the trace of the difference of functions of the
operators $H_0$ and $H$. On the continuous spectrum the SSF is connected
with the scattering matrix. However, in contrast to the latter, the concept of
SSF has content both on the continuous and on the discrete spectra.

It is convenient to introduce the SSF for a pair of selfadjoint (or unitary)
operators in terms of the perturbation determinant for this pair. Properties of
perturbation determinants are studied in §1. The theory of SSF for trace class
perturbations of selfadjoint operators is constructed in §§2–4. In §§5, 6 this
theory is carried over to the unitary case. In §§7–11 we return to selfadjoint
operators. As compared with §§2–4 in these sections a more general situation
is considered in which only the difference of suitable functions of $H_0$ and
$H$ is of trace class.

In this chapter it is assumed that the operators act in a common space and the
identification is the identity.

§1. The perturbation determinant

1. The perturbation determinant (PD) is naturally considered for arbitrary
closed operators $A_0$ and $A$ without the assumption of their selfadjointness
(or unitarity). We denote by $\rho(A)$ the set of regular points of the
operator $A$, $R(z) = (A - z)^{-1}$, $R_0(z) = (A_0 - z)^{-1}$, $V = A - A_0$.
For $\zeta \in \rho(A_0) \cap \rho(A)$ for these operators the resolvent identity (1.9.5) holds.

Suppose $\mathcal{D}(A_0) = \mathcal{D}(A)$ and $VR_0(z) \in \mathcal{S}_1$ for some point $z \in \rho(A_0)$.
Then by the Hilbert identity the operator $V R_0(z)$ is of trace class for all
$z \in \rho(A_0)$. The PD for the pair of operators $A_0, A$ is introduced by the
8. THE SPECTRAL SHIFT FUNCTION AND THE TRACE FORMULA

\[ D(z) = D_{A_{1}/A_{0}}(z) = \text{Det}[(A - z)(A_{0} - z)^{-1}] = \text{Det}[I + VR_{0}(z)], \quad z \in \rho(A_{0}). \]  

(1)

The following assertions regarding the PD follow directly from the results listed in §1.7 regarding determinants of the form \( \text{Det}(I + T) \) where \( T \in \mathfrak{H} \).

1°. The function \( D(z) \) is defined and holomorphic on the set \( \rho(A_{0}) \).

For the proof it suffices to consider the holomorphicity of the operator-valued function \( VR_{0}(z) \) in \( \mathfrak{H} \).

2°. For \( (A_{1} - A_{0})R_{0} \in \mathfrak{H} \) and \( (A_{1} - A_{0})R_{1} \in \mathfrak{H} \)

\[ D_{A_{1}/A_{0}}(z)D_{A_{1}/A_{0}}(z) = D_{A_{1}/A_{0}}(z), \quad z \in \rho(A_{0}) \cap \rho(A_{1}). \]  

(2)

In particular,

\[ D_{A_{1}/A_{0}}(z) = 1, \quad z \in \rho(A_{0}) \cap \rho(A), \]

and hence \( D_{A_{1}/A_{0}}(z) \neq 0 \) for \( z \in \rho(A_{0}) \cap \rho(A) \).

Equality (2) is a direct consequence of (1.7.13). We note that under the assumptions made \( (A_{1} - A_{0})R_{0} \in \mathfrak{H} \).

3°. For \( z \in \rho(A_{0}) \cap \rho(A) \)

\[ D^{-1}(z)D'(z) = \text{Tr}(R_{0}(z) - R(z)). \]  

(4)

Indeed, according to the rule (1.7.10) for differentiation of determinants the left-hand side of (4) is equal to \( \text{Tr}[(I + VR_{0})^{-1}VR_{0}'] \).

By (1.7.4) the bounded operator \( R_{0} \) may interchange places under the sign of the trace with the trace class operator \( (I + VR_{0})^{-1}VR_{0} \). Thus, in view of the resolvent identity (1.9.15) the operator under the trace sign is \( R_{0} - R \).

We recall (for details see [7], [11]) that an isolated point \( z_{1} \) of the spectrum of an operator \( A \) is called a normal eigenvalue of \( A \) if the projection corresponding to it

\[ P = -(2\pi i)^{-1} \int_{|z - z_{1}| = \delta} R(z) \, dz \]

is finite-dimensional. Here the integration goes over a sufficiently small circle with center at the point \( z_{1} \) traversed counterclockwise. The operator \( P \) projects onto the subspace of eigen- and associated vectors of the operator \( A \) corresponding to the eigenvalue \( z_{1} \). Its dimension \( k = \dim P \) is called the algebraic multiplicity of \( z_{1} \). The subspace \( R(I - P) \) is invariant relative to \( A \), and operator \( A - z_{1}I \) is invertible there. In a neighborhood of a normal eigenvalue the resolvent of the operator \( A \) admits an expansion in a Laurent series

\[ R(z) = -\sum_{m=-k}^{\infty} C_{m}(z - z_{1})^{m}, \quad k = \dim P, \quad \text{Tr} C_{-m} = 0, \quad m \geq 2. \]  

(5)

Here \( C_{-1} = P, \ C_{-2} = (A - z_{1})P, \) and \( C_{-m} = C_{m}^{-1} \) for \( m > 2 \). The operator \( C_{-2} \) is nilpotent, \( C_{-2}^{2} = 0 \), so that, in particular, \( \text{Tr} C_{-2}^{-1} = 0 \) for \( m \geq 2 \). For selfadjoint or unitary operators \( P \) is the orthogonal projection onto the corresponding eigensubspace and \( C_{-2} = 0 \).

4°. Suppose \( z_{1} \) is a regular point or a normal eigenvalue of the operators \( A_{0} \) and \( A \) of algebraic multiplicities \( k_{0} \) and \( k \). Then at the point \( z_{1} \) the function \( D(z) \) has a pole (or zero) of order \( k_{0} - k \) (of order \( k - k_{0} \)).

Indeed, according to (4) and (5) the logarithmic derivative of \( D(z) \) has the singularity \( (k - k_{0})(z - z_{1})^{-1} \) at the point \( z_{1} \). This implies that the function \( D(z) \) itself behaves like \( (z - z_{1})^{k_{0} - k} \) as \( z \to z_{1} \).

From this it follows (cf. Theorem 1.10.7) that the eigenvalues of \( A \) can accumulate only at points of the spectrum of the operator \( A_{0} \) which are not normal eigenvalues of \( A \). This assertion, of course, is weaker than Weyl’s theorem on preservation of the essential spectrum due to the assumption that \( VR_{0}(z) \) is of trace class.

5°. Suppose \( \|VR_{0}(z)\|_{1} \to 0 \) as \( |z| \to \infty \) in some domain of \( z \). Then \( D(z) \to 1 \) in this domain.

It suffices to use the continuous dependence of \( \text{Det}(I + T) \) on the variation of \( T \) in \( \mathfrak{H} \) and consider the equality \( \text{Det} I = 1 \).

6°. For \( z \in \rho(A_{0}) \cap \rho(A), \ z' \in \rho(A_{0}) \) there is the identity

\[ D_{A_{1}/A_{0}}(z') = \text{Det}[(I - (I - VR_{0}(z))VR_{0}(z_{1} - R_{0}(z_{1})))] = \text{Det}[(I - (z' - z_{1})R_{0}(z')VR_{0}(z_{1})] = \text{Det}[(I - (z' - z_{1})VR_{0}(z')]. \]

(6)

Because of the multiplicative property (1.7.13) to prove the first of these equalities it suffices to note that

\[ (I + VR_{0}(z))[I - (I - VR(z))(R_{0}(z_{1} - R_{0}(z')))] = I + VR_{0}(z) - VR_{0}(z_{1} - R_{0}(z')) = I + VR_{0}(z_{1}). \]

We have here used the resolvent identity \( (I + VR_{0})(I - VR) = I \). Applying further the Hilbert identity, we find that

\[ (I - VR(z))[VR_{0}(z) - R(z')] = (z' - z)VR(z)R(z_{1}). \]

According to (1.7.11), from this we obtain the second equality of (6). Finally, the third equality can be obtained from the second by changing the roles of \( H_{0} \) and \( H \) and also of \( z \) and \( z' \). It is here necessary to use relation (3). The additional condition \( z' \in \rho(A) \) can be removed due to the analyticity of both sides in \( z' \in \rho(A_{0}) \).

2. We shall now make special consideration of the selfadjoint and unitary cases. For selfadjoint operators \( A_{0} = H_{0} \) and \( A = H \) the PD (1) is
holomorphic in $z$ and does not vanish in the upper and lower half-planes. To study its behavior as $|\text{Im } z| \to \infty$ we need the elementary

**Lemma 1.** If $V R_0(z_0) \in \mathcal{S}_1$, for some $z_0 \in \rho(H_0)$, then

$$\lim_{|\text{Im } z| \to \infty} \|V R_0(z)\|_1 = 0. \tag{7}$$

**Proof.** By the Hilbert identity

$$V R_0(z) = V R_0(z_0) K_0(z), \tag{8}$$

where $K_0(z) = I + (z - z_0) R_0(z)$. On the basis of the spectral theorem

$$\|K_0(z)f\|^2 = \int_{-\infty}^{\infty} \left| \frac{\lambda - z}{\lambda - z_0} \right|^2 d(E_0(\lambda)f, f). \tag{9}$$

Here the integrand is bounded uniformly with respect to $\lambda$ and $z$ and as $|\text{Im } z| \to \infty$ tends to zero for any fixed $\lambda$. Hence, by Lebesgue’s theorem the expression (9) tends to zero, i.e., $K_0(z) \to 0$ as $|\text{Im } z| \to \infty$. Relation (7) for the operator (8) now follows directly from Lemma 6.1.3. □

We note that Lemma 1 remains in force for any separable symmetrictically quasinormed ideal.

**Lemma 2.** If in the selfadjoint case $V R_0(z_0) \in \mathcal{S}_1$, for some $z_0 \in \rho(H_0)$, then

$$\lim_{|\text{Im } z| \to \infty} D(z) = 1. \tag{10}$$

If, moreover, $V \in \mathcal{B}$, then

$$\overline{D(z)} = D(\overline{z}). \tag{11}$$

**Proof.** By Lemma 1 relation (10) follows directly from property 5°. Permuting the operators under the determinant sign on the basis of identity (1.7.11), we find that

$$\overline{\text{Det}(I + VR_0(z))} = \text{Det}(I + R_0(\overline{z})V) = \text{Det}(I + VR_0(\overline{z})).$$

This gives (11). □

In Part 3 it is shown that equality (11) is valid also without the superfluous condition $V \in \mathcal{B}$.

In the unitary case $A_0 = U_0$, $A = U$ we denote the spectral parameter by $\zeta$. The PD $D(\zeta)$ is now holomorphic in $\zeta$ and does not vanish inside or outside the unit circle. The role of Lemma 2 is played by

**Lemma 3.** Suppose the operators $U_0$, $U$ are unitary and $U - U_0 \in \mathcal{S}_1$. Then $D(\zeta) \to 1$ as $|\zeta| \to \infty$ and

$$\overline{D(\zeta)} = D^{-1}(0) D(\overline{\zeta}), \quad \overline{\zeta} = \overline{\zeta}^{-1} \tag{12}$$

in particular, $|D(0)| = 1$.

**Proof.** Since

$$\|VR_0(\zeta)\|_1 \leq \|V\| \|\zeta - 1\|^{-1},$$

the first assertion follows directly from property 5°. Further, by equalities (1.7.12) and (1.7.13)

$$\overline{D(\zeta)} = \text{Det}[(U_0^{-1} - \overline{\zeta}^{-1})(U^{-1} - \overline{\zeta}^{-1})] = \text{Det}[U_0 \{U_0 - \zeta \}^{-1} (U - \zeta) U^{-1}] = \text{Det}[U_0 - \zeta^{-1} (U - \zeta) \text{Det}(U^{-1} U_0)].$$

In view of (1.7.12) the first factor on the right-hand side is equal to $D(\zeta)$, while the second is equal to $D_{U_0 U_0^{-1}}(U) = D^{-1}(0)$. Finally, letting $\zeta \to 0$ in (12) and noting that $D(\zeta) \to 1$, we obtain the equality $|D(0)| = 1$. □

3. We shall now discuss a generalization of the concept of PD to the case of operators for which only the difference of the resolvents is of trace class. We here restrict ourselves to the consideration of selfadjoint operators $H_0, H$. We denote by

$$U = \mu_1 (H - a)(H - \overline{a})^{-1}, \quad U_0 = \mu_1 (H_0 - a)(H_0 - \overline{a})^{-1}, \quad |\mu_1| = 1, \quad \text{Im } a > 0,$$

their Cayley transforms corresponding to an arbitrary fractional-linear mapping

$$\zeta = \mu_1 (z - a)(z - \overline{a})^{-1} \tag{14}$$

of the upper half-plane onto the unit disk. It is clear that

$$U - U_0 = -2 \mu_1 i \text{ Im } a [R(\overline{a}) - R_0(\overline{a})] \in \mathcal{S}_1. \tag{15}$$

The generalized PD for the pair $H_0, H$ can be introduced by the equality

$$\overline{D_0(z)} := D_{U_0 U_0^{-1}}(\zeta) = \text{Det}[(H - \overline{a}^{-1})(H - z)(H_0 - z)^{-1}(H_0 - \overline{a})], \tag{16}$$

where $\zeta$ and $z$ are connected by relation (14). The expression (16), of course, does not depend on the number $\mu_1$. It can be verified by direct transformations that the operator under the Det sign in (16) is equal to $I + (z - \overline{a}) R(\overline{a}) V R_0(z)$, and hence

$$\overline{D_0(z)} = \text{Det}[I + (z - \overline{a}) R(\overline{a}) V R_0(z)]. \tag{17}$$

Sometimes $\overline{D_0(z)}$ is also written as the ordinary PD for the pair $\tilde{H}_0 = R_0(\overline{a}), \tilde{H} = R(\overline{a})$:

$$\overline{D_0(z)} = D_{\tilde{H}_0 \tilde{H}_0^{-1}}(\tilde{z}), \quad \tilde{z} = (z - \overline{a})^{-1}. \tag{18}$$

The properties of generalized PD are essentially the same as those of ordinary determinants. Thus, by its definition (see, for example, (17)) the function $\overline{D_0(z)}$ is holomorphic on the set $\rho(H_0)$ and, in any case, for $\text{Im } z \neq 0$. According to (13), (14)

$$\text{Tr}[U_0 - \zeta^{-1} - (U - \zeta^{-1})] = i(2 \mu_1 \text{ Im } a)^{-1} (z - \overline{a})^2 \text{Tr}[R_0(z) - R(z)].$$
It is clear that assumptions (21) are more flexible than the condition \( V R_0(z) \in \Theta_1 \). They may also be more convenient since it is simpler to verify that some operator belongs to the class \( \Theta_2 \) than to the trace class.

For the functions \( \tilde{D}(z) \) the majority of properties of ordinary PD are preserved. In particular, in the selfadjoint case the function \( \tilde{D}(z) \) is, as before, holomorphic for \( \text{Im} z \neq 0 \). Expression (4) for the logarithmic derivative also holds. Indeed, according to (1.7.10),

\[
\tilde{D}^{-1}(z)\tilde{D}'(z) = \text{Tr}[I + G_0 R_0(z) G^*]^{-1} G_0 R_0^2(z) G^*.
\]

By (1.7.4) and (1.9.15) the expression under the trace sign is equal to

\[
R_0(z)G^* [I + G_0 R_0(z) G^*]^{-1} G_0 R_0(z) = R_0(z) - R(z).
\]

Using the expression for a regularized determinant (1.7.14) it is possible to introduce a regularized PD

\[
D_p(z) = \text{Det}_p(I + V R_0(z)), \quad p = 2, 3, \ldots.
\]

This definition is good for \( VR_0(z) \in \Theta_p \).

We note that by equality (1.7.17) for \( VR_0 \in \Theta_1 \) the regularized perturbation determinant can be expressed in terms of the ordinary determinant as follows

\[
D_p(z) = D(z) \exp \left( \sum_{k=1}^{p-1} \frac{1}{k!(1-k)} \text{Tr}(VR_0(z))^k \right).
\]

According to the results of Part 4 of §1.7 many properties of ordinary determinants carry over to regularized determinants. Thus, the generalization of (4) has the form

\[
D_p^{-1}(z)D_p'(z) = (-1)^{p-1} \text{Tr}(R(z)(VR_0(z))^p)
\]

\[
= - \text{Tr} \left[ R(z) - \sum_{k=0}^{p-1} (-1)^k R_0(z)(VR_0(z))^k \right]
\]

A proof of this can easily be obtained with the help of formula (1.7.18).

§2. The SSF in the selfadjoint case. Trace class perturbation

1. The spectral shift function (SSF) \( \xi(\lambda) = \xi(\lambda; H, H_0), \lambda \in \mathbb{R} \), for a pair of selfadjoint operators \( H_0, H \) in a Hilbert space \( H \) arises in connection with the relation

\[
\text{Tr}[f(H) - f(H_0)] = \int_{-\infty}^{\infty} \xi(\lambda) df(\lambda),
\]

called the trace formula. Here \( f \) is an arbitrary function of some suitable class, and the real-valued function \( \xi \) in any case must be locally summable. If the function \( f \) is absolutely continuous on some interval (for example, on the entire axis \( \mathbb{R} \)), when integrating over this interval \( df \) can, of course, be
replaced by \( f'(\lambda) d\lambda \). It is clear that the validity of (1) for any \( f \in C^\infty_0(\mathbb{R}) \) fixes \( \zeta \) up to an additive constant.

In view of the spectral theorem the SSF can be found formally from the equality

\[
\zeta(\lambda) = \text{Tr}(E_\lambda - E(\lambda)).
\]

Moreover, if \( E(\lambda) - E_\lambda \in \Theta_1 \) for a.e. \( \lambda \) and the function (2) belongs to \( L^1_{\text{loc}} \), then \( f(H) - f(H_0) \in \Theta_1 \) for any \( f \in C^\infty_0(\mathbb{R}) \) and formula (1) holds. Actually, in the theory of trace class perturbations the operator \( E(\lambda) - E_\lambda \), as a rule, is not of trace class. The reason for this is the nonsmoothness of the characteristic function (of the interval \( (-\infty, \lambda) \)). Nevertheless, relation (1) for some function \( \zeta \) turns out to be satisfied. The SSF can hereby be constructed in terms of the perturbation determinant \( D_{H/H_0}(z) \). The properties of the function \( \zeta \) and the “volume” of the class of admissible functions \( f \) depend on the “closeness” of the operators \( H_0 \) and \( H \).

Under broad assumptions the SSF is connected with the scattering matrix \( S(\lambda) = S(\lambda; H, H_0) \) by the Birman-Krein formula

\[
\det S(\lambda) = \exp[-2\pi i \zeta(\lambda)], \quad \text{a.e. } \lambda \in \partial_0.
\]

In §§2-4 we consider the SSF in the case of trace class perturbations. In this section the SSF is constructed in terms of the perturbation determinant, and its properties are studied. The trace formula (1) is established in §3, while relation (3) is established in §4.

2. Suppose \( V = H - H_0 \in \Theta_1 \) and the perturbation determinant \( D(z) = D_{H/H_0}(z) \) is given by equality (1.1). For \( \Im z \neq 0 \) the function \( D(z) \) is holomorphic and \( D(z) \neq 0 \). According to (1.10), single-valued branches of the function \( \ln D(z) \) are fixed in the upper and lower half-planes by the condition \( \ln D(z) \to 0 \) as \( |\Im z| \to \infty \). From (1.11) it follows that for such a choice

\[
\ln D(\overline{z}) = \overline{\ln D(z)}.
\]

The SSF is constructed in the following theorem of M. G. Krein.

**Theorem 1.** For \( V \in \Theta_1 \) there is the representation

\[
\ln D(z) = \int_{-\infty}^{\infty} \zeta(\lambda)(\lambda - z)^{-1} d\lambda, \quad \Im z \neq 0,
\]

where

\[
\zeta(\lambda) = \pi^{-1} \lim_{\epsilon \to 0^+} \arg D(\lambda + i\epsilon).
\]

For a.e. \( \lambda \in \mathbb{R} \) the limit in (5) exists and

\[
\int_{-\infty}^{\infty} |\zeta(\lambda)| d\lambda \leq \|V\|_1,
\]

\[
\int_{-\infty}^{\infty} \zeta(\lambda) d\lambda = \text{Tr} V.
\]

Moreover, \( \zeta(\lambda) \leq k_+ (\zeta(\lambda) \geq -k_-) \) for a.e. \( \lambda \in \mathbb{R} \) if the operator \( V \) has only \( k_+ \) positive (\( k_- \) negative) eigenvalues. In particular \( \zeta(\lambda) \geq 0 \) (respectively, \( \zeta(\lambda) \leq 0 \)) for \( V \geq 0 \) (respectively, \( V \leq 0 \)).

**Proof.** The proof of this theorem is broken into three steps. The first of them consists in considering one-dimensional perturbations, the second in passing to perturbations of arbitrary finite rank, and the third in extending the results to general trace class perturbations. Let us note preliminarily that, once the representation (4) with a function \( \zeta = \zeta \in L_1(\mathbb{R}) \) has been established, equality (5) and the existence of the limit in it follow from Theorem 1.2.5.

Step 1. For \( V = \gamma(\cdot, v) v, \|v\|_1 = 1 \), the perturbation determinant is equal to

\[
D(z) = \det(I + \gamma(\cdot, R_0(\overline{z}))v) v
= 1 + \gamma R_0(z) v, v
= 1 + \gamma \int_{-\infty}^{\infty} (\lambda - z)^{-1} d(E_\lambda(\overline{z}) v, v).
\]

Suppose, for example, \( \gamma > 0 \). Then

\[
\Im D(z) = \gamma \|R_0(z) v\|^2 \Im z > 0
\]
in the upper half-plane. Moreover, for \( z = iy \), \( y \to +\infty \), for the function (8) we have

\[
D(iy) = 1 - (iy)^{-1} \gamma + o(y^{-1}).
\]

This relation follows directly from Lebesgue’s theorem on passage to the limit under the integral sign. We now consider the function \( \mathfrak{g}(z) = \ln D(z) \) holomorphic in the upper half-plane. Since \( \arg D(z) \to \infty \) as \( \Im z \to \infty \) and \( \Im D(z) > 0 \), it follows that \( 0 < \arg D(z) < \pi \) for all \( \Im z > 0 \). Further, from (9) it follows that

\[
\lim_{y \to +\infty} y \ln D(iy) = iy.
\]

Thus, the function \( \mathfrak{g}(z) \) has a positive, bounded imaginary part \( \Im \mathfrak{g}(z) = \arg \mathfrak{g}(z) = \arg D(z) \) and satisfies condition (1.2.9). Therefore, according to Theorem 1.2.9 and Corollary 1.2.10, for \( \mathfrak{g}(z) \) the representation (4) holds, where the function \( \zeta(\lambda) \) is defined by equality (5). The limit in (5) exists and, moreover, \( \zeta \in L_1(\mathbb{R}) \). From the condition \( 0 < \arg D(z) < \pi \) it follows that \( 0 \leq \zeta(\lambda) \leq 1 \). Comparing now equalities (4) and (10), we find that the integral in (7) is equal to \( \gamma \). This ensures relations (6) and (7). The representation (4) is carried over to \( z \) in the lower half-plane by passing to the complex conjugate. Similarly, for \( \gamma < 0 \) we have \( -\pi < \arg D(z) < 0 \), and hence \( -1 \leq \zeta(\lambda) \leq 0 \).

Step 2. The assertion proved extends to arbitrary finite-dimensional perturbations by means of the multiplication theorem (1.2) for perturbation
determinants. Let

\[ V_m = \sum_{\nu=1}^{m} \gamma_{\nu}(\nu, v) \nu_{\nu}, \quad H_m = H_0 + V_m. \]  

(11)

Then the difference \( H_m - H_{m-1} = \gamma_{\nu}(\nu, v) \nu_{\nu} \) is one-dimensional, and from (1.2) it follows that for \( H = H_n \)

\[ \ln D_{H/H_0}(z) = \sum_{m=1}^{n} \ln D_{H_m/H_{m-1}}(z). \]

(12)

To each term on the right we apply the representation (4), already proved for a one-dimensional perturbation. We denote the corresponding SSF by \( \xi^{(m)} \) and set

\[ \xi(\lambda) = \xi^{(m)}(\lambda), \quad \xi^{(m)} = \xi(H_m, H_{m-1}). \]

(13)

From equality (12) it then follows that the representation (4) with the SSF (13) holds also for \( \ln D_{H/H_0}(z) \). By (13) it is clear that \(-k_{-} \leq \xi(\lambda) \leq k_{+}\), while the representation (7) for \( \text{Tr} V_n \) can be obtained by addition of the corresponding "one-dimensional" equalities. Moreover, since \( \|\xi^{(m)}\|_{L_1} = \|\gamma_{\nu}\| \), this implies that

\[ \|\xi^{(m)}\|_{L_1} \leq \sum_{m=1}^{n} \|\xi^{(m)}\|_{L_1} = \|V_n\|, \]

(14)

which proves (6).

Step 3. Suppose now that \( V \) is an arbitrary trace class operator represented by the series (11) with \( m = \infty \). We define the SSF \( \xi = \xi(H, H_0) \) by the infinite series

\[ \xi(\lambda) = \sum_{m=1}^{\infty} \xi^{(m)}(\lambda), \quad \xi^{(m)} = \xi(H_m, H_{m-1}). \]

(15)

According to (14), this series converges absolutely in \( L_1 \), and for the function (15) inequality (6) holds. From this it also follows that (15) converges absolutely for a.e. \( \lambda \in \mathbb{R} \).

To prove the representation (4) we take a limit as \( n \to \infty \) in the representation (4) for \( \ln D_{H/H_0}(z) \). By the continuity of the determinant in \( \Theta_1 \), we have that \( D_{H/H_0}(z) \to D_{H/H_0}(z) \) as \( \|V_n - V\| \to 0 \). On the right-hand side of (4) the integral of \( \xi^{(m)}(\lambda - z)^{-1} \) converges to the integral of \( \xi(\lambda)(\lambda - z)^{-1} \), since by construction \( \|\xi^{(m)} - \xi\|_{L_1} \to 0 \) as \( n \to \infty \). Equality (7) for \( \text{Tr} V \) can also be obtained by a limiting procedure from the analogous equality for \( \text{Tr} V_n \).

Summarize, finally, in an expansion of the form (11) for the operator \( V \) only \( k_{+} \) of the numbers \( \gamma_{\nu} \) are positive. Then, as shown at Step 1, in the series (15) all the functions \( \xi^{(m)} \) are nonpositive except for \( k_{+} \) terms each of which does not exceed 1. From this, of course, it follows that the sum of the convergent series (15) also does not exceed \( k_{+} \), i.e., \( \xi(\lambda) \leq k_{+} \).

Remark 2. Under the conditions of Theorem 1, for the SSF there is a representation (15) in the form of a series composed from the SSF for one-dimensional perturbations. This series converges absolutely in \( L_1(\mathbb{R}) \) and therefore absolutely for a.e. \( \lambda \in \mathbb{R} \).

Remark 3. The SSF \( \xi \) is uniquely determined from the representation (4). This, in particular, determines the constant up to which the SSF is determined by the trace formula (1) considered on the class \( C_{0}\mathcal{C}(\mathbb{R}) \). An equivalent normalization can be obtained by requiring that \( \xi \in L_1(\mathbb{R}) \), which is automatically satisfied if \( \xi \) is defined by equality (5).

Remark 4. It follows from Theorem 1.2.5 that the integral (4) has boundary values as \( z \to \lambda \pm i0 \) for a.e. \( \lambda \). Therefore, the entire perturbation determinant \( D(\lambda \pm i\varepsilon) \) (and not only its argument) has a limit as \( \varepsilon \to 0 \) for a.e. \( \lambda \in \mathbb{R} \), and this limit cannot be equal to zero on a set of positive measure.

3. We continue the study of the properties of the SSF. For it there is an "addition formula."

**Proposition 5.** For \( H_1 - H_0 \in \Theta_1 \) and \( H - H_1 \in \Theta_1 \) for a.e. \( \lambda \in \mathbb{R} \)

\[ \xi(\lambda; H, H_1) + \xi(\lambda; H_1, H_0) = \xi(\lambda; H, H_0); \]

(16)

in particular,

\[ \xi(\lambda; H, H_0) = -\xi(\lambda; H_0, H). \]

**Proof.** It suffices to add representations (4) for \( \ln D_{H/H_0}(z) \) and \( \ln D_{H/H_1}(z) \) and apply the multiplication theorem for perturbation determinants (1.2). In addition one has to take into account (see Remark 3) that \( \xi \) is unique.

Comparing (16) with the estimate (6), we establish the continuity of the SSF in \( L_1(\mathbb{R}) \) as the operator arguments vary in \( \Theta_1 \). More precisely, we have

**Proposition 6.** For \( H_1 - H_0 \in \Theta_1 \) and \( H - H_0 \in \Theta_1 \)

\[ \|\xi(H, H_0) - \xi(H_1, H_0)\|_{L_1(\mathbb{R})} \leq \|H - H_1\|_1. \]

By Proposition 5, this already implies the joint continuity of the SSF, and

\[ \|\xi(H, H_0) - \xi(H_1, H_0)\|_{L_1(\mathbb{R})} \leq \|H - H_1\|_1 + \|H_0 - H_0\|_1. \]

In a similar way, comparing (16) with the last assertion of Theorem 1, we obtain

**Proposition 7.** For \( H_1 - H_0 \in \Theta_1 \), \( H - H_1 \in \Theta_1 \), and \( H \geq H_1 \) for a.e. \( \lambda \in \mathbb{R} \)

\[ \xi(\lambda; H, H_0) \geq \xi(\lambda; H_1, H_0). \]

(17)

We shall now discuss the behavior of the SSF as a function of \( \lambda \) on the "discrete spectrum."
Proposition 8. On component intervals of the set $\rho_0 \cap \rho$ of common regular points of the operators $H_0$ and $H$ the SSF $\xi(\lambda)$ assumes constant integral values. If $\lambda$ is an isolated eigenvalue of finite multiplicity $k_0$ of the operator $H_0$ and $k$ of the operator $H$, then
\[ \xi(\lambda + 0) - \xi(\lambda - 0) = k_0 - k. \] (18)

Proof. For $\lambda = \lambda_0 \in \rho_0 \cap \rho$ the perturbation determinant $D(\lambda)$ is real and nonzero. Therefore, the values of the function (5) on $\rho_0 \cap \rho$ are integers. Moreover, the function $D(\lambda)$ is continuous on each of the component intervals of the open set $\rho_0 \cap \rho$. Thus, on these intervals the SSF $\xi(\lambda)$ is also continuous and is hence constant. We now compute the jump of $\xi(\lambda)$ on passing over $\lambda_0$. According to property 4 of §1 the perturbation determinant $D(z)$ has a pole (or zero) of order $k_0 - k$ (respectively, $k_0$) at $\lambda_0$. To prove (18) it is only necessary to apply the argument principle and to take the symmetry $D(z) = D(\bar{z})$ into account.

Suppose that in some interval $(\beta_1, \beta_2)$ the spectrum $\sigma_0$ of the operator $H_0$ is discrete. Then by Weyl’s theorem for $V \in \mathfrak{S}_0$, the spectrum $\sigma$ of the operator $H$ possesses the same property. By the way, for $V \in \mathfrak{S}_1$ this result follows also from the properties (see Part 1 of §1) of the function $D(z)$. According to (18) for some points $\alpha_1, \alpha_2 \in (\beta_1, \beta_2)$, $\alpha_j \not\in \sigma_0 \cup \sigma$, $\alpha_1 < \alpha_2$,
\[ \xi(\alpha_2) - \xi(\alpha_1) = \mathrm{dim} E_0((\alpha_1, \alpha_2)) - \mathrm{dim} E((\alpha_1, \alpha_2)). \] (19)

The right-hand side here is equal to $\mathrm{Tr}[E_0((\alpha_1, \alpha_2)) - E((\alpha_1, \alpha_2))]$, which justifies the heuristic formula (2) on the discrete spectrum. In the general case it may be assumed that the left-hand side of (19) gives a “correct regularization” of this trace.

If the operator $H_0$ is lower semibounded, then $\|VR_0(z)\|_1 \rightarrow 0$ as $|z| \rightarrow \infty$ in any sector $\arg z \in (\theta_0, 2\pi - \theta_0)$, $\theta_0 > 0$. Therefore, $D(z) \rightarrow 1$ and $\arg D(z) \rightarrow 0$ as $|z| \rightarrow \infty$ in such sectors. According to (5), from this it follows that $\xi(\lambda) = 0$ for negative $\lambda$ of sufficiently large modulus. Suppose now that the spectrum of $H_0$ is discrete to the left of some point $\beta_0$. Then for $\lambda < \beta_0$ relation (19) determines the SSF itself
\[ \xi(\lambda) = \mathrm{dim} E_0(( - \infty, \lambda)) - \mathrm{dim} E(( - \infty, \lambda)), \lambda \not\in \sigma_0 \cup \sigma, \] (20)
and not only its increment. An analogous representation for the SSF is valid also for upper semibounded operators $H_0$. If $H_0$ is bounded, then the SSF is equal to zero for sufficiently large $|\lambda|$.

§ 3. The trace formula of M. G. Krein

In this section the trace formula (2.1) is established for $V \in \mathfrak{S}_1$. The SSF is hereby defined by equality (2.5).

1. The left-hand side of (2.1) is well-defined if
\[ f(H) - f(H_0) \in \mathfrak{S}_1. \] (1)

Verification of the inclusion (1) is therefore an element of the proof of the trace formula.

We first note that formula (2.1) is trivially satisfied for $f(\lambda) = 1$, while equality (2.7) can be considered as the trace formula for $f(\lambda) = \lambda$.

To extend (2.1) to sufficiently arbitrary functions we start from the representation (2.4) for the logarithm of the perturbation determinant. We first verify (2.1) in the special case $f(\lambda) = (\lambda - z)^{-n}$, $\text{Im} z \neq 0$, $n = 1, 2, \ldots$.

Lemma 1. There are the equalities
\[ \text{Tr}[R_0(z) - R(z)] = \int_{-\infty}^{\infty} \xi(\lambda)(\lambda - z)^{-n} d\lambda, \quad \text{Im} z \neq 0, \] (2)
and, more generally,
\[ \text{Tr}[R_0^n(z) - R^n(z)] = n \int_{-\infty}^{\infty} \xi(\lambda)(\lambda - z)^{-n-1} d\lambda, \quad \text{Im} z \neq 0, \quad n \geq 1. \] (3)

Proof. We obtain (2) by differentiating (2.4) and using equality (1.4). Differentiation under the integral sign in (2.4) is, of course, legitimate, since $\xi \in L_1(\mathbb{R})$. Further differentiations of (2) lead to (3). It is hereby noted that all the derivatives of $R_0(z) - R(z)$ exist in $\mathfrak{S}_1$ by the resolvent identity.

The next class of functions for which (2.1) is verified consists of the exponentials $f(\lambda) = \exp(-it\lambda)$, $t$ a real parameter, when $f(H) = U(t)$.

Lemma 2. For $f(\lambda) = \exp(-it\lambda)$ the inclusion (1) is fulfilled and so does also the trace formula (2.1). Moreover,
\[ \|U(t) - U_0(t)\|_1 \leq |t|\|V\|_1. \] (4)

Proof. The inclusion (1) and the estimate (4) follow immediately from the obvious equality
\[ U(t) - U_0(t) = -iU(t) \int_0^t U^*(s)VU_0(s) ds. \]

According to the representation of the resolvent (1.4.4)
\[ -i(R(\mu + i\varepsilon) - R_0(\mu + i\varepsilon)) = \int_{-\infty}^{\infty} (U(t) - U_0(t)) e^{int - i\varepsilon t} dt, \quad \varepsilon > 0, \]
where by (4) the integral here converges in the trace class norm. We take the trace of both sides of this equality and note that the function $e^{-it\varepsilon}\text{Tr}(U(t) - U_0(t))$ belongs to $L_1(\mathbb{R}) \cap L_2(\mathbb{R})$. By the formula for inversion of Fourier integrals
\[ \text{Tr}(U(t) - U_0(t)) = (2\pi i)^{-1} e^{it} \int_{-\infty}^{\infty} \text{Tr}(R(\mu + i\varepsilon) - R_0(\mu + i\varepsilon)) e^{-i\varepsilon t} d\mu, \quad t > 0. \]
We now substitute the expression (2) into the right-hand side and note that under the condition $\xi \in L_1(\mathbb{R})$ the integrals over $\lambda$ and $\mu$ converge absolutely. Hence, by Fubini's theorem the order of integration can be interchanged, so that

$$
\text{Tr}(U(t) - U_0(t)) = -s(2\pi)^{-1}e^{it} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} d\lambda d\mu e^{-imt}(\lambda - \mu - it)^{-2}.
$$

(5)

By Cauchy's residue theorem the integral over $\mu$ is here equal to $-2\pi ie^{i2\pi t}\omega(\lambda)$. Therefore, equality (5) leads to the trace formula (2.1) for $f(\lambda) = \exp(-it\lambda)$, $t > 0$. The case of negative $t$ corresponds to the conjugate relation. □

It follows from Lemma 2 that the trace formula (2.1) holds for linear combinations of exponentials $\exp(-it\lambda_n)$. By closure in a suitable metric, formula (2.1) can now be extended to sufficiently arbitrary functions $f$. Of course, in doing this one should be concerned about the inclusion (1). For convergence of the integral on the right in (2.1) for $\xi \in L_1(\mathbb{R})$ it is natural to suppose that $f' \in L_1(\mathbb{R})$. It is known (see [80]), however, that for $H - H_0 \in \mathfrak{E}_1$ the latter condition still does not ensure the inclusion (1). Therefore, to prove (2.1) more restrictions on $f$ are required.

**Theorem 3.** Suppose $V \in \mathfrak{E}_1$ and the function $f$ is continuously differentiable while its derivative admits the representation

$$
f'(\lambda) = \int_{-\infty}^{\infty} \exp(-it\lambda) \, ds(m)(t), \quad |s(m)(t)| < \infty,
$$

(6)

with a finite (complex) measure $m$. Then the inclusion (1) is fulfilled and the trace formula (2.1) holds for the SSF (2.5).

**Proof.** From (6) it follows that

$$
f(\lambda) = f_0 - \int_{-\infty}^{\infty} (\exp(-it\lambda) - 1)(it)^{-1} \, ds(m)(t) - f_0 = \text{const},
$$

and hence by the spectral theorem

$$
f(H) - f(H_0) = \int_{-\infty}^{\infty} (U(t) - U_0(t))(it)^{-1} \, ds(m)(t).
$$

(7)

By (4) the integral here converges in the trace norm, so that the inclusion (1) holds. We now take the trace of both sides of (7) and replace $\text{Tr}(U(t) - U_0(t))$ by its expression (2.1). This gives the equality

$$
\text{Tr}[f(H) - f(H_0)] = \int_{-\infty}^{\infty} ds(m)(t) \int_{-\infty}^{\infty} \xi(\lambda) \exp(-it\lambda) \, d\lambda.
$$

By Fubini's theorem the order of the absolutely convergent integrals can be interchanged. According to the representation (6) for $f'$ from this we obtain formula (2.1). □

We note that under condition (6) there is the estimate

$$
|\|f(H) - f(H_0)\|_1| \leq C\|H - H_0\|_1, \quad C = |m(\mathbb{R})|.
$$

(8)

The store of functions $f$ given by Theorem 3 is rather large. In any case it contains the Schwartz class $\mathcal{S}(\mathbb{R})$ and hence also $C_0^\infty(\mathbb{R})$. It is also clear that the function $f'$ satisfies (6) if $f'$ belongs to the Sobolev class $W_2^2(\mathbb{R})$ for $2\alpha > 1$. For such $f'$ the measure $m$ in (6) is absolutely continuous and the derivative $ds(m)/dt$ is square-integrable with the weight $(1 + |t|)^{2\alpha}$, and is hence summable. Thus, locally (i.e., on any bounded interval) it is clearly sufficient that the function $f$ have two bounded derivatives.

Condition (6) does not require decay of $f$ at infinity. Thus, (6) is satisfied for any periodic function $f'$ if the series of its Fourier coefficients converges absolutely. In particular, it is possible to set $f(\lambda) = \lambda$. In general, growth at infinity can be admitted which does not exceed linear growth. For example, functions $f(\lambda)$ equal for sufficiently large $\lambda$ to the product $\lambda^\alpha \ln^\beta \lambda$, where $\alpha < 1$ and $\beta$ is arbitrary, are suitable. For such a function the measure $m$ in (6) is absolutely continuous, while the derivative $ds(m)/dt$ has a singularity as $t \to 0$ which turns out to be integrable.

2. In this part, without assuming that $V$ is of trace class, we discuss properties of the SSF following from the validity of the trace formula. We suppose that for any $f \in C_0^\infty(\mathbb{R})$ the inclusion (1) is satisfied and for some function $\xi \in L_1^{\text{loc}}(\mathbb{R})$ relation (2.1) holds. Such a function $\xi$ is fixed up to an arbitrary constant. If, moreover, (1) and (2.1) hold for at least one function $f$, for which the limits at $\pm \infty$ do not exist or they exist and $\lim_{+\infty} f(x) \neq \lim_{-\infty} f(x)$, then $\xi$ is uniquely determined.

Other restrictions are also imposed on $\xi$ by the condition that (2.1) be satisfied for all $f \in C_0^\infty(\mathbb{R})$. Thus, the SSF $\xi$ must be constant on each of the component intervals $\Lambda_\alpha$ of the set $\rho_\alpha \cap \rho$. Indeed, for $f \in C_0^\infty(\mathbb{R})$ the operators $f(H_0) = f(H) = 0$. The left-, and hence also the right-hand side, of (2.1) are equal to zero. The equality $\xi(\lambda) = \xi_\alpha$ for $\lambda \in \Lambda_\alpha$ now follows from the Du Bois-Reymond lemma. Further, if the operators $H_0$ and $H$ are bounded, then the value of $\xi$ to the left of $\beta_1 - \inf(\sigma_0 \cup \sigma)$ and to the right of $\beta_2 = \sup(\sigma_0 \cup \sigma)$ must necessarily coincide. This can easily be seen by considering $f \in C_0^\infty(\mathbb{R})$ equal to a constant $f_0$ for $\lambda \in [\beta_1, \beta_2]$. Setting $\xi(\lambda) = \xi_+ \lambda \lambda < \beta_1$, and $\xi(\lambda) = \xi_+ \lambda \lambda > \beta_2$, we find that the right-hand side of (2.1) is equal to $(\xi_--\xi_+)/f_0$. Since the left-hand side of (2.1) is equal to zero and $f_0$ is arbitrary, this implies that $\xi_\alpha = \xi_+$. In the semibounded case, where $\beta_1 > -\infty$, we always assume that $\xi_\alpha = 0$.

When the properties of the function $\xi$ are improved, the integral on the right in (2.1) may retain its meaning even without the condition $f' \in L_1(\mathbb{R})$. For example, if the SSF $\xi$ is continuous on some segment, then it suffices to suppose that the function $f$ has bounded variation there. In particular, if
the finite segment \( \Lambda = [\alpha_1, \alpha_2] \) does not intersect \( \sigma_0 \cup \sigma \), then
\[
\int_{\Lambda} \xi(\lambda) \, df(\lambda) = \xi(\lambda_0) [f(\alpha_2) - f(\alpha_1)],
\]
(9)
where \( \xi_0 \) is the constant value of \( \xi(\lambda) \) on \( \Lambda \).

This equality makes it possible to extend the definition of the integral (2.1) to functions that are defined and sufficiently smooth only in a neighborhood of the union of the spectra of the operators \( H_0 \) and \( H \). We shall assume that the set \( \sigma_0 \cup \sigma \) is covered by a finite collection \( \Omega = \bigcup \Omega_n \) of nonintersecting intervals \( \Omega_n, n = 1, \ldots, N \), (one or two of which may be infinite). For any function \( f \) continuous on \( \Omega \), the integral (2.1) over \( R(\Omega) \) is defined by equality (9). Here, if the segment \( \Lambda \subset R(\Omega) \) is infinite, we may set \( \xi(\lambda) = 0 \) for \( \lambda \in \Lambda \), and hence the integral (9) is equal to zero. In other words,
\[
\int_{-\infty}^{\infty} \xi(\lambda) \, df(\lambda) := \int_{-\infty}^{\infty} \xi(\lambda) \, d\tilde{f}(\lambda),
\]
(10)
where \( \tilde{f} \) is any extension of \( f \) to the entire axis which has locally bounded variation. In particular, it may be assumed that \( \tilde{f}(\lambda) = 0 \) for \( \lambda \in R(\Omega) \).

The left-hand side of (2.1) is determined only by the values of \( f \) at points of the spectra \( \sigma_0 \) and \( \sigma \). For the definition adopted of the integral on the right in (2.1), equality (2.1) itself can also be extended to functions defined only on \( \Omega \). Namely, we have

**Lemma 4.** Suppose \( f \) is defined on \( \Omega \) and for some extension of it to the entire axis (2.1) holds. Then the trace formula (2.1) is true also for \( f \) itself.

**Proof.** It suffices to use the trace formula for \( \tilde{f} \) and note equality (10). \( \square \)

Finally, we show that it suffices to verify the trace formula (2.1) for functions \( f \) defined only on \( \Omega \). We restrict our consideration to functions of the class \( C^0_\infty \). The next assertion is in some sense opposite to Lemma 4.

**Lemma 5.** Suppose for some \( \xi_\infty \in L^{(0)}_1(\Omega) \) the trace formula (2.1) holds for all \( f \in C^0_\infty(\Omega) \). Then there exists a function \( \xi \in L^{(0)}_1(R) \) such that the trace formula holds for any \( f \in C^0_\infty(R) \).

**Proof.** By the Du Bois-Reymond lemma we see first of all that \( \xi_\infty \) is constant in neighborhoods of the end points of \( \Omega \). We further note that on each of the intervals \( \Omega_n \), it is possible to add a constant to \( \xi_\infty \) without violating (2.1) for \( f \in C^0_\infty(\Omega) \). For a suitable choice of these constants it can be arranged for \( \xi_\infty \) to have the same value at contiguous end points of each pair of adjacent intervals \( \Omega_n \) and \( \Omega_{n+1} \). We extend the function \( \xi_\infty \) to the interval between \( \Omega_n \) and \( \Omega_{n+1} \) by this common constant. The function \( \xi \) so constructed is constant on the component intervals of the set \( \rho_0 \cup \rho \). For it formula (2.1) already holds for any \( f \in C^0_\infty(R) \). Indeed, \( f \) can be decomposed into the sum \( f = f_1 + f_2 \), where \( f_1 \in C^0_\infty(\Omega) \) and \( f_2 \in C^0_\infty(R) \) is nonzero only on the intervals of constancy of the function \( \xi \).

\[\text{§ 3. The Trace Formula of M. G. Krein} \]

For \( f_1 \) formula (2.1) holds by hypothesis, while for \( f_2 \) both sides of (2.1) are obviously equal to zero. \( \square \)

3. We return to considering trace class perturbations. The next assertion is obtained by combining Theorem 3 and Lemma 4.

**Theorem 6.** Suppose \( f \) is defined on \( \Omega \) and admits an extension to a function satisfying on \( \mathbb{R} \) the conditions of Theorem 3. Then for \( V \in \mathcal{S}_1 \) the trace formula (2.1) holds.

Theorem 6 shows that off the spectra admissible functions \( f \) may have arbitrary singularities. If \( \inf \Omega > -\infty \) (or \( \sup \Omega < \infty \)), then the behavior of \( f \) as \( \lambda \to -\infty \) (or \( \lambda \to +\infty \)) also remains arbitrary.

As concerns (local) smoothness on \( \Omega \), under the conditions of Theorem 6 the function \( f \) is necessarily continuously differentiable. As already noted in part 1, the existence of two bounded derivatives suffices for the validity of (2.1). Relaxation of the last condition is a substantial problem.

This problem was considered by M. Sh. Birman and M. Z. Solomyak [45, 46] and V. V. Peller [72]. The main element in the extension of (2.1) to a broader class of functions (as compared with Theorem 3) consists in verifying the inclusion (1) for this class. Sufficient, but close to necessary, conditions for the validity of the inclusion (1) and of inequality (8) have been found in [72] in terms of some Besov class \( B \) (for a precise formulation for unitary operators see [54]). Extension of (2.1) to \( f \in B \) is obtained by a suitable approximation of \( f \) by smooth and compactly supported (or at least satisfying the conditions and bounds) functions \( f_n \).

A somewhat different approach to the construction of the SSF, which extends, in particular, the store of admissible functions, was proposed in the work of M. Sh. Birman and M. Z. Solomyak [46]. This approach is based on considering families of operators \( H = H_0 + sV \), where \( s \in [0, 1] \), and their spectral measures \( E_s = E_{\lambda_s} \). It was shown in [45, 46] that for a rather broad class of functions \( f \) in trace norm there exists a continuous derivative which can be represented in the form of a double operator integral (see (6.8))

\[
\frac{df(H_\lambda)}{ds} = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(\mu) - f(\lambda) \, dE_s(\mu) V \, dE_s(\lambda).
\]

Moreover,
\[
\text{Tr} \frac{df(H_\lambda)}{ds} = \int_{-\infty}^{\infty} [f(\lambda) \, d\text{Tr}(E_s(\lambda) V)].
\]

By integrating the last equality over \( s \in [0, 1] \), for \( \text{Tr}(f(H_\lambda) - f(H_0)) \) we obtain a representation of the form (2.1) where, however, the role of \( \xi(\lambda) \) is played by the measure \( d\Xi(\lambda) \).

\[
\Xi(X) = \int_{-\infty}^{\infty} \text{Tr}(E_s(\lambda)X) \, ds.
\]

Its absolute continuity can be established by means of Theorem 1. Namely, comparing the two representations of \( \text{Tr}(U(t) - U_0(t)) \), by the uniqueness theorem for Fourier-Stieltjes integrals, we find that \( d\Xi(\lambda) = \xi(\lambda) \, d\lambda \). Here (11) gives a new expression for the SSF.

We further note that the classes considered in [45, 46, and 72] include functions \( f \) for which the derivatives \( f' \) satisfy a Hölder condition (uniformly on the entire axis) with some positive exponent and \( f' \in L^p(R) \) for some \( p \in [1, \infty) \). These conditions are convenient for verifying the validity of inclusion (1) and the trace formula (2.1). However, this store of functions does not contain (and is not contained in) the store of functions in Theorem 3.
§4. THE SCATTERING MATRIX AND BIRMAN-KREIN FORMULA

4. Connection with the scattering matrix.

The Birman-Krein formula

1. The basic result in this section is

THEOREM 1. For \( V \in \mathcal{S}_1 \) relation (2.3) holds.

We shall present two proofs of this theorem. The first of them, just as the proof of Theorem 2.1, follows the “step” method.

PROOF. For a one-dimensional perturbation \( V = \gamma(v, v)v \), \( \|v\| = 1 \), the scattering matrix was computed explicitly in §6.7. Namely, by equalities (6.7.9), (6.7.10)

\[
\text{Det} S(\lambda) = 1 - 2\pi i \gamma D^{-1}(\lambda + i0) D(\lambda - i0),
\]

where \( D(z) \) is defined by relation (6.7.2) and, hence, coincides with (2.8). Equalities (1.11) and (2.5) show that the right-hand side of (1) is equal to \( \exp(-2\pi i \xi(\lambda)) \).

We decompose a finite-dimensional perturbation \( V = V_n \) into a sum of one-dimensional operators \( \gamma(v, v)v_i \), \( i = 1, \ldots, n \), and use the notation (2.11). The SSF can then be represented by the sum (2.13). On the other hand, by equality (7.1.5),

\[
S(\lambda; H, H_0) = \tilde{S}(\lambda; H, H_{m-1}) \cdots \tilde{S}(\lambda; H_2, H_1) S(\lambda; H_1, H_0),
\]

where the operators \( \tilde{S}(\lambda; H_m, H_{m-1}) \) are unitarily equivalent (the realization of this equivalence depends on \( m \)) to the scattering matrices \( S(\lambda; H_m, H_{m-1}) \).

From this it follows that

\[
\text{Det} S(\lambda; H, H_0) = \prod_{m=1}^{n} \text{Det} S(\lambda; H_m, H_{m-1}).
\]

Therefore, relation (2.3) for the finite-dimensional case can be obtained by multiplying such relations for one-dimensional perturbations.

An arbitrary operator \( V \in \mathcal{S}_1 \) is approximable in the trace norm by finite-dimensional operators \( V_n \). According to Remark 2.2, for the SSF we have \( \xi(\lambda; H_n, H_0) \rightarrow \xi(\lambda; H, H_0) \) as \( n \rightarrow \infty \) for a.e. \( \lambda \in \mathbb{R} \). Convergence of the scattering matrices

\[
\lim_{n \rightarrow \infty} S(\lambda; H_n, H_0) - S(\lambda; H, H_0) = 0, \quad \text{a.e.} \quad \lambda \in \mathbb{R},
\]

along some sequence \( \{n_k\} \) follows from Proposition 7.6.7 (see relation (7.6.9)). From (3) it follows that

\[
\lim_{n_k \rightarrow \infty} \text{Det} S(\lambda; H_{n_k}, H_0) = \text{Det} S(\lambda; H, H_0).
\]

Thus, passing to the limit as \( n_k \rightarrow \infty \) in equality (2.3) for the pair \( H_0, H_0 + V \), we obtain (2.3) for the pair \( H_0, H_0 + V \).

PROOF. The second proof of Theorem 1 is based on the identity

\[
\text{Det}[I - (V - V R(z)V) R_0(z) - R_0(\mathcal{F})] = D^{-1}(z) D(z),
\]

which follows from (1.6) for \( z' = z \). In (6) we pass to the limit on \( z = \lambda + i\varepsilon \) as \( \varepsilon \rightarrow 0 \). According to (1.11) and (2.5) the right-hand side of (6) is equal to \( \exp[-2\pi i \xi(\lambda)] \), and its limit is equal to \( \exp[-2\pi i \xi(\lambda)] \). It remains to show that

\[
\lim_{\varepsilon \rightarrow 0} \text{Det}[I - (V - V R(z)V) R_0(z) - R_0(\mathcal{F})] = \text{Det} S(\lambda), \quad z = \lambda + i\varepsilon.
\]

Suppose \( V = G^* G \), where \( G_0 \) and \( G \) are Hilbert-Schmidt operators. By property (1.7.11) the determinant on the left in (7) is equal to

\[
\text{Det}[I - (I - G_0 R(z)G^*)] G_0(R_0(z) - R_0(\mathcal{F})) G^*.
\]

According to Theorem 6.1.9 and Corollary 6.1.10, the operator \( \tilde{B}(\lambda + i\varepsilon) = G_0 R(\lambda + i\varepsilon) G^* \) has a limit in the norm of \( \mathcal{S}_1 \) as \( \varepsilon \rightarrow 0 \) for a.e. \( \lambda \). Similarly, according to Theorem 6.1.5 and Corollary 6.1.6, there exists the limit

\[
\lim_{\varepsilon \rightarrow 0} G_0(R_0(\lambda + i\varepsilon) - R_0(\mathcal{F})) G^* = 2\pi i Z_0^* (\lambda; G_0) Z_0(\lambda; G), \quad \text{a.e.} \quad \lambda,
\]

in the norm of \( \mathcal{S}_1 \), the equality itself being a consequence of (1.5.7). We recall (see Part 1 of §7.5) that the operators \( Z_0(\lambda; G_0) \) and \( Z_0(\lambda; G) \) act from \( \mathcal{F}^* \) to \( h_0(\lambda) \) and belong to the Hilbert-Schmidt class. Thus, by the continuous dependence of \( \text{Det}(I + A) \) on \( A \) as \( A \) varies in the norm of \( \mathcal{S}_1 \), from this it follows that the limit in (7) is equal to

\[
\text{Det}[I - 2\pi i (I - \tilde{B}(\lambda + i0)) Z_0^* (\lambda; G_0) Z_0(\lambda; G)].
\]
Again using property (1.7.11), we rewrite this determinant in the form

$$\text{Det}(I - 2\pi i Z_0(\lambda; G)(I - \tilde{b}(\lambda + i0))Z_0^*(\lambda; G_0)).$$

According to the representation (5.5.7), the operator under the last determinant sign is equal to $S(\lambda).$ □

We emphasize again that the SSF takes into account the full spectra of the operators $H_0$ and $H$, while the scattering matrix can be computed directly in terms of $H_0^{(0)}$ and $H^{(0)}$. Nevertheless, relation (2.3) is not contradictory since, for example, on the discrete spectrum the variation of $\xi$ reduces, according to (2.18), to only integer jumps.

Relation (2.3) makes it possible to connect results on the behavior of the determinant of the scattering matrix and the SSF for perturbations of definite sign. Namely, we consider a family of Hamiltonians $H(\gamma) = H_1 + \gamma V$ for $H_1 - H_0 \in \mathfrak{S}_1$, $V \in \mathfrak{S}_1$, and $V \geq 0$ or $V \leq 0$. By Theorem 7.8.9 as $\gamma$ increases the point $\text{Det} S(\lambda; H(\gamma))$, $H(\gamma) \in \mathcal{T}$ for a.e. $\lambda$ moves in clockwise (counterclockwise) direction if $V \geq 0$ ($V \leq 0$). By Proposition 2.7 the SSF $\xi(\lambda; H(\gamma), H_0)$ increases for $V \geq 0$ and decreases for $V \leq 0$. Relation (2.3) shows that these two results are equivalent to one another.

2. In applications, formula (2.3) is often established in "differentiated" form. Namely, by taking the logarithm of (2.3) and differentiating formally both sides with respect to $\lambda$, we find that

$$\text{Tr} S^*(\lambda)S(\lambda) = -2\pi i \xi'(\lambda).$$

(8)

At the same time, it follows from (1.4) that

$$\frac{d}{d\lambda} \arg D(\lambda + i\epsilon) = \text{Im Tr}(R_0(\lambda + i\epsilon) - R(\lambda + i\epsilon)).$$

Comparing this equality with (2.5) and (8), we arrive at the relation

$$\text{Tr} S^*(\lambda)S(\lambda) = 2i \lim_{\epsilon \to 0} \text{Im Tr}(R(\lambda + i\epsilon) - R_0(\lambda + i\epsilon))$$

$$= 2\pi i \frac{d}{d\lambda} \text{Tr}(E(\lambda) - E_0(\lambda)).$$

(9)

We point out that the inverse passage from (9) to (2.3) is not true even formally. This is connected with the fact that on integrating (8) the function $\xi(\lambda)$ is recovered only up to a constant.

Justification of (9) requires considerable additional assumptions of "smoothness" type. Moreover, differentiation of $S(\lambda)$ with respect to $\lambda$ already presupposes that an identification of the spaces $b_0(\lambda)$ in the direct integral (2.4.2) has been chosen. However, in applications to differential operators and in the Friedrichs-Faddeev model (see the works of V. S. Buslaev and L. D. Faddeev [54] and V. S. Buslaev [52], [53]) relation (9) rather than (2.3) is usually realized. Relation (9) is then being proved for all (and not only a.e.) $\lambda$ and, with the exception of some singular points, both sides of (9) are continuous with respect to $\lambda$.

§5. The SSF in the unitary case

In this section and the next the theory developed in §§2-4 is carried over to unitary operators.

1. Let $U_0$, $U$ be unitary operators in a Hilbert space $\mathcal{H}$. The following relation, completely analogous to (2.1), is called the trace formula for the pair $U_0$, $U$ and a differentiable function $g$ on the unit circle $\mathbb{T}$:

$$\text{Tr}(g(U) - g(U_0)) = \int_{\mathbb{T}} \eta(\mu) d g(\mu), \quad \eta \in L_1(\mathbb{T}).$$

(1)

We assume everywhere that $\mathbb{T}$ and each of its arcs $(\mu_1, \mu_2)$ are traversed counterclockwise. The real function $\eta(\mu) = \eta(\mu; U, U_0)$ is called the SSF for the pair $U_0$, $U$. Sometimes formula (1) is transformed by the change of variable $\mu = e^{it}$, $t \in (-\pi, \pi]$, and the function $\hat{\eta}(i) = e^{i\theta}$ is also called SSF.

Under the condition $V = U - U_0 \in \mathfrak{S}_1$ adopted everywhere in this section the construction of the SSF can be carried out essentially as in §2. One of the differences is that in the unitary case the SSF becomes multivalued. Indeed, the SSF $\eta$ is determined up to an arbitrary additive term by relation (1), where $g$ is an arbitrary function of the class $C^\infty(\mathbb{T})$. In the selfadjoint case this ambiguity was eliminated either by the condition $\xi \in L_1(\mathbb{R})$ or by the requirement that the representation (2.4) be valid for $\ln D_{H/H_0}(x)$. Since $1 \in L_1(\mathbb{T})$, the condition $\eta \in L_1(\mathbb{T})$ does not determine the constant.

As concerns a representation of the form (2.4) for the perturbation determinant, it carries over to the unitary case and has the form

$$\ln D_{H/H_0}(\xi) = \int_{\mathbb{T}} \eta(\mu)(\mu - \xi)^{-1} d \mu, \quad |\xi| \neq 1.$$  

(2)

Here, of course, single-valued branches of the analytic function $\ln D(z)$, $D = D_{H/H_0}$, for $|\xi| > 1$ and $|\xi| < 1$ are assumed. Such branches exist, since $\ln D(\xi)$ is holomorphic and nonzero for $|\xi| \neq 1$, and are determined by specifying the value of $\ln D(\xi)$ at any two points $\xi_1$, $|\xi_1| > 1$, and $\xi_2$, $|\xi_2| < 1$. In the selfadjoint case relation (2.4) automatically fixes the branches of $\ln D(\xi)$ in both half-planes. In the unitary case the exterior and interior of the unit circle play a different role. Similarly to §2, the representation (2) can be satisfied in the domain $|\xi| > 1$ only if $\ln D(\xi) \to 0$ as $|\xi| \to \infty$. This determines the branch of $\ln D(\xi)$ for $|\xi| > 1$ but does not eliminate the ambiguity of the function $\eta$, since change of it by a constant value does not change the value of the integral (2) for $|\xi| > 1$. On the other hand, for $|\xi| < 1$ a choice of the branch of $\ln D(\xi)$ fixes the function $\eta$ uniquely. However, the representation (2) may be considered for any of the branches of $\ln D(\xi)$. Note that, as soon as (2) is satisfied for one branch of $\ln D(\xi)$. 
this representation is also valid for any other branch, if a suitable integer is added to η.

Thus, in the unitary case the SSF is defined only up to an integer additive term. The multivaluedness of the SSF cannot be removed in principle. More precisely, as we shall see in the next section, an attempt to select a unique branch η(U, U₀) runs into contradiction with the continuity of η(U, U₀) with respect to U (or U₀).

2. In this part we shall establish the representation (2) for some real-valued function η ∈ L₁(𝕋). We recall that by Lemma 1.3 for U − U₀ ∈ 𝕀 the perturbation determinant D(ξ) = D_U(ξ) → 1 as |ξ| → ∞ and satisfies the symmetry relation (1.12). We define the branch of ln D(ξ) for |ξ| > 1 by the condition arg D(ξ) → 0, |ξ| → ∞, while for |ξ| < 1 we choose the branch of ln D(ξ) arbitrarily. For any such branch

\[ \ln D(ξ) = \ln D(ξ) + \ln D(0). \]

In the theory of unitary operators the perturbation is naturally introduced multiplicatively (from the left or right) (see §1.11). Namely, together with V we consider the operator N = V U₀⁻¹. Then the (left) multiplicative perturbation M = I + N = U U₀⁻¹ is a unitary operator. Since, moreover, N ∈ 𝕀, by the spectral theorem for some orthonormal sequence 𝜙ₘ and 𝜙ₘ(⋅; 𝜙ₘ) 𝜙ₘ there is the expansion

\[ N = \sum_{m=1}^{∞} \tau_m P_m, \quad \sum_{m=1}^{∞} |\tau_m| = \|N\| < ∞. \]

The spectrum 1 + 𝜙ₘ = 𝑒𝑖θₘ of the operator M lies on the unit circle and can accumulate only at the point 1. We set

\[ U_n = (I + \tau_1 P_1) U_{n-1} = \left( I + \sum_{m=1}^{n} \tau_m P_m \right) U_0. \]

From (4) it follows that \(\|U_n - U\| → 0\) as \(n → ∞\).

To prove the existence of the SSF we again use the "step" method. Here we must ensure the convergence of the corresponding series of η(Uₙ, U₀⁻¹) in L₁(𝕋) which requires a correct normalization of these functions. One of the possible normalizations of the SSF is fixed by the condition

\[ \int_π η(μ; U, U₀) μ⁻¹ dμ = i \sum_{m=1}^{∞} θ_m, \quad θ_m ∈ (-π, π]. \]

where 𝑒𝑖θₘ are the eigenvalues of the operator U U₀⁻¹. For such a normalization it turns out that η(U₀, U₀) = 0, and

\[ \int_{-π}^{π} |η(e^{it}; U, U₀)| dt ≤ \sum_{m=1}^{∞} |θ_m|. \]

Since \(2|θ_m| ≤ π|τ_m|\) for \(θ_m ∈ (-π, π]\), for \(V ∈ 𝕀\) the series in (6) and (7) converge.

The role of Theorem 2.1 in the theory of perturbations of unitary operators is played by the following assertion.

**Theorem 1.** For \(U - U₀ ∈ 𝕀\), the representation (2) holds, where

\[ η(μ) = μ⁻¹ \lim_{r⁻→0} \arg D(rμ) - (2π)⁻¹ \arg D(0) \]

and the limit in (8) exists for a.e. \(μ ∈ 𝕋\) and η ∈ L₁(𝕋). If the operator \(UU₀⁻¹\) has only a finite number of eigenvalues on the open upper (lower) semicircle, then for a.e. μ ∈ 𝕋 the function η is upper (lower) semibounded.

**Proof.** We note first of all that by Theorem 1.2.11 from the representation (2) with a function η = η ∈ L₁(𝕋) it follows that the limit on the right in (8) exists. Moreover, according to the first inversion formula (1.2.15) and the relation |D(0)| = 1 the function η can be recovered by equality (8). It suffices to establish the representation (2) itself with η = η ∈ L₁(𝕋) for |ξ| < 1. Comparing equalities (1.2.14) and (3), we then find that also for |ξ| > 1 the function ln D(ξ) coincides with the integral on the right side of (2).

We again commence the proof of (2) for |ξ| < 1 with the consideration of a one-dimensional perturbation \(N = τ(⋅, 𝜙) 𝜙, \quad τ = e^{iθ} - 1, \quad θ ∈ (-π, π], \quad ∥𝜙∥ = 1\), where (cf. (2.8))

\[ D(ξ) = \det[I + τ(⋅, U₀ R(ξ) 𝜙) 𝜙] = 1 + τ(R(ξ) U₀ 𝜙, 𝜙), \quad R(ξ) = (U₀ - ξ)⁻¹. \]

This equality can also be written in the form

\[ D(ξ) = 1 + 2⁻¹ τ + 2⁻¹ τΨ(ξ) = e^{iθ/2} [\cos(θ/2) + i \sin(θ/2)]Ψ(ξ), \]

where

\[ \Psi(ξ) = ((U₀ + ξ)(U₀ - ξ)⁻¹ 𝜙, 𝜙), \quad \Re Ψ(ξ) = (1 - |ξ|^2)∥(U₀ - ξ)⁻¹ 𝜙∥² > 0. \]

Since \(D(0) = 1 + τ = e^{iθ}\), the branch of arg D(ξ) can be fixed by the condition \(\arg D(0) = θ ∈ (-π, π]\). We shall demonstrate that the function arg D(ξ) is bounded for |ξ| < 1. We set \(Φ(ξ) = e^{iθ/2} D(ξ)\). For this function

\[ \arg Φ(ξ) = \arg D(ξ) - 2⁻¹ \arg D(0), \quad \arg Φ(0) = θ/2 \]

and

\[ \Im Φ(ξ) = \sin(θ/2) \cdot \Re Ψ(ξ). \]

By (9) \(\Im Φ(ξ) > 0\) for \(θ ∈ (0, π]\), and hence in this case \(0 < \arg Φ(ξ) < π\). Similarly, \(\Im Φ(ξ) < 0\) and \(-π < \arg Φ(ξ) < 0\) for \(θ ∈ (-π, 0]\). The
boundedness of \( \arg D(\zeta) \) now follows from relation (10). Thus, the function \( \ln D(\zeta) \) satisfies all the conditions of Corollary 1.2.13. Therefore, for \( |\zeta| < 1 \) we have the representation (2) with the function \( \eta \) defined by equality (8).

According to (10), for the adopted normalization we have the inequalities \( 0 \leq \eta(\mu) \leq 1 \) if \( \theta \in [0, \pi] \) and \( -1 \leq \eta(\mu) \leq 0 \) if \( \theta \in (-\pi, 0) \). Finally, for \( \zeta = 0 \) relation (2) shows that the integral on the left in (6) is equal to \( i\theta \).

From this, in turn, it follows that the integral in (7) is equal to \( |\theta| \).

For an arbitrary \( V \in \text{Sf} \), we consider the expansion of the operator \( N \) in the series (4) and use the notation (5). Since the operators \( U_m \) and \( U_{m-1} \) differ by a one-dimensional operator, as already shown, the representation (2) holds for \( \ln D_u U_{m-1}(\zeta) \) with the function \( \eta^{(m)}(\mu) \). By construction

\[
\sum_{m=1}^{\infty} \int_{-\pi}^{\pi} |\eta^{(m)}(e^{it})| dt = \sum_{m=1}^{\infty} |\theta_m| \leq 2^{-1} \pi \| V \|_1 < \infty.
\]

We set

\[
\eta(\mu) = \sum_{m=1}^{\infty} \eta^{(m)}(\mu).
\]

In view of (11) this series converges absolutely in \( L_1(\mathbb{T}) \) and hence for a.e. \( \mu \in \mathbb{T} \). Equality (6) is a consequence of the corresponding equalities for each of the functions \( \eta^{(m)} \) separately.

We show that relation (2) holds for the function (12) if the branch of \( \ln D(\zeta) \) is fixed by the condition

\[
\arg D(0) = \sum_{m=1}^{\infty} \theta_m.
\]

We first consider the determinant \( D_n = D_{U_n U_0} \) corresponding to a finite-dimensional perturbation. In this case the function \( \eta_n = \eta(U_n, U_0) \) consists only of the first \( n \) terms in (12). According to the theorem of multiplication of determinants (1.2) under condition (13), \( \ln D_n(\zeta) \) is the sum of \( \ln D_{U_n U_{m-1}}(\zeta) \) over \( m = 1, \ldots, n \). Therefore, the representation (2) for \( \ln D_n \) is obtained by direct summation of the representations for \( \ln D_{U_n U_{m-1}} \).

In the general case (2) is established by approximation of \( \ln D(\zeta) \) by the functions \( \ln D_n(\zeta) \). Namely, \( D_n(\zeta) \rightarrow D(\zeta) \) for any \( |\zeta| < 1 \), since \( \|U_n - U\|_{\infty} \rightarrow 0 \) as \( n \rightarrow \infty \). Since, moreover, according to (13), \( \arg D_n(0) \rightarrow \arg D(0) \), it follows that \( \ln D_n(\zeta) \rightarrow \ln D(\zeta) \) as \( n \rightarrow \infty \), as well. We further note that \( \eta_n \rightarrow \eta \) in \( L_1(\mathbb{T}) \). Thus, in the representation (2) for \( \ln D_n(\zeta) \) for any \( \zeta, |\zeta| < 1 \), it is possible to pass to the limit as \( n \rightarrow \infty \). This completes the verification of relation (2) for all \( |\zeta| \neq 1 \).

Suppose now that the operator \( U U_0^{-1} \) has only \( k_+ \) eigenvalues on the upper semicircle (the point 1 is excluded from it, while the point -1 is included). Our consideration of the one-dimensional case shows that then in the series (12) for \( \eta(\mu; U, U_0) \) all the terms are nonpositive, except for \( k_+ \), terms, each of which does not exceed 1. Since this series converges almost everywhere, its sum \( \eta(\mu; U, U_0) \leq k_+ \) for a.e. \( \mu \in \mathbb{T} \). This inequality, proved for the special normalization (6), in invariant terms implies only that the SSF is upper semibounded. The case where the spectrum of \( U U_0^{-1} \) is finite on the lower semicircle is, of course, treated similarly. \( \square \)

Remark 2. The SSF can also, of course, be expressed in terms of the limit values of \( \arg D(\zeta) \) by approaching \( \mathbb{T} \) from the domain \( |\zeta| > 1 \). Namely, according to (1.2.15),

\[
\eta(\mu) = -\pi^{-1} \lim_{r \to 1+0} \arg D(r\mu) + (2\pi)^{-1} \arg D(0) = (2\pi)^{-1} \left[ \lim_{r \to 1+0} \arg D(r\mu) - \lim_{r \to 1-0} \arg D(r\mu) \right].
\]

Remark 3. A function \( \eta \) constructed in the proof of Theorem 1 satisfies the normalization (6). Such a SSF is unique and, moreover, satisfies inequality (7). Relations (6), (7) can be written also in a form analogous to (2.7), (2.6). Let us define (for example, by the spectral theorem) the branch of \( \ln M = i \arg M \) of the unitary operator \( M \) by the condition \( \arg \mu \in (-\pi, \pi) \).

In terms of the multiplicative perturbation \( M = U U_0^{-1} \) relations (6), (7) can be rewritten in the form

\[
\int_{\mathbb{T}} \eta(\mu; U, U_0)^{-1} \mu^{-1} d\mu = \text{Tr} \ln M,
\]

\[
\|\eta(U, U_0)\|_{L_1(\mathbb{T})} \leq \|\ln M\|_1 \leq 2^{-1} \pi \| V \|_1.
\]

In the last estimate we have taken account of the inequality \( 2|\theta_m| \leq \pi |e^{i\theta_m} - 1| \). Moreover, for this branch

\[
-k_- \leq \eta(\mu) \leq k_+,
\]

where \( k_- \) is the number of eigenvalues of \( M \) on the upper semicircle \((1, -1)\), and \( k_+ \) on the lower semicircle \((-1, 1)\).

We further note that, as in the selfadjoint case, augmentations to Theorem 1 similar to Remarks 2.2 and 2.4 are valid in the unitary case.

3. On the discrete spectrum the behavior of the SSF \( \eta \) is completely analogous to the selfadjoint case. Namely, from the second equality of (14) it follows that for \( \mu \in \rho(U_0) \cap \rho(U) \cap \mathbb{T} \) the function \( \eta(\mu) \) assumes only integer values and is constant on each of the component intervals of this set. We shall find the jump \( 2\pi(i\eta(\mu_1 + 0) - \eta(\mu_1 - 0)) \) of the SSF on passing in a counterclockwise direction through the isolated eigenvalue \( \mu_1 \) of the operator \( U_0 \) (of the operator \( U \)) of finite multiplicity \( k_0 \) (of multiplicity \( k \)). According to the second equality of (14) the quantity \( \eta(\mu_1 + 0) - \eta(\mu_1 - 0) \) is equal to the increment of \( \arg D(\zeta) \) on passing (in a counterclockwise direction, as always) around the point \( \mu_1 \). On the basis of the argument principle and property 4 of §1 from this it follows that

\[
\eta(\mu_1 + 0) - \eta(\mu_1 - 0) = k_0 - k.
\]
For the SSF \( \eta \) it is, of course, also possible to write out an analogue of equality (2.20).

Unless specifically stipulated, the SSF \( \eta(U, U_0) \) is assumed to be defined only up to an arbitrary integer. The multivaluedness of the SSF must always be born in mind when considering various relations for the SSF. Thus, from the theorem of multiplication of perturbation determinants (1.2) it follows that for \( U - U_0 \in \Theta_1 \) and \( U - U_1 \in \Theta_1 \) there are the equalities

\[
\eta(U, U_1) + \eta(U, U_0) = \eta(U, U), \quad \eta(U_0, U) = -\eta(U, U_0)
\]

(19)

which are understood modulo integers. The representation (12) in the form of a series absolutely convergent in \( L_1(\mathbb{T}) \) remains in force for any branch \( \eta \); on the right-hand side it is here necessary to assume that, with the exception of a finite number of terms, the functions \( \eta^{(m)} \) are normalized as indicated in Theorem 1. Similarly, modulo \( 2\pi \) equality (6) is also preserved. We put off the discussion of continuity of \( \eta(U, U_0) \) with respect to \( U \) to the next section (cf. Proposition 2.6).

4. We now proceed to a derivation of the trace formula (1). With the help of Fourier series this formula can be obtained even more simply than in the selfadjoint case. First of all, differentiating the representation (2) with respect to \( \zeta \) and using equality (1.4), we find that Lemma 3.1 carries over to the unitary case. In other words, formula (1) is satisfied for \( g(\mu) = (\mu - \zeta)^{-n} \), \( \zeta \neq 1 \), \( n = 1, 2, \ldots \). In particular, setting \( \zeta = 0 \), we establish (1) for \( g(\mu) = \mu^{-n} \). Passing to the complex conjugate, we obtain (1) also for \( g(\mu) = \bar{\mu}^{-n} \). We thus have

**Lemma 4.** For \( U - U_0 \in \Theta_1 \) the representation (1) for \( \text{Tr}(U^n - U_0^n) \) is valid for all integers \( n \).

To extend (1) to a broader class of functions \( g \) we need

**Lemma 5.** For any integer \( n \) there is the estimate

\[
\|U^n - U_0^n\|_1 \leq \|U - U_0\|_1 \|U^n - U_0^n\|_1.
\]

(20)

**Proof.** In view of the inequality

\[
\|U^{n+1} - U_0^{n+1}\|_1 = \|U^n(U - U_0) + (U^n - U_0^0)U_0\|_1 \leq \|U - U_0\|_1 \|U^n - U_0^n\|_1 + \|U^n - U_0^n\|_1
\]

for positive integers \( n \) (20) can be obtained by induction on \( n \). For negative \( n \) (20) is obtained by passage to adjoint operators. \( \square \)

**Theorem 6.** Suppose \( U - U_0 \in \Theta_1 \), while the function \( g \) is continuously differentiable and \( g'(\mu) \), \( |\mu| = 1 \), can be expanded in an absolutely convergent Fourier series. Then the operator \( g(U) - g(U_0) \) is of trace class, and for the SSF (8) the trace formula (1) holds.

**5.** The SSF is connected with the scattering matrix by the relation

\[
\text{Det} S(\mu, U, U_0) = \exp[-2\pi i \eta(\mu; U, U_0)], \quad \mu \in \Phi(U_0),
\]

(23)

which can also be established in full analogy to the selfadjoint case. We first note that although the SSF is defined up to an integer additive term this indeterminacy does not affect the right-hand side of (23). The second proof of Theorem 4.1, for example, carries over directly to the unitary case.

Indeed, from (1.6) we obtain the identity

\[
\text{Det}[-(V - V' R(\zeta)V')R_0(\zeta' - R_0(\zeta))] = D^{-1}(\zeta)D(\zeta'), \quad \zeta = \zeta^{-1},
\]

(24)

\( R = R_U, \quad R_0 = R_{U_0} \), which plays the role of (4.6). We here set \( \zeta = \mu \) and pass to the limit as \( r \to 1 \). Since by Lemma 1.3 \( |D(\zeta)| = |D(\zeta')| \), the right-hand side of (24) is equal to

\[
\exp[i \arg D(\mu^{-1}) - i \arg D(\mu)].
\]

According to (14), this expression converges to \( \exp(-2\pi i \eta(\mu)) \).
8. THE SPECTRAL SHIFT FUNCTION AND THE TRACE FORMULA

We shall show that the left-hand side of (24) converges to \( \text{Det} S(\mu) \). We consider the factorization of \( V = G^*G_0 \) into Hilbert-Schmidt operators. We shall now need analogues of the results of \( \S 6.1 \) for the unitary case. They can be obtained in an entirely elementary way by passing to Cayley transforms. Indeed, suppose some point \( \mu \in \mathbb{T} \) is not an eigenvalue of the operator \( U \). There then exists a selfadjoint operator \( H \) in terms of which \( U \) can be expressed by relation (1.13). In terms of it

\[
\tilde{B}(\zeta) := G_0R_\mu(\zeta)G^* = (2i\mu, \text{Im} a)^{-1}(z - \bar{a})G_0(I + (z - \bar{a})R_H(z))G^*,
\]

where \( z \) and \( \zeta \) are connected by equality (1.14). From Theorem 1.6.9 and Corollary 1.6.10 it now follows that the operator \( \tilde{B}(\zeta) \) for \( \zeta = r\mu \) and \( r - 1 - 0 \) has a limit in \( \Theta_1 \) for a.e. \( \mu \in \mathbb{T} \). In an entirely similar way, with the help of Theorem 1.6.5 and Corollary 1.6.6 we see that the operator \( G_0(R_\mu'(\zeta) - R_\mu(\zeta'))G^* \) for \( \zeta = r\mu \) and \( r - 1 - 0 \) has a limit in \( \Theta_1 \) for a.e. \( \mu \in \mathbb{T} \). Moreover, this limit is equal to the right-hand side of (7.2.11). We note that, just as in the selfadjoint case, the operators \( Z_0(\mu; G_0) \) and \( Z_0(\mu; G) \) belong to the Hilbert-Schmidt class. Using the possibility of permuting the operators under the determinant sign (see equality (1.7.11)) and the continuity of the determinant in \( \Theta_1 \), we now find that the limit of the left-hand side of (24) is equal to

\[
\text{Det}(I - 2\pi i Z_0(\mu; G)(I - \tilde{B}(\mu)))Z_0(\mu; G_0)).
\]

Because of the representation (7.2.10) the operator in square brackets is \( S(\mu) \). We have thus established

**Theorem 7.** For \( U - U_0 \in \Theta_1 \) equality (23) holds.

§6. Continuity of the SSF

with respect to an operator parameter. Multivaluedness

In the preceding section the SSF \( \eta(\mu; U, U_0) \) was determined up to an arbitrary integer. Discussion of its continuity with respect to the parameter \( U \) (for fixed \( U_0 \) and \( U - U_0 \in \Theta_1 \)) makes sense only if a particular branch of \( \eta(\mu; U, U_0) \) is selected. The problem of constructing a single-valued and continuous SSF is solvable locally. However, on the entire set of operators \( U \), differing from \( U_0 \) by a trace class operator, the requirement of continuity of the SSF necessarily leads to a multivalued function. This function depends not only on \( U_0 \) and \( U \), but also on the path connecting them. Its multivaluedness has roughly the same meaning as in the case of the function \( \ln z \) of a complex variable \( z \).

1. In the proof of Theorem 5.1 a unique branch of the SSF was selected by condition (5.6), which also admits the form (5.15). In this section we shall use this condition only for \( \|U - U_0\| < 2 \) when the operator \( UU_0^{-1} \) has no eigenvalues equal to \(-1\). Namely, we adopt

**Definition 1.** Suppose \( U - U_0 \in \Theta_1 \) and \( \|U - U_0\| < 2 \). Then we call the branch of the SSF satisfying condition (5.6) regular and denote it by \( \eta(U, U_0) \).

We shall discuss properties of the regular branch. We first note that \( \eta(U_0, U_0) = 0 \) and by inequality (5.16) the function \( \eta(U, U_0) \) converges to zero in \( L_1(\mathbb{T}) \) as \( \|U - U_0\| \to 0 \). Moreover, for the regular branch, holds inequality (5.17), where \( k_+ \text{ and } k_- \) is the number of eigenvalues of the operator \( UU_0^{-1} \) lying on the open upper (lower) semicircle. Of course, inequality (5.17) has content only if one of the numbers \( k_+ \) or \( k_- \) is finite.

The transfer of the first relation (5.19) to regular branches requires additional assumptions. We shall here need the elementary

**Lemma 2.** Suppose the point \(-1\) does not belong to the spectra of the unitary operators \( U \) and \( U' \). We fix a branch of the logarithm by the condition \( \arg \mu \in (-\pi, \pi) \). Then there is the estimate

\[
\|\ln U' - \ln U\|_1 \leq C\|U' - U\|_1.,
\]

where the number \( C \) depends only on the distance of the point \(-1\) to the spectra of \( U \) and \( U' \).

**Proof.** We represent \( \ln U \) (and \( \ln U' \)) by the Riesz integral

\[
\ln U = -(2\pi i)^{-1}\int_G \ln(\zeta - \zeta)^{-1}d\zeta,
\]

where the simple closed contour \( G \) encloses the spectrum of the operator \( U \) and does not contain the point zero inside it. Then by the resolvent identity

\[
\|\ln U' - \ln U\|_1 \leq (2\pi)^{-1}\int_G \ln |\zeta||U - \zeta^{-1}||U' - \zeta^{-1}||d\zeta|\cdot\|U - U'\|_1.
\]

It remains to note that the integrand is bounded by a number depending only on \( G \). 

**Proposition 3.** Suppose \( U_i - U_0 \in \Theta_1 \), \( \|U_i - U_0\| < 2 \) and the quantity \( \|U_i - U_0\|_1 \) is sufficiently small. Then

\[
\eta(U_i, U_0) + \eta(U_0, U_i) = \eta(U_i, U_0).
\]

Moreover,

\[
\eta(U, U_0) = -\eta(U, U_0).
\]

**Proof.** Under the condition \( \|U_i - U_0\| + \|U_0 - U_i\| < 2 \) the operators \( UU_0^{-1}, UU_0^{-1}, U_0^{-1}, U_0^{-1}, U_0^{-1} \) have no eigenvalues equal to \(-1\). Thus, all the terms in (1) are well defined. In view of (5.19) relation (1) is clearly satisfied modulo integers. Therefore, we need only to show that the integrals of \( \eta(\mu; U, U_i) \) and of \( \eta(\mu; U_0, U_0) \) with respect to \( d\mu \), \( \mu \in \mathbb{T} \), differ in modulus by less than \( 2\pi \). The first of these integrals, according to (5.16), does not exceed \( 2\pi^{-1}\|U - U_0\|_1 \), while the second, according
Lemma 6. Suppose $U_1 - U_0 \in \Theta_1$ and $\mathcal{M}$ is some connected subset of $\mathcal{L}(U_1)$ containing $U_1$. Suppose that two SSF $\mathcal{M}_1(U_1, U_0)$, $j = 1, 2$, are single-valued and continuous in $L_1(\mathbb{T})$ with respect to $U \in \mathcal{M}$. Then if $\mathcal{M}_1(U_1, U_0) = \mathcal{M}_1(U_1, U_0)$ these functions coincide for all $U \in \mathcal{M}$.

Proof. As soon as the difference $\delta = \eta_1 - \eta_2$ does not depend on $\mu$, it follows that

$$n(U, U_0) := \eta_1(\mu; U, U_0) - \eta_2(\mu; U, U_0) = (2\pi)^{-1} \int_\mathbb{T} (\eta_1(\mu; U, U_0) - \eta_2(\mu; U, U_0)) d\mu.$$

Thus, the integer-valued function $n(U, U_0)$ is continuous with respect to $U \in \mathcal{M}$. In view of the connectedness of $\mathcal{M}$, from this it follows that $n(U, U_0) = n(U_1, U_0)$ for all $U \in \mathcal{M}$. □

Together with the set $\mathcal{L}(U_1)$, we consider its subset $\mathcal{L}_0(U_1)$ consisting of those operators $U \in \mathcal{L}(U_1)$ for which $\|U - U_1\| < 2$. We shall see that both sets $\mathcal{L}$ and $\mathcal{L}_0$ are connected.

Lemma 7. Any operator $U \in \mathcal{L}(U_1)$ (operator $U \in \mathcal{L}_0(U_1)$) can be connected with $U_1$ by a continuous (in $\Theta_1$) curve $\Gamma = \{U(s)\}$, $0 \leq s \leq 1$, belonging to the set $\mathcal{L}(U_1)$ (the set $\mathcal{L}_0(U_1)$).

Proof. Suppose $M = UU_1^{-1} = \sum e^{i\theta_m} p_m$, where $\theta_m \in (-\pi, \pi]$. It is then possible, for example, to set $M(s) = \sum e^{i\theta_m} p_m$ and $U(s) = M(s)U_1$. It is clear that the $U(s)$ are unitary, $U(0) = U_1$, $U(1) = U$. By the inequality

$$\|M(s) - M(s')\| \leq \sum |\theta_m| |s - s'|$$

the family $U(s)$ depends continuously in $\Theta_1$ on $s \in [0, 1]$ and $U(s) - U_1 \in \Theta_1$. Moreover, $\|U(s) - U_1\| \leq \|U - U_1\|$, so that $U(s) \in \mathcal{L}_0(U_1)$ for $U \in \mathcal{L}_0(U_1)$. □

Thus, Problem 5 can have only one solution. However, as we shall see in the next part, Problem 5 actually has no solution. At the same time on the set $\mathcal{L}_0(U_1)$ the analogous problem can easily be solved.

Theorem 8. Suppose $U_1 - U_0 \in \Theta_1$ and the value of the SSF is given at $U = U_1$. Then on the set $\mathcal{L}_0(U_1)$ a single-valued SSF $\eta(U, U_0)$ exists, is unique, and can be constructed by the formula

$$\eta(U, U_0) = \eta_1(U, U_1) + \eta(U_1, U_0).$$

Proof. For a given value of $\eta(U_1, U_0)$ uniqueness follows directly from Lemmas 6 and 7. At the same time Proposition 4 shows that the function $\eta(U, U_0)$ is continuous on $\mathcal{L}_0(U_1)$ and assumes the value $\eta(U_1, U_0)$ at $U = U_1$. □

3. A SSF function continuous in $U$ on the entire set $\mathcal{L}(U_1)$ can be constructed if its value $\eta(U_1, U_0)$ depends on the curve $\Gamma \in \mathcal{L}(U_1)$ connecting the operators $U_1$ and $U$. This construction uses
8. The Spectral Shift Function and the Trace Formula

Lemma 9. Suppose \( U_1 - U_0 \in \mathcal{E}_1 \), \( U(0) = U_1 \), \( U(1) = U_2 \), and the family \( U(s) - U_0 \) is continuous in \( \mathcal{E}_1 \) with respect to \( s \in [0, 1] \). Then there exists a unique SSF \( \eta(U(s), U_0) \) which in \( L^2(\mathbb{T}) \) depends continuously on \( s \in [0, 1] \) and assumes the given value \( \eta(U(s), U_0) \) at \( s = 0 \).

Proof. The uniqueness of such a SSF follows from Lemma 6. To prove its existence we consider for each \( s_0 \in [0, 1] \) a neighborhood \( \Omega_{s_0} \) such that \( \| U(s) - U(s_0) \| < 1 \) for \( s \in \Omega_{s_0} \). Then \( U(s') \in \mathcal{L}_2(U(s')) \) for any \( s', s'' \in \Omega_{s_0} \). By the Heine-Borel lemma the segment \( [0, 1] \) can be covered by a finite number of intervals \( \Omega_{s_0} \). On each of them the existence of a continuous function \( \eta(U(s), U_0) \) follows from Theorem 8. \( \square \)

We note that due to the parametrization by \( s \) in Lemma 9 the curves \( \Gamma = \{ U(s) \} \) with self-intersections are admitted.

The value of the SSF depending on \( \Gamma = \{ U(s) \} \) can now be defined by the equality \( \eta(U, U_0) := \eta(U(1), U_0) \). If \( \Gamma \subset \mathcal{L}_2(U_0) \), then the SSF \( \eta(U, U_0) \) is defined by formula (3) and hence does not depend on \( \Gamma \). In the general case there is such a dependence. Namely, for the family of operators \( U(s) \), \( U(0) = U(1) = U_1 \), the SSF \( \eta(U(s), U_0) \), varying continuously with respect to \( s \), at the point \( s = 1 \) may acquire an integral increment as compared with \( s = 0 \). We shall present such an example for the case \( U_0 = U_1 \) and a one-dimensional perturbation.

Example 10. Suppose \( U(s) = U_0(I + \tau(s)P) \) where \( \tau(s) = s^{2\pi i s} - 1 \) and \( P \) is a one-dimensional orthogonal projection, so that \( U(0) = U(1) = U_0 \). Then for \( s = 0 \) and \( s = 1 \) both SSF \( \eta(U; U(0), U_0) \) and \( \eta(U; U(1), U_0) \) are equal to integer constants. We suppose that for \( s = 0 \) the SSF is equal to zero. From (5.6) it follows that modulo integers there is the equality

\[
(2\pi i)^{-1} \int_T \eta(\mu; U(s), U_0) \mu^{-1} d\mu = s. \tag{4}
\]

Since the left-hand side is continuous with respect to \( s \in [0, 1] \) and is equal to zero for \( s = 0 \), equality (4) is actually precise. Therefore, \( \eta(\mu; U(1), U_0) = 1 \) while \( \eta(\mu; U(0), U_0) = 0 \).

This example becomes entirely transparent in the special case \( U_0 = I \), \( U(s) = I + \tau(s)P \) when, according to (5.18),

\[
\eta(\mu; U(s), U_0) = \begin{cases} 1, & \mu \in (1, \exp(2\pi i s)) \\ 0, & \mu \in (\exp(2\pi i s), 1). \end{cases}
\]

As \( s \) increases the arc \( (1, \exp(2\pi i s)) \), where \( \eta(\mu) = 1 \), fills out a greater part of \( \mathbb{T} \) and for \( s = 1 \) we find that \( \eta(\mu; U(1), U_0) = 1 \) for all \( \mu \in \mathbb{T} \).

Example 10 shows that Problem 5 has no solution. Thus, in the unitary case the attempt to define globally a SSF \( \eta(U, U_0) \) continuous in \( U \) leads to a multivalued function.

§7. The SSF in the Selfadjoint Case

1. In this section we construct the SSF for a pair of selfadjoint operators \( H_0 \) and \( H \) under the condition

\[
R(z) - R_0(z) \in \mathcal{E}_1, \quad z \in \rho(H_0) \cap \rho(H). \tag{1}
\]

We recall that by the identity (6.4.8) this condition is satisfied for all regular points simultaneously. According to equality (1.15) the Cayley transforms \( U_0 \) and \( U \) of the operators \( H_0 \) and \( H \) differ by a trace class operator. In Part 3 of §6.5 such a connection between the operators \( H_0 \), \( H \) and \( U_0 \), \( U \) was used to prove the existence of the WO in the unitary case. Here, on the contrary, we apply the results of §§ 5, 6 to construct the SSF in the selfadjoint case.

Suppose the Cayley transforms \( U_0 = U_0(a) \) and \( U = U(a) \) of the operators \( H_0 \) and \( H \) are defined by equality (1.13). Since \( U - U_0 \in \mathcal{E}_1 \), by the results of §5 the SSF \( \eta(\mu) = \eta(\mu; U_0, U) \), \( \mu \in \mathbb{T} \), exists. We temporarily assume it to be defined up to an arbitrary constant.

We now consider the entire collection of Cayley transforms for two reasons. First of all, we do this in order to demonstrate the independence of the SSF of the choice of a concrete transform. Secondly, in investigating properties of the SSF it is sometimes convenient to take sufficiently large \( 1m a \). The dependence of various objects on the parameter \( a \), \( 1m a > 0 \), is not, as a rule (and on \( \mu_1, |\mu_1| = 1 \), never), reflected in the notation.

For a function \( f \) on \( \mathbb{R} \) we consider the function \( g \) on \( \mathbb{T} \) such that

\[
f(\lambda) = g(\mu), \tag{2}
\]

\[
\mu = \mu_1(\lambda - a)(\lambda - \bar{a})^{-1}. \tag{3}
\]

Then, of course, \( f(H_0) = g(U_0) \) and \( f(H) = g(U) \). Suppose that the trace formula (5.1) holds for the function \( g \). In the right-hand side of it we make the change of variable (3) and set

\[
\xi(\lambda; H, H_0) = \eta(\mu; U, U_0). \tag{4}
\]

This gives the representation (2.1) for \( \text{Tr}(f(H) - f(H_0)) \). For the function \( f \)

\[
2\text{Im} \int_{-\infty}^{\infty} [\xi(\lambda)|\lambda - a|^{-2} d\lambda = \int_{-\pi}^{\pi} |\eta(\mu)| d\mu, \tag{5}
\]

i.e., under condition (1) the SSF is integrable over \( \mathbb{R} \) only with the weight \( (\lambda^2 + 1)^{-1} \):

\[
\xi \in L_1(\mathbb{R}; (\lambda^2 + 1)^{-1}). \tag{6}
\]

It is not difficult to indicate concrete conditions on \( f \) under which the corresponding function \( g \) satisfies the assumptions of §5. In recoupling it is necessary to bear in mind that \( \mu \to \mu \), counterclockwise (clockwise) as \( \lambda \to +\infty \) (\( \lambda \to -\infty \)). Therefore the conditions on \( g(\mu) \) as \( \mu \to \mu \), are equivalent to assumptions regarding \( f(\lambda) \) at infinity. Outside an arbitrary neighborhood
of the point $\mu$, the function $g(\mu)$ has the same local smoothness as the function $f(\lambda)$. We note the relations between derivatives

$$
(\lambda + i\alpha)^2 f'(\lambda) = 2i\mu_1 \text{Im}a g'(\mu),
$$

$$
(\lambda + i\alpha)^2 ((\lambda + i\alpha)^2 f'(\lambda))' = -2(\mu_1 \text{Im}a)^2 g''(\mu),
$$

obtained by differentiation of equality (2).

To justify the trace formula (2.1) we will assume that the function $f$ has two locally bounded derivatives, and

$$
(\lambda + i\alpha)^2 f'(\lambda)' = O(|\alpha|^{-1+\epsilon}), \quad |\alpha| \to \infty,
$$

for some $\epsilon > 0$. It follows from (8) that the functions $\lambda^2 f'(\lambda)$ and hence also $f(\lambda)$, have limits as $\lambda \to \pm \infty$. By (2), (7) this means that the corresponding function $g(\mu)$ is twice differentiable for $\mu \neq \mu_1$, and the limits of $g'(\mu)$ and $g(\mu)$ exist as $\mu \to \mu_1$ from both sides. In order that the limits from different sides coincide with one another it is necessary to assume that

$$
\lim_{\lambda \to -\infty} f(\lambda) = \lim_{\lambda \to +\infty} f(\lambda), \quad \lim_{\lambda \to -\infty} \lambda^2 f'(\lambda) = \lim_{\lambda \to +\infty} \lambda^2 f'(\lambda).
$$

Then the function $g(\mu)$ is continuously differentiable on $\text{T}$, and according to (9) $g'(\mu)$ satisfies a Hölder condition with exponent $\epsilon$. As noted at the end of §5, under such conditions on $g$ the trace formula (5.1) holds. We have thus established

**Theorem 1.** Suppose that condition (1) is satisfied for a pair of selfadjoint operators $H_0, H$. Suppose $\eta$ is the SSF for the Cayley transforms (1.13), while the SSF $\xi$ is defined by equality (4). Then relation (6) holds for $\xi$, and the trace formula (2.1) is true for any function $f$ having two locally bounded derivatives and satisfying conditions (8) and (10) as $|\alpha| \to \infty$.

Under the conditions of Theorem 1 the function $g(\mu)$ satisfies the assumptions of Theorem 5.6 if $\epsilon = 1$ in (8). Indeed, according to (9), $g''(\mu)$ is a bounded function, and hence $g'' \in L_2(\text{T})$. Thus, the condition $\sum m^2 |\alpha|^2 < \infty$ is satisfied for the Fourier series (5.21), and hence $\sum |m\alpha|^2 < \infty$. This implies that $g'(\mu)$ can be expanded in an absolutely convergent Fourier series.

The store of functions $f$ in Theorem 1 is rather large. Of course, it contains all functions of the class $C_0^0(\mathbb{R})$ and also all functions $f(\lambda) = (\lambda - z)^{-n}, \text{Im} z \neq 0, n = 1, 2, \ldots$. Therefore, under condition (1) the equalities (3.2) and (3.3) are preserved. Now from the trace formula (2.1) the SSF $\xi$ is determined only up to a constant which remains arbitrary due to the first equality of (10). Since the function $(\lambda^2 + 1)^{-1}$ is integrable, relation (6) also does not fix this constant. Thus, in contrast to §§2–4, under the conditions of Theorem 1 an arbitrary number can be added to the SSF. It also follows from the trace formula that for $V \in \text{S}_1$, the SSF (2.5) coincides, up to a constant, with the SSF (4), defined via the Cayley transforms of $H_0$ and $H$.

2. According to (4), properties of the SSF $\xi$ can be derived from corresponding results regarding the SSF $\eta$. We must hereby specify the normalization of $\xi$. If the function $\eta_a = \eta(U(a), U_0(a))$ is fixed by equality (5.6), then the function $\xi_a$ connected with it by relation (4) for given $a$ (and $\mu_1$) is uniquely determined. By (1.15) and (5) for this function from (5.16) we obtain the inequality

$$
2 \int_{-\infty}^{\infty} \xi_a(\lambda) |\lambda - a|^{-2} d\lambda \leq \pi \text{Im} a|R(a) - R_0(a)|_1.
$$

Similarly, relation (5.15) shows that

$$
2i \text{Im} a \int_{-\infty}^{\infty} \xi_a(\lambda) |\lambda - a|^{-2} d\lambda = \text{Tr} \ln[U(a)U_0^{-1}(a)].
$$

In this part we suppose that, as adopted in §§, the SSF $\eta_a$, and hence also $\xi_a$, are defined modulo integers. Such a SSF $\xi_a$ can be expressed in terms of the generalized perturbation determinant (1.16) of the pair $H_0, H$. Namely, from the second equality of (5.14) it follows that

$$
\xi_a(\lambda) = (2\pi)^{-1} \left[ \lim_{\epsilon \to 0} \arg \tilde{D}_a(\lambda + i\epsilon) - \lim_{\epsilon \to 0} \arg \tilde{D}_a(\lambda - i\epsilon) \right],
$$

and the limits on the right exist for a.e. $\lambda \in \mathbb{R}$. To be consistent with the definitions of §§ it is here necessary to assume that $\arg \tilde{D}_a(\lambda) = 0$, while in the upper half-plane the branch of the argument is chosen in arbitrary fashion. Conversely, $\ln \tilde{D}_a(z)$ can also be found on the basis of the SSF $\xi_a$. For this it is only necessary to make the change of variable (3) in (5.2). According to (4), from this we obtain the representation

$$
\ln \tilde{D}_a(z) = (z - \alpha) \int_{-\infty}^{\infty} \xi_a(\lambda)(\lambda - z)^{-1}(\lambda - \bar{\alpha})^{-1} d\lambda.
$$

Since for $f \in C_0^0(\mathbb{R})$ the trace formula (2.1) holds for all SSF $\xi_a$, for different $a$ these functions can differ only by a constant. We shall show that they are determined by relation (4) up to an integer. To this end it is necessary to show that modulo integers the functions $\xi_a$ do not depend on $a$ (and on $\mu_1$). We use the relation (1.19) between generalized perturbation determinants corresponding to different Cayley transforms. From this it follows that

$$
\tilde{D}_a(z)\tilde{D}_b^{-1}(z) = \tilde{D}_a(z)\tilde{D}_b^{-1}(z),
$$

and hence by (13) the SSF $\xi_a$ and $\xi_b$ can differ only by an integer.

The connection of the SSF with the scattering matrix is also preserved for the normalization of $\xi$ now adopted. Indeed, by the invariance principle
unitary operators for which the fixed point $\mu_1$ is not an eigenvalue. It is nevertheless easy to construct a local continuous branch of the SSF.

**Theorem 3.** Suppose the selfadjoint operators $H_0$ and $H$ are resolvent comparable and a particular value $\xi(H_1, H_0)$ is given. Let $a$ be any complex number with $\text{Im} a > 0$. On the set $\mathcal{L}_{0,a}(H_1)$ of operators $H \in \mathcal{L}(H_1)$ satisfying the additional condition

$$\text{Im} a \| R_H(a) - R_{H_1}(a) \| < 1, \tag{17}$$

it is then possible to select a single-valued branch of the SSF $\xi(H, H_0)$ continuous with respect to $H \in \mathcal{L}(H_1)$ in the metric of $L_1(\mathbb{R}; (\lambda^2 + 1)^{-1})$. Moreover, for $H = H_1$, the function $\xi(H, H_0)$ assumes the given value $\xi(H_1, H_0)$.

**Proof.** According to (16) under condition (17) the function $\xi_a(H_1, H_0)$, constructed at the beginning of Part 2, is connected by equality (4) with the regular branch $\eta(U(a), U_1(a))$ of the SSF. We set (cf. (6.3))

$$\xi(H, H_0) = \xi_a(H_1, H_0) + \xi(H_1, H_0). \tag{18}$$

The continuity of this function in $L_1(\mathbb{R}; (\lambda^2 + 1)^{-1})$ as $H$ varies in $\mathcal{L}(H_1)$ follows from equalities of the form (5) and (16). $\square$

Theorem 3 implies that locally the SSF $\xi(H, H_0)$ can be chosen to be continuous jointly in the arguments $H_0$ and $H$. In complete analogy to the unitary case, under the conditions of Theorem 3 it can be seen (see the proof of Lemma 5 below) that on the connected component of $\mathcal{L}_{0,a}(H_1)$ containing $H_1$ a continuous branch of the SSF $\xi(H, H_0)$ is uniquely selected by prescribing its value at $H = H_1$. We do not know whether the set $\mathcal{L}_{0,a}(H_1)$ is connected. Thus, in contrast to Theorem 6.8, we say nothing regarding uniqueness of $\xi(H, H_0)$ on the entire set $\mathcal{L}_{0,a}(H_1)$. In particular, it is not excluded that the SSF (18) depends on the parameter $a$. However, for nearby $a$ there is certainly no such dependence. We shall need this result in Part 1 of §8. We emphasize that in the next assertion $\xi_a(H_0, H_0)$ denotes the SSF connected by equality (4) with the SSF $\eta(U(a), U_0(a))$ of the corresponding Cayley transforms.

**Lemma 4.** If $H \in \mathcal{L}_{0,a}(H_0)$ and the quantity $|b - a|$ is sufficiently small, then $H \in \mathcal{L}_{0,b}(H_0)$ and $\xi_b(H_0, H_0) = \xi_a(H_0, H_0)$.

**Proof.** The inclusion $H \in \mathcal{L}_{0,a}(H_0)$ follows from the continuity of the left-hand side of (17) with respect to the parameter $a$. Therefore, the SSF $\xi(H_0, H_0)$ is well defined and coincides with $\xi_a(H_0, H_0)$ modulo integers. We integrate the equality $\xi_b(\lambda) = \xi_a(\lambda) + n(b)$, where $n(b)$ is an integer and $n(a) = 0$, over $\mathbb{R}$ with weight $|\lambda - b|^{-2}$. Then according to (12)

$$2\pi \text{ln}(b) = \text{Tr} \ln[U(b)U_0^{-1}(b)] - 2i \int_{-\infty}^{\infty} |\lambda - b|^{-2} \xi_a(\lambda) d\lambda. \tag{19}$$
By the continuity of \( U(b) - U_0(b) \) in \( \mathcal{S} \), and of \( U_0(b) \) in \( \mathcal{S} \), the operator-valued function \( U(b)U_0^{-1}(b) \) is also continuous with respect to \( b \) in \( \mathcal{S} \). Therefore, it follows from Lemma 6.2 that the first term on the right in (19) depends continuously on \( b \). Moreover, the integral in (19) also is obviously a continuous function of \( b \) (for fixed \( a \)). Thus, according to (19), the integer-valued function \( n(b) \) is constant and, hence, equal to zero. \( \square \)

4. As in the unitary case, it is not hard to establish the existence and uniqueness of an extension of the SSF \( \xi(H, H_0) \) along any continuous curve. In formulation and proof the next assertion is completely analogous to Lemma 6.9.

**Lemma 5.** Suppose the operators \( H_0 \) and \( H \) are resolvent comparable, while the family of selfadjoint operators \( H(s) \) depends continuously on \( s \in [0, 1] \) in \( \mathcal{L}(H_0) \) and \( H(0) = H_1 \), \( H(1) = H \). Then there exists a unique SSF \( \xi(H(s), H_0) \) depending continuously on \( s \in [0, 1] \) in \( L_1(\mathbb{R}; (\lambda^2 + 1)^{-1}) \) and assuming the value \( \xi(H_1, H_0) \) at \( s = 0 \).

**Proof.** The existence of such a SSF can be derived directly from Theorem 3 by means of the Heine-Borel lemma. To prove uniqueness we consider the difference

\[
\xi_2(\lambda; H(s), H_0) - \xi(\lambda; H(s), H_0) =: n(s)
\]

of two different SSF. By integrating it over \( \lambda \in \mathbb{R} \) with weight \( (\lambda^2 + 1)^{-1} \), we find that the left-hand side of the equality obtained is continuous with respect to \( s \). Since, moreover, the function \( n(s) \) is integer-valued, it follows that \( n(s) = n(0) = 0 \). \( \square \)

The SSF \( \xi(H, H_0) \) is defined by continuous extension along curves on the connected component of the space \( \mathcal{L}(H_0) \) containing \( H_1 \). We cannot assert that this component exhausts \( \mathcal{L}(H_1) \). We shall show, however, that even on a connected component the SSF constructed in this manner is multivalued. To this end it suffices to indicate a closed curve \( H(s) \) in \( \mathcal{L}(H_0) \), \( H(0) = H(1) = H_0 \), on which the SSF acquires a nontrivial increment (which is, of course, an integer).

**Example 6.** Suppose, as in Example 6.10, that \( U(s) = U_0(I + \tau(s)P) \). It is then possible to set

\[
H(s) = i(I + U(s))(I - U(s))^{-1},
\]

provided that the operators \( U(s) \) have no eigenvalue equal to 1. In this case the increment of \( \xi(H(s), H_0) \) on the segment \( [0, 1] \) is equal to the increment of \( \eta(U(s), U_0) \) and is hence equal to 1. It remains to indicate a \( U_0 \) for which none of the operators \( U(s) \) has 1 as an eigenvalue.

Let \( \mathcal{F} = L_2(0, 2\pi) \), let \( U_0 \) be multiplication by \( \exp(ix) \), let \( P(\cdot, \psi) \), and let \( U = U_0(I + \tau P) \). Then the equation \( Uf = f \) is equivalent to

\[
f + \tau(f, \psi)\psi = U_0^{-1}\psi,
\]

or

\[
\tau\psi(x)\int_0^{2\pi} \overline{\psi(y)}f(y)dy = (e^{-ix} - 1)f(x).
\]

From this it follows that \( f(x) = c(e^{-ix} - 1)^{-1}\psi(x) \), and hence the inclusion \( f \in L_1(0, 2\pi) \) requires that

\[
\int_0^{2\pi} |e^{ix} - 1|^{-2} |\psi(x)|^2 dx < \infty.
\]

This condition is clearly violated if the function \( \psi \) is continuous but does not vanish at any endpoint. This occurs, in particular, for \( \psi(x) = (2\pi)^{-1/2} \).

We remark that in this example the operator \( U \) has no eigenvalues at all if the function \( \psi(x) \) is continuous and \( \psi(x) \neq 0 \) for \( x \in [0, 2\pi] \).

§8. The SSF in the selfadjoint case.

**Refinement of results**

In §§2-4 we constructed the SSF for trace class perturbations, and in §7 for resolvent comparable operators. In this section we shall consider an intermediate case.

1. The additional conditions as compared with (7.1) we shall formulate in terms of the behavior of the function \( \|R(z) - R_0(z)\| \) as \( |\text{Im} z| \to \infty \). In connection with this, we note preliminarily that by the identity (6.4.8) under the condition (7.1) alone this function is clearly bounded at infinity.

We suppose first of all that for some fixed \( \kappa \in \mathbb{R} \) and any \( \alpha \geq \alpha_0 > 0 \) there is the inequality

\[
\alpha\|R(\alpha) - R_0(\alpha)\| < 1, \quad a = \kappa + ia.
\]

We denote by \( \xi_\alpha(H, H_0) \) the SSF connected with the regular branch of the SSF \( \eta(U(a), U_0(a)) \) by relation (7.4). By Theorem 7.3 this function is continuous on the set \( \mathcal{L}_{\alpha_0}(H_0) \). Moreover, by Lemma 7.4 for different \( \alpha \) these functions coincide with one another, i.e., \( \xi_\alpha(H, H_0) = \xi(H, H_0) \).

**Theorem 1.** Suppose conditions (7.1) and (1) are satisfied. Then for any \( \tau \in (-1, 1) \) there is the inequality

\[
\int_{-\infty}^{\infty} |\xi(\lambda; H, H_0)|(1 + |\lambda - \kappa|)^{-1-\tau} d\lambda \leq C(\tau, \alpha_0) \int_{\alpha_0}^{\infty} \alpha^{-\tau} \|R(\alpha + ia) - R_0(\alpha + ia)\| d\alpha,
\]

provided the right-hand side is finite.

**Proof.** We start from inequality (7.11). Multiplying it by \( \alpha^{-\tau} \) and integrating over the interval \((\alpha_0, \infty)\), we find that
for the generalized perturbation determinant \( \ln \tilde{D}_\alpha(z) \). We recall that the branch of this function is fixed by the condition \( \ln D_\alpha(z) = 0 \). On the basis of relations (1.20) and (3), where \( \tau = 0 \), the representation (7.14) can be transformed to the form

\[
\ln D(z) - \ln D(\tilde{a}) = \int_{-\infty}^{\infty} \xi(\lambda)(\lambda - z)^{-1} d\lambda - \int_{-\infty}^{\infty} \xi(\lambda)(\lambda - \tilde{a})^{-1} d\lambda.
\]

We let \( \Im z \) tend to infinity here. Then \( \ln D(\tilde{a}) \to 0 \) by Lemma 1.2, while the second integral on the right tends to zero by Lebesgue’s theorem. In the limit we obtain (2.4). \( \square \)

3. We shall now indicate convenient conditions that make possible the selection of a single-valued branch of theSSF. Example 7.6 shows that resolvent comparability alone is insufficient for this purpose, i.e., the set \( \mathcal{D}(H) \) is too vast. At the same time in \( \xi \) a single-valued and continuous function \( \xi(H, H_0) \) was constructed on the set of operators \( H \) differing from \( H_0 \) by a trace class operator. We now establish a generalization of this result.

We denote by \( \mathcal{D}(H) = \mathcal{D}(H_1, a) \) the subset of those \( H \in \mathcal{D}(H) \) for which \( \mathcal{D}(H) = \mathcal{D}(H_1) \) for some fixed \( a \)

\[
\|VR_1(a)\| < 1, \quad V = H - H_1, \quad R_1 = R_1, \quad \Im a \neq 0.
\]

We recall (see Part 3 of §1.10) that if \( V \) is symmetric on \( \mathcal{D}(H) \) and condition (4) is satisfied, the operator \( H \) defined on \( \mathcal{D}(H_1) \) by the equality \( HF = HF + Vf \) is selfadjoint on \( \mathcal{D}(H) = \mathcal{D}(H_1) \). Since the inequalities (4) for \( a \) and \( \tilde{a} \) are equivalent to one another, it may be assumed without loss of generality that \( \Im a > 0 \).

On the basis of Lemma 7.5 uniqueness of the SSF follows from the connectedness of the set of selfadjoint operators considered.

**Lemma 3.** The set \( \mathcal{D}(H) = \mathcal{D}(H_1) \) is connected.

**Proof.** For \( H \in \mathcal{D}(H_1) \), \( V = H - H_1 \), and \( s \in [0, 1] \) the operators \( H(s) = H_1 + sV \) are selfadjoint on \( \mathcal{D}(H_1) \), and condition (4) holds for the pair \( H_1, H(s) \). It remains to verify that the operators \( H(s) \) belong to \( \mathcal{D}(H_1) \) and depend continuously on \( s \) in its metric. Setting \( R(s) = R_{H(s)}(a) \), we write the resolvent identity

\[
R(s) - R(s_1) = -s_2(s_2 - s_1)R(s_2)V R(s_1)
\]

\[
= (s_2 - s_1)[R(s_2)(H - a)(R - R_1)[(H_1 - a)R(s_1)|.\]

The factor \( R - R_1 \) belongs to \( \Theta_1 \) and does not depend on \( s \). Therefore, it is only necessary to show that the operators \( (H(s_1) - a)R(s_1) \) are bounded uniformly with respect to \( s_1 \) and \( s_2 \). Again by the resolvent identity we find that

\[
(H(s_1) - a)R(s_1) = I + (s_1 - s_2)V R(s_1)
\]

\[
= I + (s_1 - s_2)V R_1(s_2) R(s_1)^{-1}.
\]
§9. The SSF for semibounded operators

1. The SSF for resolvent comparable operators \( H_0 \) and \( H \) was constructed in §7 by means of the Cayley transform. It is also possible to define the SSF in terms of other functions of the operators which makes it possible to give up the condition of resolvent comparability. This question is considered in a general situation in §11. Here we shall consider the semibounded case. Without a shift by a constant it may be assumed that the operators \( H_0 \) and \( H \) are positive definite. In this case a natural generalization of condition (7.1) is given by the relation

\[
H^{-\tau} - H_0^{-\tau} \in \mathcal{C}_1,
\]

where \( \tau \) is some nonzero real number.

In considering fractional powers of positive operators an integral representation following from the spectral theorem is often used (for details see [4]). Namely, if \( A = A^* > 0 \) for any \( \theta \in (0, 1) \)

\[
A^0 = \pi^{-1} \sin \pi \theta \int_0^\infty t^{-\theta} A(A + t)^{-1} dt,
\]

\[
A^{-\theta} = \pi^{-1} \sin \pi (1 - \theta) \int_0^\infty t^{-\theta} (A + t)^{-1} dt,
\]

2. Construction of the SSF can now be carried out according to the scheme of §7. The operators \( h_0 = H_0^{-\tau} \), and \( h = H^{-\tau} \) now play the role of the Cayley transforms \( U_0 \) and \( U \) of the operators \( H_0 \) and \( H \). On the basis of Theorem 2.1 under condition (1) the SSF for the pair \( h_0, h \) exists, belongs to \( L_1(\mathbb{R}) \), and is equal to zero for \( \lambda \leq 0 \). For the initial pair \( H_0, H \) we define the SSF by the equalities

\[
\xi(\lambda; H, H_0) = -\text{sgn } \tau \xi(\lambda^{-\tau}; H^{-\tau}, H_0^{-\tau}), \quad \lambda \geq 0,
\]

\[
\xi(\lambda; H, H_0) = 0, \quad \lambda \leq 0.
\]

By inequality (2.6) the function (3) is absolutely integrable on \( \mathbb{R} \), with the weight \( \lambda^{-\tau - 1} \), i.e., relation (8.3) is satisfied.

We now suppose that for some function \( g(\mu) \) the trace formula (2.1) is valid for the pair \( h_0, h \), i.e.,

\[
\text{Tr}[g(h) - g(h_0)] = \int_0^{\infty} \xi(\mu; h, h_0) dg(\mu).
\]

Making the change \( \mu = \lambda^{-\tau} \) on the right, we find that for the pair \( H_0, H \) and the SSF (3), (4) the trace formula (2.1) holds if \( f(\lambda) = g(\mu) \). In particular, formula (2.1) clearly holds for \( f \in C_0^{\infty}(\mathbb{R}) \). From this it follows that if relation (1) is satisfied for two distinct values \( \tau \), then the corresponding SSF (3) are equal to one another. Moreover, if (1) holds for \( \tau = 1 \), then up to an additive constant the SSF (3), (4) coincides with the SSF (7.4).

We shall now specify the class of admissible functions \( f \). We start from the conditions indicated at the end of §3 for the validity of the trace formula for operators with a trace class difference. In the case \( \tau > 0 \) the operators \( h_0 \) and \( h \) are bounded, and under the change of variables \( \mu = \lambda^{-\tau} \) the "singular point" \( \lambda = \infty \) goes over to the point \( \mu = 0 \). In order that the function \( g(\mu) \) be Hölder continuous it suffices to require the existence for \( g(\mu) \) of two derivatives for \( \mu > 0 \) and also the estimate \( |g''(\mu)| \leq C \mu^{-1+\varepsilon}, \quad \varepsilon > 0 \), as \( \mu \to +0 \). In terms of the initial function \( f \) this means that for some \( \varepsilon > 0 \)

\[
|f(\lambda^{\tau + 1} f'')(\lambda)| \leq C \lambda^{-1+\varepsilon}, \quad \lambda \to \infty.
\]

In the case \( \tau < 0 \), in addition to uniform Hölder continuity of the function \( g(\mu) \), it is necessary to ensure the inclusion \( g' \in L_p(\mathbb{R}) \), for some \( p < \infty \). Both these conditions are satisfied if the function \( g \) has two derivatives, and \( |g''(\mu)| \leq C \mu^{-1+\varepsilon}, \quad \varepsilon > 0 \), as \( \mu \to +0 \), while \( g''(\mu) \) is uniformly bounded. In terms of \( f \) it suffices to assume that

\[
|f''(\lambda)| \leq C \lambda^{-t - 1+\varepsilon}, \quad \varepsilon > 0, \quad \lambda \to \infty.
\]

We now formulate the results obtained.

THEOREM 1. Suppose condition (1) is satisfied for positive definite operators \( H_0 \) and \( H \) for some real \( \tau \neq 0 \). We define the SSF by equalities (3), (4). Relation (8.3) holds for it. Suppose that the function \( f \) has two locally bounded derivatives. Then for \( \tau > 0 \) the trace formula (2.1) holds if some \( \varepsilon > 0 \) condition (5) is satisfied. For \( \tau < 0 \) formula (2.1) holds if some \( \varepsilon > 0 \) conditions (6) are satisfied.

3. Theorem 1 supplements and refines Theorem 7.1 in the semibounded case. For \( \tau = 1 \) relation (8.3) reduces to (7.6), while condition (6) reduces to (7.8). For \( \tau > 1 \) the operators \( H_0 \) and \( H \) are, generally speaking, not resolvent comparable. There is hereby less information regarding the behavior
of the SSF, while the conditions regarding the functions $f$ are more stringent. The case $\tau \in (-1, 1)$ occupies an intermediate position between the cases $V \in \Theta_1$ and (7.1). The information regarding $\zeta$ and the conditions on $f$ also have intermediate character. Finally, for $\tau < -1$ the SSF decays at infinity faster than for $V \in \Theta_1$, while the class of admissible functions $f$ is broader (with regard to the behavior at infinity) than in §3.

Thus, as the parameter $\tau$ in condition (1) increases, the assertion of Theorem 1 becomes weaker. This is not surprising, since condition (1) itself also becomes weaker. This follows from the next elementary assertion.

**Proposition 2.** If the inclusion (1) holds for some $\tau_1$, then it is satisfied for all $\tau \geq \tau_1$.

**Proof.** If $\tau_1 > 0$, then the operators $h_{0,1} = H_0^{-\tau_1}$ and $h_1 = H^{-\tau_1}$ are bounded. Therefore, from the identity

$$ h_1^2 - h_0^2 = h_1(h_1 - h_{0,1}) + (h_1 - h_{0,1})h_0, $$

it follows that $h_1^2 - h_0^2 \in \Theta_1$. By repeating this argument, we establish that $h_1^n - h_0^n \in \Theta_1$, $m = 2^n$, for all integers $n$. With the help of Theorem 1.6.4 we now see that $h_1^\alpha - h_0^\alpha \in \Theta_1$ for all $\alpha \geq 1$. This means that (1) is satisfied for $\tau \geq \tau_1 > 0$.

In the case $\tau_1 < 0$ we first of all verify the inclusion (1) for $|\tau| < |\tau_1|$. From the representations (2) it follows that for the operators $h_{0,1} = H_0^{-\tau_1}$, $h_1 = H^{-\tau_1}$ for $\theta \in (-1, 1)$

$$ h_1^\theta - h_0^\theta = \pi^{-1} \sin \pi \theta \int_0^\infty t^\theta [(h_{0,1} + t)^{-1} - (h_1 + t)^{-1}] dt. $$

(7)

By the resolvent identity

$$ (h_1 + t)^{-1} - (h_{0,1} + t)^{-1} = -(h_1 + t)^{-1}(h_1 - h_{0,1})(h_1 + t)^{-1} $$

(8)

the integrand in (7) is continuous in $\Theta_1$ for $t > 0$, while its norm in $\Theta_1$ does not exceed

$$ t^\theta \|h_1 - h_0\|_1 \|[(h_1 + t)^{-1} - (h_{0,1} + t)^{-1}] \|.$$  

(9)

The function $\|[(h_1 + t)^{-1} - (h_{0,1} + t)^{-1}]\|$ for $\tau_1 < 0$ is bounded (at zero) and decays like $t^{-1}$ as $t \to \infty$. Hence, the function (9) is integrable over $\mathbb{R}$ and by (7) $h_1^\theta - h_0^\theta \in \Theta_1$. This implies that (1) is satisfied for all $|\tau| < |\tau_1|$. To extend (1) to $\tau \geq -\tau_1 > 0$ it remains to use the part of Proposition 2 already proved. □

4. We return to the problem, considered in Part 1 of §8, of refining Theorem 7.1 in terms of the behavior of the function $\|R(\lambda) - R_0(\lambda)\|_1$ at infinity. Properties of the SSF were studied in Theorem 8.1. We shall now show that in the semibounded case, by imposing conditions as $z \to -\infty$,

it is possible to extend the store of admissible functions $f$. We first of all present for $\tau \in (-1, 1)$ conditions for the validity of the inclusion (1). The operators $H_0$ and $H$, as before, are assumed to be positive definite.

**Lemma 3.** Suppose condition (7.1) is satisfied and for some $\tau \in (-1, 1)$

$$ \int_0^\infty s^{-\tau} \|R(\lambda) - R_0(\lambda)\|_1 ds < \infty. $$

(10)

Then the inclusion (1) holds.

**Proof.** From the representations (2) it follows (cf. (7)) that

$$ H^{-\tau} - H_0^{-\tau} = \pi^{-1} \sin \pi \tau \int_0^\infty s^{-\tau} \|R(\lambda) - R_0(\lambda)\|_1 ds. $$

(11)

By (10) the right-hand side of (11) belongs to $\Theta_1$. □

We note that condition (10) has content only for $s \to \infty$. Combining Lemma 3 and Theorem 1, we immediately obtain the following assertion.

**Theorem 4.** Suppose conditions (7.1) and (10) are satisfied, where $\tau \in (-1, 1)$ and $\tau \neq 0$. Define the SSF by equalities (3), (4). Then the trace formula holds on the class of functions $f$ indicated for the $\tau$ given in Theorem 1.

**§10. The SSF for perturbations of definite sign**

It was shown in §2 that for $V = H - H_0 \in \Theta_1$ and $\pm V \geq 0$ the SSF satisfies the inequality $\pm \zeta(x, \lambda) \leq 0$ for $x \in \lambda \in \mathbb{R}$. In this section we shall establish a generalization of this result to a rather broad class of perturbations.

Under the condition (7.1) alone the concept of the sign of the perturbation $V = H - H_0$ has no meaning. Therefore, the very formulation of the problem of the sign of the SSF requires additional assumptions. We consider two versions of interpreting the sign of $V$. In Part 1 certain subordinacy of $V$ to $H_0$ is assumed. In Parts 2 and 3 we discuss the semibounded case where the operators are compared in terms of their quadratic forms. The results on the semibounded case are directly applicable to the multidimensional Schrödinger operator (in any dimension).

Under the conditions adopted in this section, a single-valued and continuous (in a suitable sense) branch of the SSF is selected on the entire set of operators considered. This makes it possible to speak of the sign of the SSF. For an arbitrary branch its semiboundedness follows from this.

1. In this part we base our considerations on the construction of the selfadjoint operator $H$ expounded in §1.8. For perturbations of definite sign, symmetric conditions when $\theta_0 = \theta = 1/2$ are natural. Namely, let $V = G^* V G$, where $V$ is a selfadjoint contraction in an auxiliary space $\mathfrak{S}$.
and $G: \mathcal{H} \to \mathcal{V}$. We suppose that the operator $G$ is defined on $\mathcal{D}(H_{1}^{1/2})$ and for some $a$ it satisfies the estimate

$$
\delta = ||G||_{H_{1}^{1/2}} < 1, \quad \Im a \neq 0.
$$

We note that the estimates (1) for $a$ and $\overline{a}$ are equivalent to one another. According to Proposition 1.0.4 under condition (1) there exists a (unique) selfadjoint operator $H$ corresponding to the sum $H_{1} + V$ in the sense of Definition 1.9.2. For its resolvent the various identities of §7.9 hold, where it is necessary to set $G_{0} = \mathcal{V}G$. As in §7.8, we assume that $V \geq 0$ ($V \leq 0$) if $\mathcal{V} \geq 0$ ($\mathcal{V} \leq 0$).

We suppose that

$$
G_{1}(z) \in \Theta_{2}, \quad z \in \rho(H_{1}).
$$

Then by the identity (1.9.15) the resolvents of the operators $H_{1}$ and $H$ differ by a trace class operator, i.e., $H \in \mathcal{L}(H_{1})$. We denote by $\mathcal{L}_{2}(z) = \mathcal{L}_{2}(z)(H_{1})$ the set of those $H$ for which conditions (1), (2) are satisfied and $\pm \mathcal{V} \geq 0$. The sets being considered now are, of course, analogous to the set of operators $\mathcal{L}_{2}(z)$ on which the SSF was constructed in Part 3 of §8. Now, however, there is additionally an argument regarding the sign of the perturbation. The sets $\mathcal{L}_{2}(z)$ are connected. Indeed, for a perturbation $V_{s} = s\mathcal{V}$, $s \in [0, 1]$, and $G_{s} = s^{1/2}G$ conditions of the form (1) and (2) are obviously satisfied. Therefore, $H(s) = H_{1} + s\mathcal{V} \in \mathcal{L}_{2}(z)$, and by (1.9.15) the curve $H(s)$ is continuous with respect to $s \in [0, 1]$ in the metric of $\mathcal{L}(H_{1})$. A detailed proof of this fact can be obtained in analogy to Lemma 8.3.

Simultaneously with the proof of inequality (2.17) (or of the opposite inequality) it becomes clear that condition (7.17) is satisfied on the set $\mathcal{L}_{2}(z)$. According to Theorem 7.3, this also distinguishes a single-valued continuous branch of the SSF. Moreover, from the connectedness of $\mathcal{L}_{2}(z)$ it follows that by giving the value of $\xi(H_{1}, H_{0})$ the SSF is fixed on $\mathcal{L}_{2}(z)(H_{1})$ uniquely.

**Theorem 1.** Suppose the selfadjoint operators $H_{0}$ and $H_{1}$ are resolvent comparable and a particular value $\xi(H_{1}, H_{0})$ is given. Let $a$ be some complex number with $\Im a \neq 0$. Then on the set of operators $H \in \mathcal{L}_{2}^{(a)}(H_{1})$ there exists and is unique a single-valued branch of the SSF $\xi(H, H_{0})$ which is continuous in $L_{1}(\mathbb{R}; (a^{2} + 1)^{-1})$, and

$$
\pm(\xi(\lambda; H_{1}, H_{0}) - \xi(\lambda; H_{1}, H_{0})) \geq 0, \quad a.e. \lambda \in \mathbb{R}.
$$

**Proof.** In order that the SSF be well defined it is first of all necessary to verify inequality (7.17). In place of the resolvents $R(a)$ and $R_{1}(a)$ it is now more convenient for us to work with the Cayley transforms $U = U(a)$ and $U_{1} = U_{1}(a), \Im a > 0$, of the operators $H$ and $H_{1}$. To prove (7.17) it is sufficient to show that $-1$ is not an eigenvalue of the operator $UU_{1}^{-1}$. Then, according to Theorem 7.3, the SSF is determined by equality (7.18), where

$$
\xi_{Q}(H, H_{1}) \text{ is connected with the regular branch } \eta_{Q}(U, U_{1}) \text{ by a relation of the form (7.4). From this it follows that inequality (3) is equivalent to } \pm \eta_{Q}(U, U_{1}) \geq 0. \text{ By Remark 5.3 (see inequality (5.17)) or Proposition 6.4, to prove the last inequalities it suffices to show that}
$$

$$
\pm\text{Im}(UU_{1}^{-1}) \geq 0, \quad H \in \mathcal{L}_{2}^{(a)}(H_{1}).
$$

Suppose, to be specific, that $H \in \mathcal{L}_{2}^{(a)}(H_{1})$. Replacing $G$ by $\mathcal{Y}^{1/2}G$ if necessary, it may be assumed that $\mathcal{V} = I$. We calculate the operator $UU_{1}^{-1}$. Applying the resolvent identity, we find that for $R_{1} = R_{1}(a), R = R(a), a = \kappa + i\alpha,$

$$
\overline{\alpha}_{1}U = I - 2i\alpha R = I - 2i\alpha R_{1} + 2i\alpha RV R_{1},
$$

and hence

$$
UU_{1}^{-1} = I + 2i\alpha RV R_{1} = I + 2i\alpha RV R_{1},
$$

Thus, according to (1.9.14), for $V = G^{*}G$

$$
UU_{1}^{-1} = I + 2i\alpha(G_{R_{1}}(a))^{*}(I + GR_{1}(a)G^{*})^{-1}(GR_{1}(a)).
$$

From this it follows that for $g = (I + GR_{1}(a)G^{*})^{-1}(GR_{1}(a))$

$$
\text{Im}(UU_{1}^{-1}, f, f) = 2\alpha \Re(1 + GR_{1}(a)G^{*})g, g \geq 2\alpha(1 - \delta^{2})\|g\|^{2}
$$

by condition (1). Since $\delta < 1$, this proves (4).

It remains to verify that $-1$ is not an eigenvalue of the operator $UU_{1}^{-1}$. Otherwise, for $UU_{1}^{-1}f = -f$ from (6) it follows that $g = 0$. At the same time, according to (5),

$$
UU_{1}^{-1}f = f + 2i\alpha(G_{R_{1}}(a))^{*}g
$$

and hence $UU_{1}^{-1}f = f$ for $g = 0$. Together with the initial assumption $UU_{1}^{-1}f = -f$ this shows that $f = 0$. □

By roughing up somewhat the assertion obtained, it is possible to formulate it without selecting a branch of the SSF. Namely, for $H_{1} \in \mathcal{L}(H_{0})$ and $H \in \mathcal{L}_{2}^{(a)}(H_{1})$ the left-hand side of (3) is lower semibounded.

In Theorem 1 it is not possible to drop condition (1). In the next example all the assumptions of Theorem 1 are satisfied with the exception of (1). However, the operator $|V|^{1/2}H_{0}^{-1}|V|^{1/2}$ remains in it bounded.

**Example 2.** Suppose that $H_{0}$ is a positive operator with discrete spectrum and such that $H_{0}^{-1} \in \Theta_{1}.$ Then for $H = -H_{0}$ the difference $H^{-1} - H_{0}^{-1} \in \Theta_{1}, \mathcal{D}(H) = \mathcal{D}(H_{0})$, and $H \leq H_{0}$. The SSF $\xi(\lambda) = \xi(\lambda; H, H_{0})$ for this problem is completely determined by formula (2.20). Assuming that
of decrease of their absolute values). We denote by \( \psi_m \) the eigenvectors of \( A - A_0 \) corresponding to \( \alpha_m \) and set

\[
A_n = A_0 + \sum_{m=1}^{n} \alpha_m (\cdot, \psi_m) \psi_m.
\]

According to Theorem 9.1 under condition (7) the SSF is defined by the relations

\[
\xi(\lambda; A, A_0) = \xi(\lambda^2; A', A'_0), \quad \lambda \geq 0,
\]

and \( \xi(\lambda; A, A_0) = 0 \) for \( \lambda \leq 0 \). For this SSF we shall establish an expansion of the form (2.15). We emphasize that the inequality \( A_0 \geq A \) (or \( A_0 \geq A_0 \)) is not assumed now. The proof of the next assertion is put off to Part 3.

**Theorem 4.** Suppose condition (7) for some \( \tau > 1 \) is satisfied for some pair of positive bounded operators \( A_0, A \). Then the operator \( A - A_0 \) belongs to the class \( \Theta_\rho \) for any \( p > \tau \), and the SSF admits the series representation

\[
\xi(\lambda; A, A_0) = \sum_{n=1}^{\infty} \xi(\lambda; A_n, A_{n-1})
\]

converging in \( L_1 \) with weight \( \lambda^{\tau-1} \), where \( l \) is the first odd integer greater than \( \tau \).

We point out that in Theorem 4 it is implied that \( L_1 \) is a space of functions defined on some finite interval \( (-\epsilon, \epsilon) \). Therefore, the weight function \( \lambda^{\tau-1} \) is bounded but vanishes as \( \lambda \to 0 \). Thus for larger \( l \) the convergence assertion becomes weaker. It follows from Theorem 4 that the series (10) converges to zero for negative \( \lambda \). For perturbations \( A - A_0 \) of definite sign the operators \( A_n \geq 0 \) for all \( n \), and hence all the terms on the right-hand side of (10) are equal to zero for \( \lambda \leq 0 \). We also emphasize that, in contrast to \( \xi \), in Theorem 4 absolute convergence is not asserted.

In deriving from (10) information on the sign of the SSF \( \xi(\lambda) = \xi(\lambda; A, A_0) \), it is necessary to note that, by Theorem 2.1, \( 0 \leq \pm \xi(\lambda; A_n, A_{n-1}) \leq 1 \) for \( \pm \alpha_n \geq 0 \). Therefore, Theorem 4 has the

**Corollary 5.** Suppose under the conditions of Theorem 4 the operator \( A - A_0 \) has only a finite number \( k_+ \) of positive \( (k_- \) of negative) eigenvalues. Then \( \xi(\lambda) \leq k_+ \) (respectively, \( \xi(\lambda) \geq k_- \) ) for a.e. \( \lambda \in \mathbb{R} \). The series (10) thereby converges absolutely in \( L_1 \) with the weight \( \lambda^{\tau-1} \) and, hence, for a.e. \( \lambda \in \mathbb{R} \).

Theorem 3 follows immediately from this. It is only necessary to use the fact that by relations (9.3) and (9)

\[
\xi(\lambda; H, H_0) = -\xi(A^{-1}; A, A_0)
\]

and \( A \geq A_0 \) (\( A \leq A_0 \)) for \( H \leq H_0 \) (for \( H \geq H_0 \)).
LEMMA 6. Under the conditions of Theorem 4 for any \( s \geq 0 \)
\[
A' (A - A_0) \in \Theta_r,
\] (11)
where \( r > \tau (s + 1)^{-1} \) if \( s \leq \tau - 1 \), and \( r = 1 \) if \( s > \tau - 1 \).

PROOF. Set \( A_0' = h_0', A' = h, \tau^{-1} = \theta, s \theta^{-1} = \alpha \). It may be assumed without loss of generality that \( \alpha < 1 \). We start from relation (9.7). We multiply it on the left by \( h^{\alpha} \) and set
\[
\Omega(t) = h^{\alpha} (h + t)^{-1} (h - h_0) (h_0 + t)^{-1}.
\]
To estimate \( \Omega(t) \), we consider the obvious inequalities
\[
\| (h + t)^{-1} \| \leq t^{-1}, \quad \| h^{\alpha} (h + t)^{-\alpha} \| \leq 1.
\] (12)
Since \( h - h_0 \in \Theta_1 \), by the first of them \( \| \Omega(t) \| \leq C t^{-\theta} \). Thus, the integral of \( t^\theta \Omega(t) \) over \( t \in [1, \infty) \) converges in the trace norm for any \( \alpha \geq 0 \) and \( \theta \in (0, 1) \). For \( t \leq 1 \) with the help of the second equality of (12) we find that
\[
\| \Omega(t) \|_1 \leq C t^{\alpha - 2}.
\] (13)
Therefore, for \( \alpha + \theta > 1 \) the integral of \( t^\theta \Omega(t) \) over the interval \((0, 1)\) also converges in \( \Theta_1 \). In this case
\[
h^{\alpha} (h^\theta - h_0^\theta) \in \Theta_r
\] (14)
for \( r = 1 \). In the case \( \alpha + \theta \leq 1 \) the integral converges only in a weaker metric. Estimating the norm of the operator (9.8) by \( 2 t^{-1} \), we find that \( \| \Omega(t) \| \leq C t^{-\theta} \). By (13) it follows from this that
\[
\| \Omega(t) \|_1 \leq \| \Omega(t) \|_1 \| \Omega(t) \|^{-1} \leq C t^{\alpha - 2} t^{-1}.
\]
Hence, the inclusion (14) is satisfied for \( \alpha + \theta r > 1 \). In the original notation this means that the inclusion (11) holds for \( r = 1 \) if \( s > \tau - 1 \), and for \( r > \tau - s \) if \( s \leq \tau - 1 \).

The assertion regarding the case \( s \leq \tau - 1 \) can be strengthened by interpolating it between the two extreme values \( s_1 = 0 \) and \( s_2 = \tau - 1 \). To this end we consider the operator-valued function \( \Phi(s) = A' (A - A_0) \) holomorphic in the strip \( s_1 < \text{Re } s < s_2 \). This function is uniformly bounded in operator norm, and on the boundary lines \( \text{Re } s = s_j \) its values belong to the classes \( \Theta_{r_j} \), where \( r_1 \) and \( r_2 \) are any integers greater than \( \tau \) and 1, respectively. Hence, according to Theorem 1.6.4, for any \( s \in [s_1, s_2] \) the inclusion \( \Phi(s) \in \Theta_r \) holds for \( r > \tau (s + 1)^{-1} \). \( \square \)

§ 10. THE SSF FOR PERTURBATIONS OF DEFINITE SIGN

LEMMA 7. Suppose \( A, A_0, \) and \( A_n \) are bounded operators and \( A \geq 0 \). We denote by \( r_\ell \) any continuous function of the parameter \( t \geq 1 \) satisfying the conditions \( r_\ell \geq 1 \) and
\[
r_\ell^{-1} \leq r_\ell^{-1} + r_\ell^{-1}.
\] (15)
Suppose that the inclusions
\[
A' (A' - A_0') \in \Theta_{r_\ell}, \quad s = 0, 1, 2, \ldots
\] (16)
are satisfied for \( l = 1 \). Then these inclusions are satisfied for all positive integers \( l \) (and all \( s \)). Moreover, the validity of the relations
\[
\lim_{n \to \infty} \| A' (A' - A_0') \|_{r_\ell} = 0, \quad s = 0, 1, 2, \ldots,
\] (17)
for \( l = 1 \) implies that these relations are satisfied for all positive integers \( l \) (and all \( s \)).

PROOF. We use induction on \( l \) and justify the passage from \( l \) to \( l + 1 \). We apply the obvious identity
\[
A' (A' - A_0') = A' (A' - A_0') + A' (A - A_0) A' - A' (A - A_0) A' = A' (A - A_0) A' + A' (A - A_0) A'.
\] (18)
The first term on the right belongs to the class \( \Theta_{r_\ell} \), by the induction hypothesis. Consider the second term. By hypothesis the operator \( A' (A - A_0) \), and therefore also its adjoint \( (A - A_0) A' \), belong to the class \( \Theta_{r_\ell} \). By Theorem 1.6.4 applied to the operator-valued function \( A' (A - A_0) A' \) in the strip \( 0 \leq \text{Re } s \leq s + l \), the operator \( A' (A - A_0) A' \) belongs to the same class \( \Theta_{r_\ell} \). For the third term on the right in (18) we use Proposition 1.6.3.

Since, by hypothesis, \( A' (A - A_0) \in \Theta_{r_\ell} \), and \( A' - A' \in \Theta_{r_\ell} \), it follows from (15) that their product belongs to \( \Theta_{r_\ell} \). The proof of relation (17) is completely similar. We again use induction on \( l \), and apply the identity (18) to the pair \( A, A_0 \). \( \square \)

REMARK 8. Actually, to prove (16) and (17) for any \( l > 1 \) and \( s = 0 \) it suffices to require that they be satisfied for \( l = 1 \) and \( 0 \leq s \leq l - 2 \). We further note that in Lemma 7 it is not necessary to assume that \( s \) is an integer.

It is now not hard to prove Theorem 4.

PROOF OF THEOREM 4. From Lemma 6 for the case \( s = 0 \) it follows immediately that \( A - A_0 \in \Theta_r \) for any \( r > \tau \). Therefore, the operators (8) are well defined. Let us show that
\[
\lim_{n \to \infty} \| A' - A_0' \|_1 = 0.
\] (19)
We choose a parameter \( \varepsilon > 0 \) so that \( l \geq \tau + \varepsilon \), and set \( r_\ell = (\tau + \varepsilon)^{-1} \) for \( \sigma \leq \tau + \varepsilon \) and \( r_\ell = 1 \) for \( \sigma \geq \tau + \varepsilon \). Such a function clearly satisfies condition (15). On the basis of Lemma 6 for any \( s \geq 0 \) we have the inclusion
A^t(A - A_n) \in \mathcal{E}_{n+1}^t. We denote by \( P_n \) the orthogonal projection onto the subspace spanned by the vectors \( \psi_1, \ldots, \psi_n \). Then \( P_n \rightarrow I \) strongly as \( n \rightarrow \infty \), and, hence, by Lemma 6.1.3 for all \( s \geq 0 \) the operators

\[
A^t(A - A_n) = A^t(A - A_0)(I - P_n)
\]

converge to zero in the norm of the class \( \mathcal{E}_{n+1} \). Relation (19) now follows from Lemma 7.

Finally, we establish the expansion (10). According to Proposition 2.6, from (19) it follows that

\[
\lim_{n \to \infty} \int_{-\infty}^{\infty} \left[ \xi(\mu; A^t, A_0^t) - \xi(\mu; A_n, A_0) \right] d\mu = 0.
\]

Here we make the change of variable \( \mu = \lambda^t \) and note that for \( \tau = l \) equality (9) holds. Moreover, an analogous equality holds (for a.e. \( \lambda \in \mathbb{R} \)) for the pair \( A^t, A_n \). To prove the expansion (10) it remains to observe that by Proposition 2.5 the SSF \( \xi(\lambda; A_n, A_0) \) is equal to the sum of the SSF \( \xi(\lambda; A_m, A_{m-1}) \) over \( m = 1, \ldots, n \).

Remark 9. Under the assumptions of Corollary 5 it can be chosen to be the first integer greater than \( \tau \), i.e. \( l \) is not necessarily odd. Indeed, in the change of variable \( \mu = \lambda^t \) in the proof of Theorem 4 it was required that \( l \) be odd only because the operators \( A_n \), in general, could have nontrivial negative parts. In the conditions of Corollary 5, however, \( A_n \geq 0 \) for sufficiently large \( n \).

§11. Further information on the SSF

1. We shall now consider the case where the condition of resolvent comparability (7.1) is violated. Just as for semibounded operators (see §9), further extension of the conditions for the validity of the trace formula (2.1) can be achieved by first constructing the SSF for suitable functions of the operators \( H_0 \) and \( H \). Indeed, suppose for a given pair of self-adjoint operators \( H_0 \) and \( H \) for some real function \( \varphi \) the operators \( h_0 = \varphi(H_0) \) and \( h = \varphi(H) \) are well defined and there is the inclusion

\[
\varphi(H) - \varphi(H_0) = \mathcal{E}_1.
\]

Then by Theorem 2.1 for the new pair \( h_0, h \) there exists the SSF \( \xi(\mu; h, h_0) \) constructed in terms of the perturbation determinant \( D_{h/h_0}(\xi) \) by a formula of the form (2.5). For any function \( g \) of the classes indicated in §3 (for example, in Theorem 3.3) a representation of the form (2.1) for \( \text{Tr}[g(h) - g(h_0)] \) holds. In terms of the original pair this means that we have

Lemma 1. Suppose relation (1) is fulfilled, and the function \( g \) satisfies the conditions of Theorem 3.3. Then for the function \( f(\lambda) = g(\varphi(\lambda)) \) there is the representation

\[
\text{Tr}[f(H) - f(H_0)] = \int_{-\infty}^{+\infty} \frac{\xi(\mu; h, h_0)}{\varphi(\mu)} d\mu.
\]

Transformation of relation (2) to the form (2.1) requires additional assumptions. We adopt

Condition 2. Suppose the spectra \( \sigma \) and \( \sigma \) of the operators \( H_0 \) and \( H \) are covered by a finite union \( \Omega \) of nonintersecting intervals \( \Omega_n \), one or two of which may be infinite, \( n = 1, \ldots, N \), and the function \( \varphi \) is defined and sufficiently smooth (say, twice continuously differentiable) on \( \Omega \). Suppose on each of the \( \Omega_n \) the derivative \( \varphi'/(\lambda) \) has constant sign (and does not vanish), and the function \( \varphi(\lambda) \) is one-to-one on the entire set \( \Omega \). We moreover assume the function \( \varphi \) to be bounded.

We denote by \( \varphi(\Omega) = \bigcup_n \varphi(\Omega_n) \) the image of the set \( \Omega \) under the mapping \( \varphi \) and by \( \psi(\Omega) \rightarrow \varphi(\Omega) \) the mapping inverse to \( \varphi \). In the integral (2) over the domain \( \varphi(\Omega) \) it is possible to make the change of variable \( \mu = \varphi(\lambda) \). Then (2) can be written in the form

\[
\text{Tr}[f(H) - f(H_0)] = \int_{\Omega} \xi(\lambda; H, H_0) d\psi(\lambda) + \int_{\mathbb{R}\backslash\varphi(\Omega)} \xi(\mu; h, h_0) d\varphi(\mu),
\]

where the SSF

\[
\xi(\lambda; H, H_0) = \text{sgn} \varphi'(\lambda) \xi(\varphi(\lambda); \varphi(H), \varphi(H_0)), \quad \lambda \in \Omega.
\]

It is clear that on each of the intervals \( \Omega_n \) the coefficient \( \text{sgn} \varphi'(\lambda) = \pm 1 \) is constant, and the function (4) is locally integrable. We further note the estimate

\[
\int_{\Omega} \|\xi(\lambda; H, H_0)||\varphi'(\lambda)|| d\lambda \leq \|\varphi(H) - \varphi(H_0)\|,
\]

which follows directly from inequality (2.6).

Above, the conditions for the validity of relations (2) and (3) were formulated in terms of the function \( g \) of the variable \( \mu = \varphi(\lambda) \). We now start from the function \( f \) of the variable \( \lambda \). In deriving the trace formula for it we simultaneously indicate conditions for the second integral on the right in (3) to drop out and, consequently, for (3) to be transformed to the usual form (2.1). Suppose first that \( f \in C^\infty(\varphi(\Omega)) \). Set \( g(\mu) = f(\varphi(\Lambda)) \) for \( \mu \in \varphi(\Omega) \) and extend it by zero to \( \mathbb{R}\backslash\varphi(\Omega) \). By differentiating the equality \( g(\mu) = f(\lambda) \), we find that

\[
g'/(\mu) = \varphi'(\lambda)^{-1} f'(\lambda), \quad g''/(\mu) = \varphi'(\lambda)^{-1} f'(\lambda) f''(\lambda).
\]

Therefore, \( g \in C^\infty(\varphi(\Omega)) \), and hence the function \( g \) satisfies the conditions of Theorem 3.3. Moreover, the integral in (3) over \( \mathbb{R}\backslash\varphi(\Omega) \) is equal to zero. From (3) we thus obtain directly
Lemma 3. Suppose the function $\varphi$ satisfies Condition 2 and the inclusion (1) holds. Define the SSF by equality (4). Then for $f \in C_0^\infty(\Omega)$ the trace formula (2.1) holds.

2. We postpone extension of the store of admissible functions in (2.1) to the next part. As pointed out in Part 2 of §3, this requires coordination of the values of $\xi$ on the different intervals $\Omega_n$. Of course, this coordination is unnecessary if $\Omega$ consists of only one interval and also in some other cases (see Part 4).

Conditions regarding the behavior of $f$ at infinity are also systematically discussed in the next part. Here, of course, only the case where the set $\sigma_0 \cup \sigma$ is unbounded from at least one side is of interest.

The connection (2.3) of the SSF (4) with the scattering matrix $S(\lambda; H, H_0)$ is preserved. Indeed, by Theorem 6.5.3 under the conditions of Lemma 3 the invariance principle holds for the WO. In terms of scattering matrices this implies (see §2.6) that (under the natural correspondence of the direct integrals) $S(\lambda; H, H_0) = S(\varphi(\lambda); h, h_0)$ if $\varphi'(\lambda) > 0$, and $S(\lambda; H, H_0) = S'(\varphi(\lambda); h, h_0)$ if $\varphi'(\lambda) < 0$. It is therefore only necessary to use equality (2.3) for the operators $h_0, h$ at the point $\varphi(\lambda)$.

Of course, the definition of the SSF by formula (4) does not depend on the specific choice of the function $\varphi$. More precisely, suppose two functions $\varphi_1$ and $\varphi_2$ satisfy Condition 2 and relation (1) is fulfilled for them. Then for $\lambda \in C_0^\infty(\Omega)$ the trace formula (2.1) holds for the two SSF $\xi_1$ and $\xi_2$ constructed from $\varphi_1$ and $\varphi_2$. According to the Du Bois-Reymond lemma it follows from this that on each interval $\Omega_n$ the difference $\xi_1 - \xi_2$ is constant (but may depend on $n$). Moreover, by an equality of the form (2.3)

$$\exp(-2\pi i \xi_1(\lambda)) = \exp(-2\pi i \xi_2(\lambda))$$

and, hence, the function $\xi_1 - \xi_2$ is necessarily integer-valued. In particular, the following assertion holds. In it the SSF for the pairs $H_0, H$ and $\varphi(H_0), \varphi(H)$ are defined independently of one another in terms of the determinants $D_{H_0H_0}(\varepsilon)$ and $D_{HH_0}(\xi)$.

Lemma 4. Suppose $H - H_0 \in \Theta_1$, the function $\varphi$ satisfies Condition 2 and the inclusion (1) holds. Then the SSF for the pairs $H_0, H$ and $\varphi(H_0), \varphi(H)$ are connected by formula (4) up to a function assuming constant integer values on the intervals $\Omega_n$.

3. The coordination of the values of the SSF on different intervals $\Omega_n$ is realized by means of the procedure used in the proof of Lemma 3.5. We assume that the intervals $\Omega_1, \ldots, \Omega_n$ are enumerated in the direction from left to right. By Proposition 2.8 applied to the pair $h_0, h$ the function (4) assumes constant integer values in neighborhoods of the end points of $\Omega_n$. Therefore, by changing the values of the SSF on each of the $\Omega_n$ by an integer constant $a_n$, it is possible to arrange that the values of the SSF

$$\zeta(\lambda; H_0) = \sgn \varphi' \xi(\varphi(\lambda); \varphi(H), \varphi(H_0)) + a_n, \quad \lambda \in \Omega_n$$

at the contiguous endpoints of $\Omega_n$ and $\Omega_{n+1}$ are equal to one another for all $n = 1, \ldots, N - 1$. It is now only necessary to extend $\xi$ to the interval between $\Omega_n$ and $\Omega_{n+1}$ by this common constant. The procedure presented determines the SSF up to an integer constant which is the same for the entire line. In particular, in the semibounded case where

$$\beta_1 = \inf(\sigma_0 \cup \sigma) > -\infty,$$

as always, we may put $\xi(\lambda) = 0$ for $\lambda > \beta_1$. The SSF constructed assumes constant integer values on the component intervals of the set $\rho_0 \cap \rho$, while on the absolutely continuous spectrum the connection (2.3) with the scattering matrix is preserved.

We now take up the extension of the store of admissible functions. We note first that in Lemma 3 in place of the infinite differentiability of the function $f$ it suffices to suppose that $f$ has two bounded derivatives. The corresponding function $g(\mu) = f(\lambda)$ extended by zero to $\mathbb{R}|\varphi(\Omega)$ will, as before, satisfy the conditions of Theorem 3.3. By Lemma 3.5 the coordination of the values of the SSF we have performed on different intervals $\Omega_n$ extends (2.1) automatically to all compactly supported and twice differentiable functions $f$ on $\Omega$.

The conditions on $f$ at infinity depend in an essential way on the properties of the function $\varphi$ and require special discussion. We first note that since $\varphi$ is bounded and monotone its derivative $\varphi'(\lambda)$ is integrable at infinity. We suppose that for $\lambda \to \pm \infty$, $\lambda \in \Omega$, for some $\epsilon > 0$ there are the estimates

$$|\varphi'(\lambda)|^{-1} f' \lambda), f'(\lambda)| \leq C|\lambda|^{-1+\epsilon},$$

$$|\varphi'(\lambda)|^{-1} f' \lambda), f'(\lambda)| \leq C|\varphi'(\lambda)||\varphi(\lambda) - \varphi(\pm \infty)|^{-1+\epsilon}. \tag{7}$$

From the first of them it follows that the function $\varphi'(\lambda) f'(\lambda)$ has a finite limit as $\lambda \to \pm \infty$. Thus, the function $f'(\lambda)$ is integrable, and hence $f(\lambda)$ converges to a finite limit as $\lambda \to \pm \infty$. According to (5), this means that $g(\mu)$ and $g'(\mu)$ tend to finite limits as $\mu \to \mu_\pm = \varphi(\pm \infty)$. These limits are equal to

$$f_{\pm} = \lim_{\lambda \to \pm \infty} f(\lambda), \quad f'_{\pm} = \lim_{\lambda \to \pm \infty} f'(\lambda) \tag{8}$$

respectively. Moreover, the second estimate of (7) shows that

$$|g''(\mu)| \leq C|\mu - \mu_\pm|^{-1+\epsilon}.$$

Therefore, the function $g(\mu)$ is Hölder continuous with exponent $\epsilon$ in the corresponding one-sided neighborhood of the point $\mu_\pm$.

To obtain the trace formula (2.1) we start from relation (3). We first suppose that the function $f(\lambda)$ is nonzero only in neighborhoods of $+\infty$ and
Again decomposing \( f \) into a sum of functions, one of which is compactly supported and the other is zero for \( \lambda \in [\gamma_1, \gamma_N] \), we see that this relation is preserved even without the assumption \( f(\lambda) = 0 \) for \( \lambda \in [\gamma_1, \gamma_N] \). As in the case \( \mu_+ = \mu_- \), from (10) it follows necessarily that
\[
a_1 + \xi_- = a_N + \xi_+ =: \xi_0. \tag{11}
\]
Passing now to the new SSF \( \xi(\lambda) = \xi_0 \) (this does not contradict the coordination adopted), we obtain the usual form (2.1) of the trace formula. Since for \( \mu_+ \neq \mu_- \) it is admitted that \( f(+\infty) \neq f(-\infty) \), the SSF is uniquely determined by the trace formula.

In the semibounded case, \( B_0 = \inf (\sigma_0 \cup \sigma) > -\infty \), the SSF is also uniquely fixed by the condition \( \xi(\lambda) = 0 \) for \( \lambda < B_1 \), and the values of the limits \( f_\pm \) and \( f'_\pm \) remain arbitrary. We formulate the final result.

**Theorem 5.** Suppose the function \( \varphi \) satisfies Condition 2 and the inclusion (1) holds. On \( \Omega_0 \), we define the SSF \( \xi(\lambda) \) by equality (6), and we choose the numbers \( a_n \) so that the function \( \xi(\lambda) \) extends by constants to a continuous function on intervals of \( \mathbb{R} \setminus \varphi(\Omega) \). Moreover, in the case where the set \( \sigma_0 \cup \sigma \) is not bounded from both sides and \( \varphi(+\infty) \neq \varphi(-\infty) \), the values of the SSF \( \xi(\lambda) \) are determined independently by (11). Suppose that the function \( f \) is twice differentiable on \( \Omega_0 \), the second derivative is locally bounded, and the estimates (7) hold at \(+\infty\). If the set \( \sigma_0 \cup \sigma \) is not bounded above (below), moreover, if \( \sigma_0 \cup \sigma \) is not bounded from both sides and \( \varphi(+\infty) = \varphi(-\infty) \), then it is necessary to assume additionally that \( f_\pm = f'_\pm \) and \( f'_\pm \) remain arbitrary. We formulate the final result.

The conditions on \( f(\lambda) \) at infinity are determined only by the behavior of the function \( \varphi(\lambda) \) as \( |\lambda| \to \infty \). Suppose, for example, that \( \varphi(\lambda) = \lambda^{\tau-1} \), \( \tau \) an odd positive integer, for sufficiently large \( |\lambda| \). Then conditions (7) reduce to the single estimate (9.5) (satisfied as \( |\lambda| \to \infty \)). Moreover, the coordination conditions (8) are also needed, where \( \varphi'(\lambda) = -\tau \lambda^{\tau-1} \). For \( \tau = 1 \) they reduce, of course, to (7.10).

4. We shall illustrate Theorem 5 by simple examples. In applications the case is most often considered where the operators \( H_0 \) and \( H \) have a common regular point, for example, \( \lambda = 0 \). The function \( \varphi(\lambda) \) equal to \( \lambda^{\tau-1} \) (\( \tau \) an odd positive integer) then satisfies Condition 2 for all \( \lambda \). Here the set \( \Omega \) consists of the two intervals \( \Omega_0 = (-\infty, -e) \) and \( \Omega_0 = (e, \infty) \) for some \( e > 0 \). Suppose the inclusion (9.1) holds, and the SSF \( \xi(\lambda) = \xi(\lambda; H, H_0) \) for all \( \lambda \in \mathbb{R} \) is defined by equality (9.3). Since \( \xi(\lambda; h, h_0) = 0 \) for sufficiently large \( |\mu| \), the SSF \( \xi(\lambda) = 0 \) in a two-sided neighborhood of the point zero, so that additional coordination of the values of \( \xi(\lambda) \) on \( \Omega_1 \) and \( \Omega_2 \) is not required. As in the semibounded case, the function \( \xi(\lambda) \) satisfies relation (8.3). The trace formula (2.1) holds if \( f \) satisfies conditions (9.5) and (8).
In particular, the functions \( f(\lambda) = (\lambda - z)^{-\tau}, \quad z \in \rho_0 \cap \rho \), an integer \( s \geq \tau \), are suitable.

For \( \tau = 1 \) the result obtained, of course, follows from Theorem 7.1, where regularity of the point \( \lambda = 0 \) is not assumed. As compared with that theorem, here a somewhat different construction of the SSF is given. We further note that we can consider without changes the case where \( \varphi(\lambda) = \lambda^{-\tau} \) for \( \lambda > 0 \), \( \varphi(\lambda) = -|\lambda|^{-\tau} \) for \( \lambda < 0 \), and \( \tau \) is an arbitrary positive number.

We now suppose that the inclusion (1) holds for the function \( \varphi(\lambda) \) equal to \( \lambda^{-\tau} + \varphi_0 \) for \( \lambda > 0 \), and to \( \lambda^{-\tau} \) for \( \lambda < 0 \); here \( \tau \) is odd, as before. In this case condition (9.5) at infinity is preserved, but for \( \varphi_0 \neq 0 \) the equalities \( f_\tau = f_\tau \) and \( f_\tilde{\tau} = f_\tilde{\tau} \) need not be required.

If the spectra of the operators \( H_0 \) and \( H \) cover the entire axis, then for the function \( \varphi \) it is convenient to take \( \varphi(\lambda) = \tan^{-1}(\lambda^\tau) \) for some odd \( \tau \). If the inclusion (1) holds for it, then the trace formula (2.1) is valid for functions \( f \) satisfying condition (9.5) as \( |\lambda| \to \infty \). Since \( \varphi(\infty) \neq \varphi(-\infty) \) the limits \( f_\xi \) and \( f_{\tilde{\xi}} \) remain arbitrary.

Above we have assumed that the function \( \varphi(\lambda) \), for which the inclusion (1) holds, is bounded. If \( \varphi(\lambda) \to \infty \) as \( \lambda \to \infty \), then admissible functions \( f(\lambda) \) may also tend to infinity as \( \lambda \to \infty \). For example, according to Theorem 3.3, functions \( f(\lambda) \), equal to \( \varphi(\lambda)^\alpha \), \( \alpha \in (0, 1) \), for sufficiently large \( \lambda \), are suitable.

5. In scattering theory, assumptions of local character are very convenient (see §6.4). Local criteria for the existence of the SSF were obtained in the work of L. S. Koplienko [61]. They are somewhat more restrictive as compared to the conditions for the existence and completeness of the wave operators. Nevertheless, new effective conditions of global nature can be obtained from the local criteria as before. Thus, in [61] the existence of the SSF was established and a description of the class of admissible functions \( f \) was given under the condition that

\[
(R(z) - R_0(z))K_0(z) \in \Theta_1, \quad \text{Im } z \neq 0,
\]

for some \( \kappa \geq 0 \) and, in addition, the first factor (the difference of the resolvents) belongs to \( \Theta_p \) for some \( p < \infty \).

It is also possible to carry the theory of the SSF to a pair of spaces. Thus, in the work [62] L. S. Koplienko obtained a generalization of Theorem 9.1. The condition \( H^{-\tau} - \Omega H_0^{-\tau} \in \Theta_1 \), \( \tau \) a positive integer, now plays the role of the inclusion (9.1). The following assumptions regarding the operator \( \Sigma \) were also required:

\[
(\Sigma^* - I)H_0^{-\tau} \in \Theta_1, \quad H^{-\tau}((\Sigma^* - I)) \in \Theta_1.
\] (12)

Under these conditions, it was proved in [62] that there exists a SSF \( \xi = \xi(H, H_0; \Sigma) \) for which the trace formula for a pair of spaces

\[
\text{Tr}(f(H) - \Sigma f(H_0)\Sigma^*) + \text{Tr}((\Sigma^* - I)f(H_0)) = \int_{-\infty}^{\infty} \xi(\lambda)f'(\lambda) \, d\lambda
\] (13)

holds. For the function \( \xi(\lambda) \), as before, inclusion (8.3) is fulfilled, and for the normalization \( \xi(0) = 0 \) the connection (2.3) with the scattering matrix \( S(\lambda; H, H_0; \Sigma) \) is preserved as well. Functions \( f \) admissible in (13) must satisfy condition (9.5) as \( |\lambda| \to \infty \) and also \( f(\lambda) \to 0 \) as \( |\lambda| \to \infty \). Moreover, it was shown in [62] that under the additional assumption of the existence of a bounded operator \( \Sigma^{-1} \) the left-hand side of (13) can be replaced by

\[
\text{Tr}[f(H) - \Sigma f(H_0)\Sigma^{-1}].
\]

Instead of (12) it is here sufficient to suppose that \( (\Sigma^* - I)H_0^{-\tau} \in \Theta_\infty \), while the condition \( f(\lambda) \to 0 \) as \( |\lambda| \to \infty \) can be dropped.

Generalization of the theory to the case of nontrace class perturbations was investigated in the paper of L. S. Koplienko [63] (see also the paper of Neidhardt [128]). Under the condition \( V \in \Theta_2 \) it was established in [63] that there exists a function \( \xi(\lambda) \) such that for some class of functions \( f \)

\[
\text{Tr}(f(H_0 + V) - f(H_0) - df(H_0 + \varepsilon V)) / d\varepsilon |_{\varepsilon = 0} = -\int_{-\infty}^{\infty} \xi(\lambda)f''(\lambda) \, d\lambda.
\]

Note that for \( f \) considered the operator on the left side is of trace class.
Review of the Literature

Without striving for bibliographic completeness, we have tried to nevertheless distinguish the most essential works. Works are also noted which determine the exposition of the basic text of the book. Of course, the personal tastes of the author could not help but affect the choice of literature. A great deal of additional literature can be found in the detailed bibliography in the book of Baumgärtel and Wollenberg [30] where, however, many papers are not annotated. A brief historical survey of general nature precedes the bibliographic references according to chapters.

The beginning of the construction of scattering theory can apparently be related to the work of Friedrichs [96]. In [96] the perturbation of the operator of multiplication by an integral operator with a smooth and small kernel was studied. The method developed by Friedrichs in connection with the construction of perturbation theory for this model turned out to be closely related to the stationary approach in scattering theory. Actually, at that time the concepts of scattering theory did not exist.

The time-dependent definition of the wave operators (WO) at a formal level was given by Møller [126]. Still earlier, without the use of WO, the scattering operator was introduced in the works of Wheeler [139] and Heisenberg [101]. The stationary representation for the scattering matrix appeared in the physics literature in the works of Lippmann and Schwinger [125] and Gell-Mann and Goldberger [98].

The first mathematical proof of the existence (and unitarity) of the WO was obtained by Friedrichs [97] within the framework of the model he had proposed earlier. This required refinement of the meaning of passage to the limit in the definition of the WO. The work of Friedrichs to considerable extent determined the further development of scattering theory.

A large number of new important concepts and considerations was introduced in scattering theory in connection with the investigation of differential operators. In the one-dimensional case expansion in eigenfunctions of the continuous spectrum was constructed already in the classical paper of Weyl [138]. The multidimensional case is essentially more difficult. Here the decisive breakthrough occurred in the pioneering work of A. Ya. Povzner [73],

325
who established the existence of solutions of the scattering problem for the Schrödinger equation. The construction of such solutions was based in [73] on preliminary investigation by means of the Fredholm alternative of an integral equation for the resolvent of the Schrödinger operator. This made it possible to dispense with the condition of smallness of the perturbation adopted in [97]. In [74] A. Ya. Povzner proved a theorem on expansion in solutions of the scattering problem. This theorem is basically equivalent to the construction of the stationary WO and the proof of their isometricity and completeness. In [104] Ikebe established that in this problem the time-dependent WO also exist and coincide with the stationary WO.

O. A. Ladyzhenskaya and L. D. Faddeev [68] and L. D. Faddeev [79] succeeded in combining the approaches of Friedrichs and Povzner. Existence and completeness of the WO in the Friedrichs model was established in [68] and [79] without the assumption of smallness of the perturbation. The axiomatization of the approach developed in [73], [74], [68], [79] led to the creation of a unitarily invariant theory of smooth perturbations. In connection with the smooth method in scattering theory we mention primarily the works of Kato [109], [112].

In a substantially more complicated situation the approach of the works [68] and [79] was used by L. D. Faddeev [22] in the construction of scattering theory for a system of three pairwise interacting particles. A characteristic feature of the three-particle Hamiltonian is that its total potential does not decay at infinity in the configuration space of the system. Perturbation by such a potential is not even relatively compact. This leads to a violation of the completeness of the WO (scattering theory becomes multichannel), and correct account of the additional scattering channels arising requires qualitatively new considerations.

The trace class approach to perturbation theory of the continuous spectrum arose within the framework of abstract operator theory. It was originally developed independently of smooth methods and of requirements of applications. The theorem on the existence (and completeness) of the WO under a trace class perturbation was obtained in the works of Kato [106], [107] and Rosenblum [136]. Development of the trace class method to the level at which it could be applied to the theory of differential operators was realized in the works of Kuroda, M. Sh. Birman, Kato himself, and many others. We note primarily the works of M. Sh. Birman, where the invariance principle was found [38], [39] and a local technique was developed [40]. Apparently, Kuroda [118], [119] first applied the trace class theory to differential operators—to the Schrödinger operator. A very broad class of differential operators was considered by M. Sh. Birman in [41] on the basis of machinery he developed in [39], [40].

Further historical and bibliographic notes will be given according to chapters.

Chapter 1

The material of this chapter is standard. In addition to the literature indicated in the main text we mention the works of Dunford and Schwartz [9] and Riesz and B. Sz-Nagy [19] in which much additional material can be found. We shall also make some specific remarks.

In connection with the classification of the spectrum in §3, we mention that further decomposition (in some respects natural) of the absolutely continuous spectrum into components can be made. In this regard see the paper of Avron and Simon [88].

It is difficult to cite exactly the authors of the results in §8 on inversion of a holomorphic operator-valued function. Apparently, Theorem 2 on inversion inside a domain of analyticity is due to I. Ts. Gokhberg (see [7]). Theorem 3 on inversion up to the boundary is presented in explicit form in the paper of Kuroda [120].

Construction of a selfadjoint operator in terms of its resolvent (see §10) is often applied in some contexts. As one of the first works where this technique was used we cite the note of V. A. Yavryan [81].

Perturbation theory of unitary operators was developed by M. G. Krein but was, apparently, not published.

Chapter 2

Within the framework of abstract operator theory the basic concepts of scattering theory were formulated in connection with the investigation of perturbations of trace class type. The case $H_0 = H$, $I = I$ was first considered. A precise definition of the WO was given in Kato's paper [106], where the necessity of introducing the projection onto the absolutely continuous subspace was noted. Moreover, elementary properties of the WO were studied in [106].

The generalization of the concepts to the case of a pair of spaces was also proposed by Kato [110]. Such a generalization is important, since to a certain extent it takes scattering theory beyond the framework of perturbation theory. For example, introduction of a nontrivial identification is useful in multichannel problems [82], [92]. The idea of connecting the WO with an interval of the spectral axis (local WO) is due to M. Sh. Birman [40], who also gave local conditions for the existence of such WO.

An abstract definition of the scattering matrix was proposed in the work of M. Sh. Birman and M. G. Krein [43].

The condition for the existence of the WO in §5 was obtained by Cook [91] in considering the WO for the Schrödinger operator. The assumptions regarding the potential were sharpened in [100], [105].

The invariance principle (IP) was discovered by M. Sh. Birman [38], [39] in connection with criteria of trace class type. His considerations were essentially stationary. The term "invariance principle" itself was introduced by
Kato [108], who obtained a time-dependent proof of it. In connection with the extension of the store of admissible functions we mention the note of A. Yu. Konstantinov [59]. We remark that the IP can be established also under conditions of the type of Cook's criterion—see [90] and Volume 3 of the course [18].

Stationary unitarily invariant representations for the basic objects of scattering theory appeared (for \( J = I \)) in the context of the trace class approach in the papers of M. Sh. Birman and S. B. Entina [49]. Somewhat different, but formula equivalent, representations were obtained in [79] within the framework of the Friedrichs-Faddeev model. Later in “axiomatic” schemes formula representations were separated from the concrete conditions for their validity. Generalization to arbitrary \( J \) was achieved by means of the results of [50]. In this regard, see the remarks to Chapter 5.

Chapter 3

The technique used in Part 2 of §1 of introducing an auxiliary identification \( J \) is sometimes called the technique of Kupsch-Sandhas [116]. In the problem of the existence of the WO it makes it possible to “cut off” the singularities of the perturbation. The material of §2 is borrowed from [83]. The exposition in §4 of the connection of scattering theories for equations of first and second orders in time follows the paper of M. Sh. Birman [39].

Theorem 3.1 on the absolute form of the IP for abelian WO is due to Wollenberg [130], [140], [141], but the proof given in §5 differs from the original proof. In connection with this, we emphasize that in §5 Theorem 3.1 was derived from Theorem 5.2 which does not properly belong to scattering theory. Apparently, Theorem 5.2 has not been published previously. Wollenberg's proof can be found in the monograph [30].

Chapter 4

As already noted, the proof of existence and completeness of the WO for perturbation of the operator of multiplication by an integral operator with smooth kernel was given by Friedrichs [96], [97] under the assumption of smallness of the kernel. A general theorem was constructed by L. D. Faddeev in [79] where the technique of dealing with singular integrals was developed. The term “Friedrichs model” was introduced by L. D. Faddeev. Our exposition in §§1, 2 follows [79] rather closely. By the way, consideration of the case where the exponent of Hölder continuity of the kernel is less than or equal to \( \frac{1}{2} \) could not be found in the literature. The original method of Friedrichs and works developing it (we note, for example, the papers of Rejto [134], [135]) are treated in the book [25].

The concept of smoothness relative to a selfadjoint operator was introduced by Kato [109], [111]. Theorem 5.1 on the existence of the WO for relatively smooth perturbations is due to him. A generalization of Kato’s theorem to local WO was found by Lavine [123].

The commutator conditions of smoothness were set forth in the papers of Kato [111] and Lavine [122]-[124]. Theorem 4.1 presented in the text was established by A. F. Vakulenko [55]. A somewhat different type of smoothness conditions, also formulated in terms of commutators, was obtained in the works of Mourre [127] and Perry, Sigal, and Simon [132]. This type is especially convenient in multiparticle problems.

The concept of strong \( H_0 \)-smoothness and the method expounded in §6 of verifying \( H \)-smoothness for relatively compact perturbations arose as a result of the analysis of more concrete methods of the works of A. Ya. Povzner [73], [74] and L. D. Faddeev [79]. The merit of a correct axiomatization of the approaches of these works belongs basically to Kato and Kuroda [109], [112], [114], [120]. Our exposition is apparently closest to later papers of Kuroda [121]. In application to the Schrödinger operator a similar approach was developed also by Agmon [84]. In this context we mention also Kuroda's lectures [33].

Chapter 5

The general stationary scheme of constructing scattering theory arose as a result of the comparison of works on the trace class [49], [40], [37] and smooth [73], [74], [79] methods. The direction described in Chapter 5 is sometimes called the axiomatic or conditional approach. Actually, the “advanced” axiomatization of Chapter 5 arose in the same works of Kuroda [120] and Kato and Kuroda [114] where the concrete method of §4.6 was developed.

The technique of [120], [114] is based on the factorization of the perturbation applied in trace class theory. Howland [102] and Rejto [135] also obtained results closely related to [120], [114]. However, in [102], [135] the conditions on the perturbation were formulated in terms of an auxiliary Banach space. Such a technique is essentially equivalent to the factorization method but is relatively closer to the Friedrichs-Faddeev model.

Expositions of the axiomatic scheme, more or less closely related to one another, can be found in the literature in great number. In addition to the works already cited, we mention the work of Kato and Kuroda [115] and the book of Baumgärtel and Wollenberg [3]. A generalization of axiomatic scattering theory to the case of a pair of spaces was given by V. G. Deich [58].

Our exposition is based on the paper of M. Sh. Birman and D. R. Yafaev [50], where formula representations for the WO and the scattering operator and matrix were developed in detail. Moreover, in [50] conditions for the existence of the strong time-dependent WO are separated from conditions for their isometricity, which is essential in the case of a pair of spaces.
Chapter 6

Consideration of trace class perturbations occupies a central place in the framework of abstract operator theory. Many concepts and propositions of scattering theory arose just in connection with the development of the trace class method.

Briefly, its history is as follows. For finite-dimensional perturbations the existence of the WO was established by Kato [106], while for a difference \( H - H_0 \) of trace class and the additional assumption of absolute continuity of \( H_0 \) and \( H \) it was established by Rosenblum [136]. Kato again [107] eliminated the last condition by completing in 1957 the proof of Theorem 2.1. In the works of Kato and Rosenblum the stationary method was combined with the time-dependent approach. One-dimensional perturbation was considered by stationary means by the explicit connection (see §7) between the resolvents of the operators \( H_0 \) and \( H \). Passage to finite-dimensional perturbations was realized on the basis of the theorem of multiplication of the WO. Finally, the time-dependent Lemma 3.1 (of Rosenblum) was used in the approximation of an arbitrary trace class operator by finite-dimensional operators.

Generalization of the Kato-Rosenblum theorem to the case of a pair of spaces and an arbitrary identification \( \mathcal{I} \) (Theorem 2.3) was obtained by Pearson [131] only in the year 1978. Pearson’s method is purely time-dependent; a stationary proof of Theorem 2.3 was found in [50]. Introduction of the operator “parameter” \( \mathcal{I} \) made the Kato-Rosenblum theorem considerably more flexible. This made it possible to easily obtain from it criteria for existence of the WO convenient in applications, including local criteria. The technique used in §§4, 5 of passing to an auxiliary identification was applied already in Volume 3 of the course [18].

Up to the appearance of the paper [131], generalizations of the Kato-Rosenblum theorem needed for applications in the theory of differential operators were obtained without introducing an auxiliary identification \( \mathcal{I} \). The stationary technique developed by M. Sh. Birman and S. B. Entina [49], and in a local version by M. Sh. Birman [40], was used. In particular, in [49] Theorem 1.9 was established which served as the analytic basis of the stationary method in the theory of trace class perturbations. This made it possible to obtain a purely stationary proof of the Kato-Rosenblum theorem and to find natural generalizations of it in [49], [40]. Local criteria [40] for the existence (and completeness) of the WO turned out to be especially useful. The concept of subordinacy of operators was introduced in [40], and it was established on the basis of it that it is possible to go over from local WO to global ones. The results obtained in [49], [40] included various earlier generalizations of the Kato-Rosenblum theorem (see, for example, the paper of Kuroda [119]).

Another group of criteria convenient in applications for the existence of the WO was obtained by means of the IP. Thus, existence of the WO for the trace class difference of the resolvents was first established in the work of M. Sh. Birman and M. G. Krein. Along the way in [43] the connection with scattering theory for unitary operators was established. Generalization of the result of [43] to the case where only the difference of powers of the resolvents is trace class was obtained by Kato [108]. His proof can be found in the book [11].

The first criteria for the existence of the WO in the presence of a nontrivial identification \( \mathcal{I} \) were established in the work of A. A. Belopol’ski and M. Sh. Birman [37], where the stationary technique of [40] was carried over to a pair of spaces. The existence of strong, time-dependent WO was connected in [37] and in the paper of M. Sh. Birman [42] with the question of their isometricity. Due to this existence (and completeness) of the WO in [37], [42] could be verified only under certain conditions on \( \mathcal{I} \). These conditions are, by the way, always satisfied in applications.

The material of §6 is borrowed from the paper [51]. The results of §7 on a one-dimensional perturbation are due to Aronszajn [86] and Donoghue [93]. Double Sitel’ts integral operators first appeared in the work of Yu. L. Daletskii and S. G. Krein [57]. The machinery of such integrals was systematically developed in the papers of M. Sh. Birman and M. Z. Solomyak [45], [48].

In conclusion, we note that in the addition to the literature indicated above, the method of trace class perturbations is also expounded in the monographs of Kato [11] and Putnam [35].

Chapter 7

A justification for stationary representations for the scattering matrix (SM) within the framework of smooth assumptions was given by L. D. Faddeev [79] and within the trace class framework by M. Sh. Birman and S. B. Entina [49]. The auxiliary material comprising §5 needed for this was obtained in [49].

The inclusion \( S(\lambda) - I \in \mathcal{G}_1 \), valid in the theory of trace class perturbations, was discovered in the work of M. Sh. Birman and M. G. Krein [43]. An estimate of the form (6.6) for the integral of \( |S(\lambda) - I|_p \) was also obtained there. In application to the Schrödinger operator (and without trace class assumptions) such estimates were established in the works of Amrein and Pearson [83] and Enss and Simon [95]. The effective “pointwise” estimates of §9 for \( |S(\lambda) - I|_p \) were obtained in the papers [76], [77], [137] where a realization of them for the Schrödinger operator can also be found.

Under the conditions of the trace class theory it was shown in the work [43] that in the case of perturbations of definite sign the eigenvalues of the SM accumulate at the point 1 only from one side. This assertion was extended to a broader class of perturbations (but also of trace class type) in the papers of L. S. Koplienko [143] and S. Yu. Rošfel’d [75]. Within the framework of the trace class theory the direction of rotation (on variation of the coupling
constant for perturbations of definite sign) of the spectrum of the SM was studied in the work of M. Sh. Birman and M. G. Krein [44] and within the framework of smooth perturbations in the work of Kato [113]. In the exposition of these questions in §8, following [142], we used only the structure of the stationary representation of the SM. Therefore, the results obtained are automatically valid both under smooth and trace class assumptions.

Chapter 8

The concept of a spectral shift function (SSF) first appeared (at a formal level) in the work of I. M. Lifshits [69] in connection with the quantum theory of crystals. A general theory of the SSF was constructed by M. G. Krein [64], [65] for perturbations of trace class type. A relatively detailed exposition of this theory can be found in his lectures [66]. He considered the cases $H - H_0 \in \mathcal{S}_1$ and $R(z) - R_0(z) \in \mathcal{S}_1$ as well as trace class perturbations of unitary operators. The case $VR_0(z) \in \mathcal{S}_1$ was studied in the works of V. A. Yavryan [81] and M. G. Krein and V. A. Yavryan [67]. Under conditions more general than $R(z) - R_0(z) \in \mathcal{S}_1$ (including local conditions) a SSF was constructed by L. S. Koplienko [60], [61].

The connection of the SSF with the scattering matrix was discovered in the work of M. Sh. Birman and M. G. Krein [43].

The results on the sign of the SSF for perturbations of definite sign were obtained in the works of M. G. Krein and L. S. Koplienko already cited.

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Subject Index

Absolutely continuous part of an operator, 24
Absolutely continuous spectrum, 24
Admissible function, 88
Analytic Fredholm alternative, 46
Birman-Krein formula, 282
Cauchy integral of a vector function, 192
Cauchy-Stieltjes transform, 19
Commutator conditions for smoothness, 135
Completeness of wave operators, 77, 103
local, 81
Conditions for selfadjointness, 56
Connection of the stationary and time-dependent approaches, 164–166
Construction of a selfadjoint operator in terms of the resolvent, 56
Continuous dependence on the perturbation of the scattering operator, 201
of the wave operator, 201
Cook's criterion for the existence of the wave operators, 84
Core of the spectrum, 25
Criteria for the existence and completeness of wave operators for trace class nuclear perturbations, 204, 208–210, 212
for trace class perturbations, 204
smooth perturbations, 138, 140, 144, 145
Determinant, 43
regularized, 44
Direct integral of Hilbert spaces, 31
Double operator integrals, 226
Dual space, 35
Elements compactly supported relative to a selfadjoint operator, 35
Equivalence of identifications, 72
Fatou's theorem on passing to the limit in inequalities, 17
Fatou's theorem on Poisson integrals, 19
Friedrichs' method of constructing a self-adjoint operator, 59
Friedrichs-Faddeev model, 113, 114
Generating function, 15
Hadamard's three-line theorem, 22
Hardy Classes, 18
Heinz' inequality, 36
Ideal E_p, 38
Intertwining property, 69
Invariance principle, 86, 90, 168, 169
for Abelian wave operators, 109
for scattering matrices, 88
for scattering operators, 88
for smooth perturbations, 142
for trace class perturbations, 194
Inversion of an operator-valued function, 46–48
3-completeness of WO, 103
Kato-Rosenblum theorem, 193
stationary proof, 194
time-dependent proof, 196
Kernel of an integral operator, 34, 171
Lebesgue's theorem on decomposition of a measure, 14
on passing to the limit under the integral sign, 16
Lidskii's theorem, 42
Limit values of the resolvent in the theory of trace class perturbations, 191
Luzin-Privalov uniqueness theorem, 18
Operator adjoint, 49
bounded relative to another operator, 49
compact, 35
differential of first order, 83
finite-dimensional, 36
Fourier, 41
Fredholm, 37
Hilbert-Schmidt, 38
identification, 67
integral, 33, 169
inverse to Fredholm operator, 37
multiplication, 26
scattering, 82
Schrödinger, 10
trace class, 38
integral, 241
Parseval equality for vector-valued functions, 91
Pearson's theorem, 193
stationary proof, 193
time-dependent proof, 196
Perturbation
by an integral operator, 123
by the boundary condition, 97
multiplicative, 106
of a unitary operator, 62
of Fourier type, 215
one-dimensional, 219
relatively compact, 183
small, 122, 146
trace class, 193
Perturbation determinant, 265
for unitary operators, 268
generalized, 269
modified, 270
Perturbation determinant integral representation of, 271
Poisson integral, 20
theorem on convergence in $L_1$, 20
Polar decomposition, 36
Polarization identity, 23
Radon-Nikodým theorem, 13
Resolvent, 28
Resolvent equation, 50
Resolvent of comparably operators, 211
Scattering amplitude, 7
Scattering cross section, 12, 260
Scattering matrices
for perturbations of definite sign, 256
the spectrum of, 253
Scattering matrix, 82
for smooth perturbations, 239
for trace class perturbations, 246
for unitary operators, 234
Selfadjoint operator corresponding to a formal sum, 51, 53
Singular continuous spectrum, 24
change under one-dimensional perturbation, 223, 224
conditions for its absence, 123, 150
instability, 195
Singular numbers of a compact operator, 36
Smoothness of an operator in the Kato sense, 130
local, 134, 136
weak, 154
Spectral shift function, 271
for a trace class perturbation, 273
for perturbations of definite sign, 273,
309, 311
for resolvent comparably operators, 297
for semibounded operators, 306
in the unitary case, 285
Spectral type, maximal, 22
Spectrum, 41
absolutely continuous, 24
essential, 24
of a compact operator, 41
of a selfadjoint operator, 23
point, 24
simple, 26
singular, 24
Stationary
approach, 91, 157
wave operator, 92, 160
Stationary representation
for the scattering matrix, 176, 233, 250
for the scattering operator, 176
Stationary representation
for the scattering matrix, 94
for the scattering operator, 94, 174
for the wave operator, 93, 123, 160,
236, 237
Stone's formula, 29
Subordinacy of operators, 208
Subspace
absolutely continuous subspace, 23
singular, 23
Sufficient conditions for absolute continuity of the spectrum, 123, 150
Support of a measure, 13
Borel, 13
minimal, 15
Symmetrically normed ideals, 38
Symmetrically quasi normed ideals, 39
Theorem of multiplication
for scattering matrices, 230
for scattering operators, 229
for wave operators, 71
Theorem on preservation of the total multiplicity of the spectrum, 64
Three-line theorem for operator-valued functions, 40
Time-dependent approach, 138, 196
Time-dependent Schrödinger equation, 5
Trace formula, 271

Abelian, 76
complete, 78
in the unitary case, 135, 213
local, 74, 183
weak, 73, 74
partially isometric, 68, 71
spherical, 7
stationary, 92
strong, 68
Wave plane, 7
Weak smoothness of a Hilbert-Schmidt operator, 189, 190
Weyl-Ko Fan inequalities, 37
Weyl-von Neumann-Kuroda theorem, 194