Shibuya Keigo

If $\varphi: \mathbb{R}^d \to \mathbb{C}$ is a measurable function,
then its multiplication operator $\varphi(X)$ is densely defined in $L^2(\mathbb{R}^d)$.

Proof) Set $K_n = \{x \in \mathbb{R}^d \mid |\varphi(x)| \leq n\}$. (Then $\mathbb{R}^d = \bigcup_{n=1}^{\infty} K_n$.)

The indicator function $\chi_n = \chi_{K_n}$ is defined by $\chi_n(x) = \begin{cases} 1 & (x \in K_n) \\ 0 & (\text{otherwise}) \end{cases}$

(This function χ_n is measurable, since K_n is a measurable set.)

For any $f \in L^2(\mathbb{R}^d)$, $f_n := \chi_n \cdot f$ belongs to $L^2(\mathbb{R}^d)$.

In addition, $f_n \in D(\varphi(X))$ holds.

$\because$ $|f_n| \leq |f|$ implies $f_n \in L^2(\mathbb{R}^d)$. $\left(\Leftarrow \int_{\mathbb{R}^d} |f_n(x)|^2 dx \leq \int_{\mathbb{R}^d} |f(x)|^2 dx < \infty\right)$

$$\int_{\mathbb{R}^d} |\varphi(x)|^2 |f_n(x)|^2 dx = \int_{K_n} |\varphi(x)|^2 |f(x)|^2 dx$$
$$\leq \int_{K_n} n^2 |f(x)|^2 dx$$
$$\leq n^2 \int_{\mathbb{R}^d} |f(x)|^2 dx < \infty$$

implies $f_n \in D(\varphi(X))$. //

The sequence $\{f_n\}$ converges pointwise to f
and $|f_n - f|^2 \leq |f|^2$.

By Lebesgue's dominated convergence theorem,
$\{f_n\}$ is a sequence which converges to f in $L^2(\mathbb{R}^d)$.

Thus the claim was proved. $\square$

Shibuya Keigo

For a measurable function $\varphi : \mathbb{R}^d \to \mathbb{C}$,

φ belongs to $L^\infty(\mathbb{R}^d)$ if and only if $D(\varphi(X)) = L^2(\mathbb{R}^d)$.

Proof) It is obvious that $\varphi \in L^\infty(\mathbb{R}^d)$ implies $D(\varphi(X)) = L^2(\mathbb{R}^d)$.

Assume $\varphi \notin L^\infty(\mathbb{R}^d)$.

Set $F_n = \{x \in \mathbb{R}^d \mid n \leq |\varphi(x)| < n+1\}$.

By the assumption,

there exists a sequence $\{n_k\}_{k=1}^\infty$ such that

$$0 < n_k < n_{k+1} \text{ and } |F_{n_k}| > 0.$$

For each k,

one takes $r_k > 0$ such that $|F_{n_k} \cap B_{r_k}| > 0$

where $B_{r_k} = \{x \in \mathbb{R}^d \mid \|x\| \leq r_k\}$.

Set $E_k = F_{n_k} \cap B_{r_k}$, $a_k = |F_{n_k} \cap B_{r_k}|$.

Then $f(x) = \begin{cases} \frac{1}{\sqrt{a_k}\, n_k} & (x \in E_k) \\ 0 & (\text{otherwise}) \end{cases}$ belongs to $L^2(\mathbb{R}^d)$.

$$\because \int_{\mathbb{R}^d} |f(x)|^2 dx = \sum_{k=1}^\infty \int_{E_k} \left(\frac{1}{\sqrt{a_k}\, n_k}\right)^2 dx$$

$$= \sum_{k=1}^\infty \frac{1}{a_k \cdot n_k^2} a_k \leq \sum_{n=1}^\infty \frac{1}{n^2} < \infty \quad //$$

However, $f \notin D(\varphi(X))$ holds,

since
$$\int_{\mathbb{R}^d} |\varphi(x)|^2 |f(x)|^2 dx = \sum_{k=1}^\infty \int_{E_k} |\varphi(x)|^2 \frac{1}{a_k \cdot n_k^2} dx$$

$$\geq \sum_{k=1}^\infty \int_{E_k} n_k^2 \frac{1}{a_k \cdot n_k^2} dx$$

$$= \sum_{k=1}^\infty \frac{a_k}{a_k} = \sum_{k=1}^\infty 1 = \infty.$$

Thus "$D(\varphi(X)) = L^2(\mathbb{R}^d) \Rightarrow \varphi \in L^\infty(\mathbb{R}^d)$" was proved.

~~and~~

$\square$