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☆ Notations and Definitions

Let $\mathcal{H}$ be a Hilbert space, $\hat{\mathcal{H}}$ the projective space of $\mathcal{H}$.
For each $\phi \in \mathcal{H} \setminus \{0\}$, $\hat{\phi}$ denotes the ray $\sigma \in \hat{\mathcal{H}}$ which contains ϕ.
$P_{\mathcal{H}} : \hat{\mathcal{H}} \times \hat{\mathcal{H}} \to [0, \infty)$ is defined by $P_{\mathcal{H}}(\sigma_1, \sigma_2) = |\langle \phi_1, \phi_2 \rangle_{\mathcal{H}}|^2$ $(\sigma_1, \sigma_2 \in \hat{\mathcal{H}})$
where $\phi_1 \in \sigma_1, \phi_2 \in \sigma_2$ are unit vectors.
Let $\mathcal{K}$ be another Hilbert space, $\hat{\mathcal{K}}$ its projective space.
A bijective map $T : \hat{\mathcal{H}} \to \hat{\mathcal{K}}$ is said to be symmetric
if T preserves P, that is, if $P_{\mathcal{K}}(T\sigma_1, T\sigma_2) = P_{\mathcal{H}}(\sigma_1, \sigma_2)$ holds for any $\sigma_1, \sigma_2 \in \hat{\mathcal{H}}$.
A unitary or antiunitary operator $U : \mathcal{H} \to \mathcal{K}$ defines
a symmetric map $\hat{U} : \hat{\mathcal{H}} \to \hat{\mathcal{K}}$, $\hat{\phi} \mapsto \widehat{(U\phi)}$.
It is easy to check that $\hat{U}$ is well defined and is symmetric.

☆ Main Theorem

Wigner-Bargmann's Theorem

Any symmetric map T can be expressed by a unitary or antiunitary operator U as above.

☆ Proof of the Main Theorem

A symmetric map $T : \hat{\mathcal{H}} \to \hat{\mathcal{K}}$ induces a map $A : \mathcal{H} \to \mathcal{K}$ as follows.
One fixes a representative ψ of each $\tau \in \hat{\mathcal{K}}$ where $\|\psi\| = 1$.
Then $A0 := 0$, $A\phi := \|\phi\| \psi$ ($\psi \in T\hat{\phi}$ is the fixed representative)
define a map $A : \mathcal{H} \to \mathcal{K}$.

Lemma 1

The map A preserves the modulus of inner product.
i.e. $\forall \phi_1, \phi_2 \in \mathcal{H}$, $|\langle A\phi_1, A\phi_2 \rangle_{\mathcal{K}}| = |\langle \phi_1, \phi_2 \rangle_{\mathcal{H}}|$

sketch of proof) In the case that $\phi_1 = 0$ or $\phi_2 = 0$, the equation is valid.
If $\phi_1 \neq 0$ and $\phi_2 \neq 0$, then it can be shown by using that T preserves P. //

However, the map A is not necessarily linear or antilinear. So one may have to deform A by changing the phase of $A\phi$ for each $\phi \in \mathcal{H}$.

Case 1 : $\dim \mathcal{H} = 0$

There is nothing to prove.

Case 2 : $\dim \mathcal{H} = 1$

Fixing $\phi \in \mathcal{H} \setminus \{0\}$, one defines $U : \mathcal{H} \to \mathcal{K}$ by $U(\alpha\phi) := \alpha A\phi$ $(\alpha \in \mathbb{C})$.

Then U is unitary and $T = \hat{U}$ holds.

Case 3 : $\dim \mathcal{H} \geq 2$

One fixes a unit vector $\phi \in \mathcal{H}$, and defines $D \subset \mathcal{H}$ and $U : D \to \mathcal{K}$ as follows.

$(\phi^{\perp} := \{\phi' \in \mathcal{H} \mid \langle \phi, \phi' \rangle_{\mathcal{H}} = 0\})$

$$D := (\phi + \phi^{\perp}) \cup \phi^{\perp}$$

$$U(\phi + \phi') := c_1(\phi, \phi') A(\phi + \phi') \quad (\phi + \phi' \in \phi + \phi^{\perp})$$

$$U\phi' := c_2(\phi, \phi') A\phi' \quad (\phi' \in \phi^{\perp}).$$

where $c_1(\phi, \phi') = \dfrac{1}{\langle A\phi, A(\phi+\phi') \rangle_{\mathcal{K}}}$ $(\phi' \in \phi^{\perp})$

and $c_2(\phi, \phi') = \begin{cases} \dfrac{\|\phi'\|^2}{c_1(\phi,\phi')^* \langle A(\phi+\phi'), A\phi' \rangle} & (\phi' \in \phi^{\perp} \setminus \{0\}) \\ 1 & (\phi' = 0) \end{cases}$

Lemma 2

U also preserves the modulus of inner product.

sketch of proof) $|c_1(\phi, \phi')| = 1$, $|c_2(\phi, \phi')| = 1$ imply the claim. //

Lemma 3

$\forall \phi_1, \phi_2 \in D$, $\forall \alpha_1, \alpha_2 \in \mathbb{C}$,

$\alpha_1\phi_1 + \alpha_2\phi_2 \in D$, $\phi_1 \perp \phi_2$

$\Rightarrow \exists \beta_1, \beta_2 \in \mathbb{C}$, $|\beta_1| = |\alpha_1|$, $|\beta_2| = |\alpha_2|$ and $U(\alpha_1\phi_1 + \alpha_2\phi_2) = \beta_1 U\phi_1 + \beta_2 U\phi_2$.

sketch of proof) One takes $\theta_1, \theta_2 \in \mathbb{R}$ such that

$\langle U(\alpha_1\phi_1+\alpha_2\phi_2), U\phi_1\rangle = |\alpha_1|\,\|\phi_1\|^2 e^{i\theta_1}$ and $\langle U(\alpha_1\phi_1+\alpha_2\phi_2), U\phi_2\rangle = |\alpha_2|\,\|\phi_2\|^2 e^{i\theta_2}$ hold.

If one defines $\beta_1 = |\alpha_1| e^{-i\theta_1}$, $\beta_2 = |\alpha_2| e^{-i\theta_2}$,

then $\|U(\alpha_1\phi_1+\alpha_2\phi_2) - \beta_1 U\phi_1 - \beta_2 U\phi_2\|^2 = 0$ can be verified. //

Lemma 4

$\phi' \in \phi^\perp \Rightarrow U(\phi+\phi') = U\phi + U\phi'$ … ①

sketch of proof) By Lemma 3,

there exist $\beta_1, \beta_2 \in \mathbb{C}$ such that $U(\phi+\phi') = \beta_1 U\phi_1 + \beta_2 U\phi_2$ and $|\beta_1| = |\beta_2| = 1$.

Then $\beta_1 = \langle U\phi, U(\phi+\phi')\rangle = \langle c_1(\phi,0)A\phi, c_1(\phi,\phi')A(\phi+\phi')\rangle = 1$.

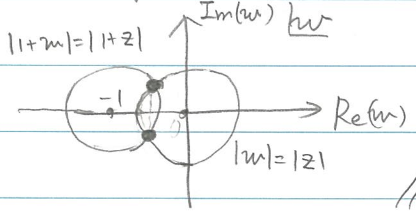
Similarly, $\beta_2 = 1$ can be proved. //

Lemma 5

$\forall z, w \in \mathbb{C}$, $|z| = |w|$ and $|1+z| = |1+w| \Rightarrow z = w$ or $z = w^*$

sketch of proof) One considers w-plane with fixing z.

Then the claim is obvious.



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Lemma 6

U is linear or antilinear on each 1-dimensional subspace V of $\phi^\perp$

sketch of proof) One takes a unit vector $\phi' \in V$.

Then for any $\alpha_1, \alpha_2 \in \mathbb{C}$, there exist $\beta_1, \beta_2 \in \mathbb{C}$ such that

$|\beta_1| = |\alpha_1|$, $|\beta_2| = |\alpha_2|$, $U\phi + U(\alpha_1\phi') \overset{\text{Lemma 4}}{=} U(\phi+\alpha_1\phi') = U\phi + \beta_2 U\phi'$ … ② and

$U\phi + U(\alpha_2\phi') = U(\phi+\alpha_2\phi') = U\phi + \beta_2 U\phi'$ … ③.

By Lemma 5, $\beta_1 = \begin{cases}\alpha_1 \\ \alpha_1^*\end{cases}$ follows from $|\alpha_1| = |\beta_1|$ and $\langle ①, ② \rangle_{\mathcal{H}}$.

$\left(\langle ①, ② \rangle_{\mathcal{H}} \Leftrightarrow \langle U(\phi+\phi'), U(\phi+\alpha_1\phi')\rangle = \langle U\phi + U\phi', U\phi + \beta_1 U\phi'\rangle \Leftrightarrow |1+\alpha_1| = |1+\beta_1| \right)$

Similarly, $\beta_2 = \begin{cases}\alpha_2 \\ \alpha_2^*\end{cases}$ follows.

In addition, $\langle ②, ③ \rangle_{\mathcal{K}}$ implies $\begin{cases} \alpha_1 = \beta_1 \\ \alpha_2 = \beta_2 \end{cases}$ or $\begin{cases} \alpha_1 = \beta_1^* \\ \alpha_2 = \beta_2^* \end{cases}$.

This shows the claim. //

Lemma 7

$\forall \phi_1, \phi_2 \in \phi^\perp$, $\|\phi_1\| = \|\phi_2\| = 1$ and $\phi_1 \perp \phi_2 \Rightarrow \forall \alpha_1, \alpha_2 \in \mathbb{C}$, $U(\alpha_1\phi_1 + \alpha_2\phi_2) = U(\alpha_1\phi_1) + U(\alpha_2\phi_2)$

sketch of proof)

$U(\alpha_1\phi_2 + \alpha_2\phi_2) = \beta_1 U\phi_1 + \beta_2 U\phi_2$ for some $\beta_1, \beta_2 \in \mathbb{C}$. ($|\beta_1| = |\alpha_1|$, $|\beta_2| = |\alpha_2|$)

On the other hand, there exist $\beta_1', \beta_2' \in \mathbb{C}$ such that

$|\beta_1'| = |\alpha_1|$, $|\beta_2'| = |\alpha_2|$, $U(\phi + \alpha_1\phi_1) = U\phi + \beta_1' U\phi_1$ and $U(\phi + \alpha_2\phi_2) = U\phi + \beta_2' U\phi_1$.

If one considers $\langle U(\phi + \alpha_1\phi_1), U(\phi + \alpha_1\phi_1 + \alpha_2\phi_2)\rangle = \langle U\phi + \beta_1' U\phi_1, U\phi + \beta_1 U\phi_1 + \beta_2 U\phi_2\rangle$,

then $\beta_1' = \beta_1$ can be checked easily by using Lemma 5.

Similarly, $\beta_2' = \beta_2$ can be also checked.

These facts imply the above equation. //

Lemma 8

U is additive on $\phi^\perp$ i.e. $\forall \phi_1, \phi_2 \in \phi^\perp$, $U(\phi_1 + \phi_2) = U\phi_1 + U\phi_2$.

sketch of proof)

If ϕ_1, ϕ_2 are linearly dependent, then it can be verified easily.

Assume ϕ_1, ϕ_2 are linearly indepent.

$\phi_2 = \alpha_1\phi_1 + \phi_3$ for some $\alpha_1 \in \mathbb{C}$ and some $\phi_3 \in \phi_1^\perp$.

Then $U(\phi_1 + \phi_2) = U(\phi_1 + \alpha_1\phi_1 + \phi_3) \overset{\text{Lemma 7}}{=} U(\phi_1 + \alpha_1\phi_1) + U\phi_3$

$\overset{\text{Lemma 6}}{=} U\phi_1 + U(\alpha_1\phi_1) + U\phi_3 = U\phi_1 + U(\alpha_1\phi_1 + \phi_3) = U\phi_1 + U\phi_2$. //

Lemma 9

$\forall \phi_1, \phi_2 \in \phi^\perp$, $\phi_1 \neq 0$, $\phi_2 \neq 0 \Rightarrow U$ is linear both $\mathbb{C}\phi_1$ and $\mathbb{C}\phi_2$ or antilinear on both $\mathbb{C}\phi_1$ and $\mathbb{C}\phi_2$.

sketch of proof) If U is linear on $\mathbb{C}\phi_1$ and antilinear on $\mathbb{C}\phi_2$,

then $\|\phi_1\|^2 + \|\phi_2\|^2 = |\langle i(\phi_1 + \phi_2), \phi_1 + \phi_2\rangle| = |\langle U(i(\phi_1 + \phi_2)), U(\phi_1 + \phi_2)\rangle|$

$\cdots = \big|\|\phi_1\|^2 - \|\phi_2\|\big|$. This contradicts $\phi_1 \neq 0$, $\phi_2 \neq 0$. //

Lemma 10

U is linear or antilinear on $\Phi^{\perp}$.

sketch of proof) For any $\phi_1, \phi_2 \in \Phi^{\perp}$, $\alpha_1, \alpha_2 \in \mathbb{C}$,

$U(\alpha_1\phi_1 + \alpha_2\phi_2) = U(\alpha_1\phi_1) + U(\alpha_2\phi_2)$ holds by Lemma 8.

If they are linearly dependent, the claim is obvious.

Assume they are linearly independent.

Then $\phi_2 = \alpha_1\phi_1 + \phi_3$ for some $\alpha_1 \in \mathbb{C}$ and some $\phi_3 \in \phi_1^{\perp}$.

By Lemma 9, U is linear on both $\mathbb{C}\phi_1$ and $\mathbb{C}\phi_3$ or antilinear on both $\mathbb{C}\phi_1$ and $\mathbb{C}\phi_3$.

This implies $U(\alpha_1\phi_1 + \alpha_2\phi_2) = \alpha_1 U\phi_1 + \alpha_2 \phi_2$ or $U(\alpha_1\phi_1 + \alpha_2\phi_2) = \overline{\alpha_1} U\phi_1 + \overline{\alpha_2} U\phi_2$ respectively.

Therefore, if U is linear on some 1-dimsional subspace of $\Phi^{\perp}$,

then U is linear on $\Phi^{\perp}$. The same is true of the case U is antilinear. //

One can extend U as a linear map if U is linear on $\Phi^{\perp}$

and as an antilinear map if U is antilinear on $\Phi^{\perp}$.

It can be checked easily that

U is untitary or antiunitary respectively

by using that T is bijective.

Futhermore, $\hat{U} = T$ can be shown

by straightforword calculations. $\square$