Report.

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Date

Hilbert-Schmidt operator

Def (finite rank operator)

A is finite rank operator $:\Leftrightarrow$ $\forall f \in \mathcal{H}$, Af can be described as

$\sum_{k=1}^{N} \langle g_k, f \rangle h_k$ where $\{g_k, h_k\}_{k=1}^{N}$ are fixed vectors in $\mathcal{H}$. $N < \infty$.

Def (compact operator)

An operator $A \in B(\mathcal{H})$ is compact

$:\Leftrightarrow$ $\exists \{A_n\}$: sequence of finite rank operators

s.t. $\|A - A_n\|_{B(\mathcal{H})} \to 0$ $(n \to \infty)$

And $\mathcal{K}(\mathcal{H}) := \{A \in B(\mathcal{H})\ ;\ A \text{ is a compact operator}\}$.

Def (Hilbert-Schmidt operator)

An operator $A \in B(\mathcal{H})$ is a Hilbert-Schmidt operator

$:\Leftrightarrow$ $\|A\|_{HS} < \infty$

What's $\|\cdot\|_{HS}$? : $\forall A \in B(\mathcal{H})$, $\|A\|_{HS} = \left[\sum_k \|Ae_k\|^2\right]^{\frac{1}{2}}$ $\{e_k\}$: an orthonormal basis of $\mathcal{H}$

$\|A\|_{HS}$ is independent of chosen orthonormal basis in $\mathcal{H}$.

$\because$ $\{\varphi_n\}$, $\{\phi_n\}$, $\{e_n\}$: orthonormal bases on $\mathcal{H}$.

First $A\varphi_n$ can be developed in $\{e_n\}$.

$$\|A\varphi_n\|^2 = \sum_j |\langle e_j, A\varphi_n \rangle|^2$$

$$\sum_n \|A\varphi_n\|^2 = \sum_n \sum_j |\langle e_j, A\varphi_n \rangle|^2$$

as $|\langle\ ,\ \rangle|^2 > 0$, we can interchange of order of sum

$$= \sum_j \sum_n |\langle A^* e_j, \varphi_n \rangle|^2$$

$$= \sum_j \|A^* e_j\|^2$$

Then $A\phi_k$ can be developed in $\{e_n\}$ in the same way.

and, $\sum_k \|A\phi_k\|^2 = \sum_j \|A^* e_j\|^2$ —(*)

So, $\sum_n \|A\varphi_n\|^2 = \sum_k \|A\phi_k\|^2$ $\therefore$ $\|A\|_{HS}$ depends only on A

And by (*), $\|A\|_{HS} = \|A^*\|_{HS}$.

We define $\mathcal{B}_2 = \mathcal{B}_2(\mathcal{H}) := \{ A \in \mathcal{B}(\mathcal{H})\ ;\ A \text{ is a Hilbert-Schmidt operator} \}$

Proposition

(a) $A \in \mathcal{B}_2 \iff A^* \in \mathcal{B}_2$

(b) $A_1, A_2 \in \mathcal{B}_2 \Rightarrow A_1 + A_2 \in \mathcal{B}_2$

(c) $\|A\| \le \|A\|_{HS}$

(d) $A \in \mathcal{B}_2,\ B \in \mathcal{B}(\mathcal{H})$

$\Rightarrow AB \in \mathcal{B}_2,\ BA \in \mathcal{B}_2,\ \|AB\|_{HS} \le \|B\|\,\|A\|_{HS},\ \|BA\|_{HS} \le \|B\|\,\|A\|_{HS}$

(e) $\mathcal{B}_2(\mathcal{H}) \subset \mathcal{K}(\mathcal{H})$

∵ (a) It's chacked by $\|A\|_{HS} = \|A^*\|_{HS}$

(b) 〃 direct calculation

(c) $\forall f \in \mathcal{H}$ $\|Af\|^2 = \sum_k |\langle Af, e_k\rangle|^2 = \sum_k |\langle f, A^* e_k\rangle|^2$ by inequality of Schwarz

$\le \sum_k \|f\|^2 \cdot \|A^* e_k\|^2$ $\|A^*\|_{HS} = \|A\|_{HS}$

$= \|f\|^2 \cdot \|A\|_{HS}^2$

$\therefore \|A\| \le \|A\|_{HS}$

(d) $\|BA\|_{HS}^2 = \sum_k \|BAe_k\|^2 \le \|B\| \sum \|Ae_k\|^2 = \|B\|^2 \|A\|_{HS}^2 < \infty$

$\therefore BA \in \mathcal{B}_2$ and $\|BA\|_{HS} \le \|B\|\,\|A\|_{HS}$

$\|AB\|_{HS} = \|B^* A^*\|_{HS} \le \|B^*\|\,\|A^*\|_{HS} = \|B\|\,\|A\|_{HS}$

(e) Let A_n be an operator such that for an orthonormal basis $\{\phi_s\}$,

A is Hilbert Schmidt op.

$$A_n f = \sum_{s=1}^{n} \langle f, \phi_s\rangle A\phi_s \qquad \forall f \in \mathcal{H}$$

Now. $\sum_{s=1}^{\infty} \|(A - A_n)\phi_s\|^2 = \sum_{s=n+1}^{\infty} \|A\phi_s\|^2$

by (c), $\|A - A_n\| \le \|A - A_n\|_{HS} = \left[\sum_{s=1}^{\infty} \|(A-A_n)\phi_s\|^2\right]^{\frac{1}{2}} = \left[\sum_{s=n+1}^{\infty} \|A\phi_s\|^2\right]^{\frac{1}{2}} < \infty$

$\therefore \left[\sum_{s=n+1}^{\infty} \|A\phi_s\|^2\right]^{\frac{1}{2}} \to 0 \qquad n \to \infty.$

Since A_n is a finite rank operator, then A is a compact operato

$\therefore \mathcal{B}_2(\mathcal{H}) \subset \mathcal{K}(\mathcal{H})$

Let $\mathcal{H}$ be $L^2(\mathcal{O}, m)$

Def (integral operator)

An operator $A \in \mathcal{B}(\mathcal{H})$ is an integral operator

$$:\Leftrightarrow \ {}^{\exists} a : \mathcal{O} \times \mathcal{O} \longrightarrow \mathbb{C} \ : \text{measurable function}$$

$$\text{s.t. } {}^{\forall} f \in D(A), \quad (Af)x = \int_{\mathcal{O}} a(x, y) f(y)\, m(dy)$$

And a is called the Kernel of A.

ex) (i) $\mathcal{H} = L^2(\mathbb{R})$, $a(x, y) = \dfrac{e^{-ixy}}{\sqrt{2\pi}}$, then A is Fourier transform $\mathcal{F}$

(ii) $\mathcal{H} = L^2(\mathbb{R}_{\geq 0})$, $a(x, y) = e^{-xy}$, then A is Laplace transform $\mathcal{L}$

example. If $a \in C([0,1] \times [0,1])$, $\mathcal{H} = L^2(0, 1)$
then A is compact. (I will prove later.)

Def (Hilbert-Schmidt integral operator)

A is a Hilbert-Schmidt integral operator

$$:\Leftrightarrow \ {}^{\exists} a \in L^2(\mathcal{O} \times \mathcal{O}, m \times m) \ \text{ s.t. } {}^{\forall} f \in D(A) \quad (Af)x = \int_{\mathcal{O}} a(x, y) f(y)\, m(dy)$$

$$\Leftrightarrow \ \underbrace{\iint_{\mathcal{O}} |a(x, y)|^2 m(dx)\, m(dy)}_{= M_a} < \infty$$

And we call ~~a~~ it Hilbert-Schmidt kernel.

Proposition $\mathcal{H} = L^2(\mathbb{R})$

$A \in \mathcal{B}_2(\mathcal{H}) \Longleftrightarrow A$ is a Hilbert-Schmidt integral operator

Fourther more $\|A\|_{HS}^2 = M_a = \|a\|^2_{L^2(\mathbb{R} \times \mathbb{R})}$

∵ ($\Leftarrow$) Let A be a H-S integral op. Let $V = \{x \in \mathbb{R} \mid \int_{\mathbb{R}} |a(x,y)|^2 dy = \infty\}$
$^\forall f \in \mathcal{H}.\ x \notin V$

$$\|Af\|_{\mathcal{H}}^2 = \int_{\mathbb{R}} |Af(x)|^2 dx = \int_{\mathbb{R}} \left| \int_{\mathbb{R}} a(x,y) f(y)\, dy \right|^2 dx$$

$$\leq \int_{\mathbb{R}} \int_{\mathbb{R}} |a(x,y)|^2 |f(y)|^2 dy\, dx$$

by using Cauchy-Schwarz inequality

$$\leq \int_{\mathbb{R}} \int_{\mathbb{R}} |a(x,y)|^2 dy\, dx \cdot \int_{\mathbb{R}} |f(y)|^2 dy$$

$$= M_a \cdot \|f\|_{\mathcal{H}}^2$$

$\therefore A \in \mathcal{B}(\mathcal{H})$ and $\|A\|_{\mathcal{B}(\mathcal{H})} \leq \sqrt{M_a}$

Let $\{e_k\}$ be an orthonormal basis of $L^2(\mathbb{R})$, then $\{\bar{e}_k\}$ is too.
And $\{e_j \bar{e}_k\}$ is an orthonormal basis of $L^2(\mathbb{R} \times \mathbb{R})$

we set $\alpha_{jk} := \langle e_j, A e_k \rangle$ so $A e_k = \sum_j \alpha_{jk} e_j$

then, $\|A\|_{HS}^2 = \sum_k \|A e_k\|^2 = \sum_k \sum_j |\alpha_{jk}|^2$

We can develop $a(\cdot,\cdot)$ in $\{e_j \bar{e}_k\}$; $a = \sum_{jk} \beta_{jk} e_j \bar{e}_k$

$$\Rightarrow \|a\|^2 = M_a = \sum_{jk} |\beta_{jk}|^2$$

So, $\alpha_{jk} = \langle e_j, A e_k \rangle_{L^2(\mathbb{R})} = \int_{\mathbb{R}} \int_{\mathbb{R}} \bar{e}_j(x)\, a(x,y)\, e_k(y)\, dx\, dy$

$$= \langle e_j \bar{e}_k, a \rangle_{L^2(\mathbb{R} \times \mathbb{R})} = \beta_{jk}$$

$$\therefore \|A\|_{HS}^2 = \sum_{kj} |\alpha_{jk}|^2 = \sum_{jk} |\beta_{jk}|^2 = M_a = \|a\|^2 < \infty$$

$\therefore A \in \mathcal{B}_2$ and $\|A\|_{HS} = \|a\|$

($\Rightarrow$) $^\forall A \in \mathcal{B}_2$ set $\alpha_{jk} = \langle e_j, Ae_k \rangle$

We define a function $a_N(x,y) = \sum_{j,k \le N} \alpha_{jk}\, e_j(x)\, \overline{e_k(y)}$

a_N is a finite linear combination of $e_j \overline{e}_k \in L^2(\mathbb{R}\times\mathbb{R})$
so a_N also belongs to $L^2(\mathbb{R}\times\mathbb{R})$

Let we consider $\iint |a_M(x,y) - a_N(x,y)|^2 dx\,dy = \sum_{j=1}^{N}\sum_{k=N+1}^{M} |\alpha_{jk}|^2 + \sum_{j=N+1}^{M}\sum_{k=1}^{N} |\alpha_{jk}|^2 \quad (M>N)$

$\| a_M - a_N \|^2_{L^2(\mathbb{R}\times\mathbb{R})}$

$$\le \sum_{j=1}^{\infty}\sum_{k=N+1}^{\infty} |\alpha_{jk}|^2 + \sum_{j=N+1}^{\infty}\sum_{k=1}^{\infty} |\alpha_{jk}|^2 \quad — (*)$$

Since $\|A\|^2_{HS} = \sum_{jk} |\alpha_{jk}|^2 < \infty$
$(*)$ also converge to 0 when $N, M \to \infty$.
Then $\{a_N\}$ is a strong Cauchy sequence in $L^2(\mathbb{R}\times\mathbb{R})$,
and it's limit a_∞ also belongs to $L^2(\mathbb{R}\times\mathbb{R})$.

Now let A_∞ be the H-S integral op. with kernel a_∞.
$^\forall e_k \in \{e_j\}$
$^\exists \gamma_{jk}$, $A_\infty e_k$ is written by $\sum_j \gamma_{jk} e_j$,

$$\gamma_{jk} = \langle e_j, A_\infty e_k \rangle = \langle e_j \overline{e_k}, a_\infty \rangle_{L^2(\mathbb{R}\times\mathbb{R})}$$
$$= \lim \langle e_j \overline{e_k}, a_N \rangle$$
$$= \alpha_{jk}$$

$$\therefore A_\infty e_k = \sum_j \alpha_{jk} e_j = Ae_k \qquad \therefore A_\infty = A$$

if $A \in \mathcal{B}_2$, then A is a H-S integral op. with kernel a_∞. $\square$

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To prove example, prepare these 3 notions.

Def. (Relatively strong compact)

normed space V, subset $A \subset V$, A is a relatively strong compact

$:\Leftrightarrow \ \forall \{x_n\} \subset A, \ \exists \underbrace{\{x_{n(i)}\}}_{\text{subsequence}} \subset \{x_n\}$ s.t. $\exists x \in V$, $\|x_{n(i)} - x\|_V \to 0 \ (i \to \infty)$

Prop. A is a compact operator $\Leftrightarrow$ $\forall \{f_n\} \subset \mathcal{H}$ bounded sequence,
$\exists \{f_{n(i)}\} \subset \{f_n\}$
s.t. $\exists y \in R(A)$. $\|Af_{n(i)} - y\| \to 0 \ (i \to \infty)$

Thm (Ascoli, Arzelà)

$A \subset C(0,1)$, A is a family of function that is uniformly bounded and equicontinuous.

Then A is a relatively strong compact subset.

Let's prove the example.

Let $\{f_n\}$ be bounded sequence in $L^2(0,1)$ $\|f_n\| \leq \rho \ (n = 1, 2, \cdots)$

Since a is continuous, $\forall \varepsilon > 0$, $\exists \delta > 0$ s.t. $|x - x'| < \delta \Rightarrow |a(x,y) - a(x',y)| < \varepsilon$ $\ 0 \leq y \leq 1$.

So, $$|Af_n(x)|^2 = \left| \int_0^1 a(x,y) f_n(y)\, dy \right|^2$$
$$\leq \int_0^1 |a(x,y)|^2 dy \int_0^1 |f_n(y)|^2 dy \leq c\rho^2 \quad c < \infty$$

This means that the family $\{Af_n\}$ is uniformly bounded.

And $(|x - x'| < \delta)$ $$|Af_n(x) - Af_n(x')|^2 = \left| \int_0^1 (a(x,y) - a(x',y)) f_n(y)\, dy \right|^2$$
$$\leq \left(\int_0^1 |a(x,y) - a(x',y)|^2 dy \right) \left(\int_0^1 |f_n(y)|^2 dy \right) < \varepsilon^2 \rho^2$$

This means $\{Af_n\}$ is equicontinuous.

By Thm (Ascoli, Arzelà), $\{Af_n\}$ is a relatively strong compact subset in $C(0,1)$

It means $\exists \{Af_{n(i)}\} \subset \{Af_n\}$ and $g \in C(0,1)$.

$\forall x \in [0,1] \quad |Af_{n(i)}(x) - g(x)| \to 0 \quad (i \to \infty)$

Then
$$\|Af_{n(i)} - g\|^2 = \int_0^1 |Af_{n(i)}(x) - g(x)|^2 dx$$
$$\leq \left(\max\{|Af_{n(i)}(x) - g(x)| \,;\, 0 \leq x \leq 1\}\right)^2$$
$$\to 0 \quad (i \to \infty)$$

$\therefore$ A is compact.

Reference :
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② 函数解析 (functional analysis)
竹之内脩 (Takenouchi Osamu)

③ 函数解析演習 (Exercise of functional analysis)
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