Chapter 4

Spectral theory for self-adjoint operators

In this chapter we develop the spectral theory for self-adjoint operators. As already seen in Lemma 2.2.6, these operators have real spectrum, however much more can be said about them, and in particular the spectrum can be divided into several parts having distinct properties. Note that this chapter is mainly inspired from Chapter 4 of [Amr] to which we refer for additional information.

4.1 Stieltjes measures

We start by introducing Stieltjes measures, since they will be the key ingredient for the spectral theorem. For that purpose, let us consider a function $F : \mathbb{R} \to \mathbb{R}$ satisfying the following properties:

(i) F is monotone non-decreasing, *i.e.* $\lambda \geq \mu \Longrightarrow F(\lambda) \geq F(\mu)$,

(ii) F is right continuous, *i.e.* $F(\lambda) = F(\lambda + 0) := \lim_{\varepsilon \searrow 0} F(\lambda + \varepsilon)$ for all $\lambda \in \mathbb{R}$,

(iii) $F(-\infty) := \lim_{\lambda \to -\infty} F(\lambda) = 0$ and $F(+\infty) := \lim_{\lambda \to +\infty} F(\lambda) < \infty$.

Note that $F(\lambda + 0) := \lim_{\varepsilon \searrow 0} F(\lambda + \varepsilon)$ and $F(\lambda - 0) := \lim_{\varepsilon \searrow 0} F(\lambda - \varepsilon)$ exist since F is a monotone and bounded function.

Exercise 4.1.1. *Show that such a function has at most a countable set of points of discontinuity. For that purpose you can consider for fixed $n \in \mathbb{N}$ the set of points $\lambda \in \mathbb{R}$ for which $F(\lambda) - F(\lambda - 0) > 1/n$.*

With a function F having these properties, one can associate a bounded Borel measure m_F on $\mathbb{R}$, called *Stieltjes measure*, starting with

$$m_F\big((a, b]\big) := F(b) - F(a), \qquad a, b \in \mathbb{R} \tag{4.1}$$

and extending then this definition to all Borel sets of $\mathbb{R}$ (we denote by $\mathcal{A}_B$ the set of all Borel sets on $\mathbb{R}$). More precisely, for any set $V \in \mathcal{A}_B$ one sets

$$m_F(V) = \inf \sum_k m_F(J_k)$$

with $\{J_k\}$ any sequence of half-open intervals covering the set V, and the infimum is taken over all such covering of V. With this definition, note that $m_F(\mathbb{R}) = F(+\infty)$ and that

$$m_F\big((a,b)\big) = F(b-0) - F(a), \qquad m_F\big([a,b]\big) = F(b) - F(a-0)$$

and therefore $m_F\big(\{a\}\big) = F(a) - F(a-0)$ is different from 0 if F is not continuous at the point a.

Note that starting with a bounded Borel measure m on $\mathbb{R}$ and setting $F(\lambda) := m\big((-\infty,\lambda]\big)$, then F satisfies the conditions (i)-(iii) and the associated Stieltjes measure m_F verifies $m_F = m$. Observe also that if the measure is not bounded one can not have $F(+\infty) < \infty$. Less restrictively, if the measure is not bounded on any bounded Borel set, then the function F can not even be defined.

Exercise 4.1.2. *Work on the examples of functions F introduced in Examples 4.1 to 4.5 of [Amr], and describe the corresponding Stieltjes measures.*

Let us now recall the three types of measures on $\mathbb{R}$that are going to play an important role in the decomposition of any self-adjoint operator. First of all, a Borel measure is called *pure point* or *atomic* if the measure is supported by points only. More precisely, a Borel measure m is of this type if for any Borel set V there exists a collection of points $\{x_j\} \subset V$ such that

$$m(V) = \sum_j m(\{x_j\}).$$

Note that for Stieltjes measure, this set of points is at most countable. Secondly, a Borel measure m is absolutely continuous with respect to the Lebesgue measure if there exists a non-negative measurable function f such that for any Borel set V one has

$$m(V) = \int_V f(x)\mathrm{d}x$$

where $\mathrm{d}x$ denotes the Lebesgue measure on $\mathbb{R}$. Thirdly, a Borel measure m is singular continuous with respect to the Lebesgue measure if $m(\{x\}) = 0$ for any $x \in \mathbb{R}$ and if there exists a Borel set V of Lebesgue measure 0 such that the support of m is concentrated on V. Note that examples of such singular continuous measure can be constructed with Cantor functions, see for example [Amr, Ex. 4.5].

The following theorem is based on the Lebesgue decomposition theorem for measures.

Theorem 4.1.3. *Any Stieltjes measure m admits a unique decomposition*

$$m = m_p + m_{ac} + m_{sc}$$

where m_p *is* a pure point measure *(with at most a countable support),* m_{ac} *is* an absolutely continuous measure *with respect to the Lebesgue measure on* $\mathbb{R}$*, and* m_{sc} *is* a singular continuous measure *with respect to the Lebesgue measure* $\mathbb{R}$*.*

4.2 Spectral measures

We shall now define a spectral measure, by analogy with the Stieltjes measure introduced in the previous section. Indeed, instead of considering non-decreasing functions F defined on $\mathbb{R}$ and taking values in $\mathbb{R}$, we shall consider non-decreasing functions defined on $\mathbb{R}$ but taking values in the set $\mathcal{P}(\mathcal{H})$ of orthogonal projections on a Hilbert space $\mathcal{H}$.

Definition 4.2.1. A spectral family*, or* a resolution of the identity*, is a family* $\{E_\lambda\}_{\lambda\in\mathbb{R}}$ *of orthogonal projections in* $\mathcal{H}$ *satisfying:*

(i) The family is non-decreasing, i.e. $E_\lambda E_\mu = E_{\min\{\lambda,\mu\}}$*,*

(ii) The family is strongly right continuous, i.e. $E_\lambda = E_{\lambda+0} = s-\lim_{\varepsilon\searrow 0} E_{\lambda+\varepsilon}$*,*

(iii) $s-\lim_{\lambda\to-\infty} E_\lambda = \mathbf{0}$ *and* $s-\lim_{\lambda\to\infty} E_\lambda = \mathbf{1}$*,*

It is important to observe that the condition (i) implies that the elements of the families are *commuting, i.e.* $E_\lambda E_\mu = E_\mu E_\lambda$. We also define *the support of the spectral family* as the following subset of $\mathbb{R}$:

$$\operatorname{supp}\{E_\lambda\} = \big\{\mu\in\mathbb{R} \mid E_{\mu+\varepsilon} - E_{\mu-\varepsilon} \neq \mathbf{0},\ \forall\varepsilon>0\big\}.$$

Given such a family and in analogy with (4.1), one first defines

$$E\big((a,b]\big) := E_b - E_a, \qquad a,b\in\mathbb{R}, \tag{4.2}$$

and extends this definition to all sets $V\in\mathcal{A}_B$. As a consequence of the construction, note that

$$E\Big(\bigcup_k V_k\Big) = \sum_k E(V_k) \tag{4.3}$$

whenever $\{V_k\}$ is a countable family of disjoint elements of $\mathcal{A}_B$. Thus, one ends up with a projection-valued map $E:\mathcal{A}_B\to\mathcal{P}(\mathcal{H})$ which satisfies $E(\emptyset)=\mathbf{0}$, $E(\mathbb{R})=\mathbf{1}$, $E(V_1)E(V_2)=E(V_1\cap V_2)$ for any Borel sets V_1,V_2. In addition,

$$E\big((a,b)\big) = E_{b-0} - E_a, \qquad E\big([a,b]\big) = E_b - E_{a-0}$$

and therefore $E\big(\{a\}\big) = E_a - E_{a-0}$.

Definition 4.2.2. *The map* $E:\mathcal{A}_B\to\mathcal{P}(\mathcal{H})$ *defined by* (4.2) *is called* the spectral measure *associated with the family* $\{E_\lambda\}_{\lambda\in\mathbb{R}}$*. This spectral measure is* bounded from below *if there exists* $\lambda_-\in\mathbb{R}$ *such that* $E_\lambda=\mathbf{0}$ *for all* $\lambda<\lambda_-$*. Similarly, this spectral measure is* bounded from above *if there exists* $\lambda_+\in\mathbb{R}$ *such that* $E_\lambda=\mathbf{1}$ *for all* $\lambda>\lambda_+$*.*

Let us note that for any spectral family $\{E_\lambda\}_{\lambda\in\mathbb{R}}$ and any $f \in \mathcal{H}$ one can set

$$F_f(\lambda) := \|E_\lambda f\|^2 = \langle E_\lambda f, f\rangle.$$

Then, one easily checks that the function F_f satisfies the conditions (i)-(iii) of the beginning of Section 4.1. Thus, one can associate with each element $f \in \mathcal{H}$ a Stieltjes measure m_f on $\mathbb{R}$ which satisfies

$$m_f(V) = \|E(V)f\|^2 = \langle E(V)f, f\rangle \tag{4.4}$$

for any $V \in \mathcal{A}_B$. Note in particular that $m_f(\mathbb{R}) = \|f\|^2$. Later on, this Stieltjes measure will be decomposed according the content of Theorem 4.1.3.

Our next aim is to define integrals of the form

$$\int_a^b \varphi(\lambda)\, E(\mathrm{d}\lambda) \tag{4.5}$$

for a continuous function $\varphi : [a, b] \to \mathbb{C}$ and for any spectral family $\{E_\lambda\}_{\lambda\in\mathbb{R}}$. Such integrals can be defined in the sense of Riemann-Stieltjes by first considering a partition $a = x_0 < x_1 < ... < x_n = b$ of $[a, b]$ and a collection $\{y_j\}$ with $y_j \in (x_{j-1}, x_j)$ and by defining the operator

$$\sum_{j=1}^{n} \varphi(y_j)E\big((x_{j-1}, x_j]\big). \tag{4.6}$$

It turns out that by considering finer and finer partitions of $[a, b]$, the corresponding expression (4.6) strongly converges to an element of $\mathscr{B}(\mathcal{H})$ which is independent of the successive choice of partitions. The resulting operator is denoted by (4.5).

The following statement contains usual results which can be obtained in this context. The proof is not difficult, but one has to deal with several partitions of intervals. We refer to [Amr, Prop. 4.10] for a detailed proof.

Proposition 4.2.3 (Spectral integrals)**.** *Let $\{E_\lambda\}_{\lambda\in\mathbb{R}}$ be a spectral family, let $-\infty < a < b < \infty$ and let $\varphi : [a, b] \to \mathbb{C}$ be continuous. Then one has*

(i) $\left\|\int_a^b \varphi(\lambda)\, E(\mathrm{d}\lambda)\right\| = \sup_{\mu\in[a,b]\cap\mathrm{supp}\{E_\lambda\}} |\varphi(\mu)|$,

(ii) $\left(\int_a^b \varphi(\lambda)\, E(\mathrm{d}\lambda)\right)^* = \int_a^b \overline{\varphi}(\lambda)\, E(\mathrm{d}\lambda)$,

(iii) For any $f \in \mathcal{H}$, $\left\|\int_a^b \varphi(\lambda)\, E(\mathrm{d}\lambda) f\right\|^2 = \int_a^b |\varphi(\lambda)|^2\, m_f(\mathrm{d}\lambda)$,

(iv) If $\psi : [a, b] \to \mathbb{C}$ is continuous, then

$$\int_a^b \varphi(\lambda)\, E(\mathrm{d}\lambda) \cdot \int_a^b \psi(\lambda)\, E(\mathrm{d}\lambda) = \int_a^b \varphi(\lambda)\,\psi(\lambda)\, E(\mathrm{d}\lambda).$$

Let us now observe that if the support $\mathrm{supp}\{E_\lambda\}$ is bounded, then one can consider

$$\int_{-\infty}^{\infty} \varphi(\lambda)\, E(\mathrm{d}\lambda) = s - \lim_{M\to\infty} \int_{-M}^{M} \varphi(\lambda)\, E(\mathrm{d}\lambda). \tag{4.7}$$

Similarly, by taking property (iii) of the previous proposition into account, one observes that this limit can also be taken if $\varphi \in C_b(\mathbb{R})$. On the other hand, if φ is not bounded on $\mathbb{R}$, the r.h.s. of (4.7) is not necessarily well defined. In fact, if φ is not bounded on $\mathbb{R}$ and if $\mathrm{supp}\{E_\lambda\}$ is not bounded either, then the r.h.s. of (4.7) is an unbounded operator and can only be defined on a dense domain of $\mathcal{H}$.

Lemma 4.2.4. *Let $\varphi : \mathbb{R} \to \mathbb{C}$ be continuous, and let us set*

$$\mathsf{D}_\varphi := \big\{f \in \mathcal{H} \mid \int_{-\infty}^{\infty} |\varphi(\lambda)|^2\, m_f(\mathrm{d}\lambda) < \infty\big\}.$$

Then the pair $\Big(\int_{-\infty}^{\infty} \varphi(\lambda)\, E(\mathrm{d}\lambda), \mathsf{D}_\varphi\Big)$ defines a closed linear operator on $\mathcal{H}$. This operator is self-adjoint if and only if φ is a real function.

Proof. Observe first that $\mathsf{D}_\varphi = \mathsf{D}_{\overline{\varphi}}$, and set $A := \int_{-\infty}^{\infty} \varphi(\lambda)\, E(\mathrm{d}\lambda)$. A is densely defined because its domain contains all elements with compact support with respect to $\{E_\lambda\}$, *i.e.* it contains all $g \in \mathcal{H}$ satisfying $g = E\big((-N,N]\big)g$ for some $N < \infty$. Thus, for $f, g \in \mathsf{D}_\varphi$ one has by the point (ii) of Proposition 4.2.3

$$\begin{aligned}\langle f, Ag\rangle &= \lim_{M\to\infty} \Big\langle f, \int_{-M}^{M} \varphi(\lambda)\, E(\mathrm{d}\lambda) g\Big\rangle \\ &= \lim_{M\to\infty} \Big\langle \int_{-M}^{M} \overline{\varphi}(\lambda)\, E(\mathrm{d}\lambda) f, g\Big\rangle = \Big\langle \int_{-\infty}^{\infty} \overline{\varphi}(\lambda)\, E(\mathrm{d}\lambda) f, g\Big\rangle.\end{aligned}$$

It thus follows that $\mathsf{D}_{\overline{\varphi}} \subset \mathsf{D}(A^*)$, and that $A^* f = \int_{-\infty}^{\infty} \overline{\varphi}(\lambda)\, E(\mathrm{d}\lambda) f$ for any $f \in \mathsf{D}_{\overline{\varphi}}$. As a consequence A^* is an extension of $\int_{-\infty}^{\infty} \overline{\varphi}(\lambda)\, E(\mathrm{d}\lambda)$, and in order to show that these two operators are equal it is sufficient to show that $\mathsf{D}(A^*) \subset \mathsf{D}_{\overline{\varphi}}$.

For that purpose, recall that if $f \in \mathsf{D}(A^*)$ there exists $f^* \in \mathcal{H}$ such that for any $g \in \mathsf{D}_\varphi$

$$\langle f, Ag\rangle = \langle f^*, g\rangle.$$

In particular this equality holds if g has compact support with respect to $\{E_\lambda\}$. One then gets for any $M \in (0, \infty)$

$$\begin{aligned}
\big\|E\big((-M,M]\big)f^*\big\| &= \sup_{g\in\mathcal{H},\|g\|=1} \big|\big\langle E\big((-M,M]\big)f^*, g\big\rangle\big| \\
&= \sup_{g\in\mathcal{H},\|g\|=1} \big|\big\langle f^*, E\big((-M,M]\big)g\big\rangle\big| \\
&= \sup_{g\in\mathcal{H},\|g\|=1} \Big|\Big\langle f, \int_{-M}^{M} \varphi(\lambda)\, E(\mathrm{d}\lambda) g\Big\rangle\Big| \\
&= \sup_{g\in\mathcal{H},\|g\|=1} \Big|\Big\langle \int_{-M}^{M} \overline{\varphi}(\lambda)\, E(\mathrm{d}\lambda) f, g\Big\rangle\Big| \\
&= \Big\| \int_{-M}^{M} \overline{\varphi}(\lambda)\, E(\mathrm{d}\lambda) f\Big\| = \Big(\int_{-M}^{M} |\varphi(\lambda)|^2 m_f(\mathrm{d}\lambda)\Big)^{1/2}.
\end{aligned}$$

As a consequence, one has

$$\sup_{M>0} \int_{-M}^{M} |\varphi(\lambda)|^2 m_f(\mathrm{d}\lambda) \le \|f^*\|^2 < \infty$$

from which one infers that $\int_{-\infty}^{\infty} |\varphi(\lambda)|^2 m_f(\mathrm{d}\lambda) < \infty$. This shows that if $f \in \mathsf{D}(A^*)$ then $f \in \mathsf{D}_{\overline{\varphi}}$.

Since A^* is always closed by (i) of Lemma 2.1.10, one infers that $\int_{-\infty}^{\infty} \overline{\varphi}(\lambda)\, E(\mathrm{d}\lambda)$ on $\mathsf{D}_{\overline{\varphi}}$ is a closed operator. So the same holds for A on D_{φ}. Finally, since $\mathsf{D}_{\varphi} = \mathsf{D}_{\overline{\varphi}}$, the second statement is a direct consequence of the first one. □

A function φ of special interest is the function defined by the identity function id, namely $\mathrm{id}(\lambda) = \lambda$.

Definition 4.2.5. *For any spectral family $\{E_\lambda\}_{\lambda\in\mathbb{R}}$, the operator $\big(\int_{-\infty}^{\infty} \lambda\, E(\mathrm{d}\lambda), \mathsf{D}_{\mathrm{id}}\big)$ with*

$$\mathsf{D}_{\mathrm{id}} := \Big\{f \in \mathcal{H} \mid \int_{-\infty}^{\infty} \lambda^2 m_f(\mathrm{d}\lambda) < \infty\Big\}$$

is called the self-adjoint operator associated with $\{E_\lambda\}$.

By this procedure, any spectral family defines a self-adjoint operator on $\mathcal{H}$. *The spectral Theorem* corresponds to the converse statement:

Theorem 4.2.6 (Spectral Theorem)**.** *With any self-adjoint operator $(A, \mathsf{D}(A))$ on a Hilbert space $\mathcal{H}$ one can associate a unique spectral family $\{E_\lambda\}$, called* the spectral family of A, *such that $\mathsf{D}(A) = \mathsf{D}_{\mathrm{id}}$ and $A = \int_{-\infty}^{\infty} \lambda\, E(\mathrm{d}\lambda)$.*

In summary, there is a bijective correspondence between self-adjoint operators and spectral families. This theorem extends the fact that any $n \times n$ hermitian matrix is diagonalizable. The proof of this theorem is not trivial and is rather lengthy. In the sequel, we shall assume it, and state various consequences of this theorem.

Extension 4.2.7. *Study the proof the Spectral Theorem, starting with the version for bounded self-adjoint operators.*

Based on this one-to-one correspondence it is now natural to set the following definition:

Definition 4.2.8. *Let A be a self-adjoint operator in $\mathcal{H}$ and $\{E_\lambda\}$ be the corresponding spectral family. For any bounded and continuous function $\varphi : \mathbb{R} \to \mathbb{C}$ one sets $\varphi(A) \in \mathscr{B}(\mathcal{H})$ for the operator defined by*

$$\varphi(A) := \int_{-\infty}^{\infty} \varphi(\lambda)\, E(\mathrm{d}\lambda). \tag{4.8}$$

Exercise 4.2.9. *For any self-adjoint operator A, prove the following equality:*

$$\operatorname{supp}\{E_\lambda\} = \sigma(A). \tag{4.9}$$

Note that part of the proof consists in showing that if $\varphi_z(\lambda) = (\lambda - z)^{-1}$ for some $z \in \rho(A)$, then $\varphi_z(A) = (A - z)^{-1}$, where the r.h.s. has been defined in Section 2.2. Let us also mention a useful equality which can be proved in this exercise: for any $z \in \rho(A)$ one has

$$\big\|(A - z)^{-1}\big\| = \big[\operatorname{dist}\big(z, \sigma(A)\big)\big]^{-1}. \tag{4.10}$$

For the next statement, we set $C_b(\mathbb{R})$ for the set of all continuous and bounded complex functions on $\mathbb{R}$.

Proposition 4.2.10. *a) For any $\varphi \in C_b(\mathbb{R})$ one has*

(i) $\varphi(A) \in \mathscr{B}(\mathcal{H})$ and $\|\varphi(A)\| = \sup_{\lambda\in\sigma(A)} |\varphi(\lambda)|$,

(ii) $\varphi(A)^* = \overline{\varphi}(A)$, and $\varphi(A)$ is self-adjoint if and only if φ is real,

(iii) $\varphi(A)$ is unitary if and only if $|\varphi(\lambda)| = 1$.

b) The map $C_b(\mathbb{R}) \ni \varphi \mapsto \varphi(A) \in \mathscr{B}(\mathcal{H})$ is a $*$-homomorphism.
c) If $\varphi \in C(\mathbb{R})$, then (4.8) defines a closed operator $\varphi(A)$ with domain

$$\mathsf{D}\big(\varphi(A)\big) = \Big\{f \in \mathcal{H} \mid \int_{-\infty}^{\infty} |\varphi(\lambda)|^2\, m_f(\mathrm{d}\lambda) < \infty\Big\}. \tag{4.11}$$

In the point (iii) above, one can consider the function $\varphi_t \in C_b(\mathbb{R})$ defined by $\varphi_t(\lambda) := \mathrm{e}^{-i\lambda t}$ for any fixed $t \in \mathbb{R}$. Then, if one sets $U_t := \varphi_t(A)$ one first observes that $U_tU_s = U_{t+s}$. Indeed, one has

$$\begin{aligned} U_tU_s &= \int_{-\infty}^{\infty} \mathrm{e}^{i\lambda t}\, E(\mathrm{d}\lambda) \int_{-\infty}^{\infty} \mathrm{e}^{-i\lambda s}\, E(\mathrm{d}\lambda) = \int_{-\infty}^{\infty} \mathrm{e}^{-i\lambda t}\, \mathrm{e}^{-i\lambda s}\, E(\mathrm{d}\lambda) \\ &= \int_{-\infty}^{\infty} \mathrm{e}^{-i\lambda(t+s)}\, E(\mathrm{d}\lambda) = U_{t+s}. \end{aligned}$$

In addition, by an application of the dominated convergence theorem of Lebesgue, one infers that the map $\mathbb{R} \ni t \mapsto U_t \in \mathscr{B}(\mathcal{H})$ is strongly continuous. Indeed, since $\big| \mathrm{e}^{-i\lambda(t+\varepsilon)} - \mathrm{e}^{-i\lambda t} \big|^2 \leq 4$ one has

$$\|U_{t+\varepsilon}f - U_t f\|^2 = \int_{-\infty}^{\infty} \big| \mathrm{e}^{-i\lambda(t+\varepsilon)} - \mathrm{e}^{-i\lambda t} \big|^2 m_f(\mathrm{d}\lambda) \;\to\; 0 \quad \text{as} \quad \varepsilon \to 0.$$

As a consequence, such a family $\{U_t\}_{t\in\mathbb{R}}$ is called *a strongly continuous unitary group.* Note that since $\mathrm{e}^{-i\lambda t} = \sum_{k=0}^{\infty} \frac{(-i\lambda t)^k}{k!}$ one also infers that whenever A is a bounded operator

$$U_t = \sum_{k=0}^{\infty} \frac{(-itA)^k}{k!} \tag{4.12}$$

with a norm converging series. On the other hand, if A is not bounded, then this series converges on elements $f \in \cap_{k=0}^{\infty} \mathsf{D}(A^k)$. In particular, it converges strongly on elements of $\mathcal{H}$ which have compact support with respect to the corresponding spectral measure.

Let us now mention that the above construction is only one part of a one-to-one relation between strongly continuous unitary groups and self-adjoint operators. The proof of the following statement can be found for example in [Amr, Prop. 5.1].

Theorem 4.2.11 (Stone's Theorem)**.** *There exists a bijective correspondence between self-adjoint operators on $\mathcal{H}$ and strongly continuous unitary groups on $\mathcal{H}$. More precisely, if A is a self-adjoint operator on $\mathcal{H}$, then $\{\mathrm{e}^{-itA}\}_{t\in\mathbb{R}}$ is a strongly continuous unitary group, while if $\{U_t\}_{t\in\mathbb{R}}$ is a strongly continuous unitary group, one sets*

$$\mathsf{D}(A) := \Big\{ f \in \mathcal{H} \mid \exists\, s - \lim_{t\to 0} \frac{1}{t}[U_t - 1]f \Big\}$$

and for $f \in \mathsf{D}(A)$ one sets $Af = s - \lim_{t\to 0} \frac{i}{t}[U_t - 1]f$, and then $\big(A, \mathsf{D}(A)\big)$ is a self-adjoint operator.

Exercise 4.2.12. *Provide a precise proof of Stone's theorem.*

Let us close with section with two important observations. First of all, the map $\varphi \mapsto \varphi(A)$ can be extended from continuous and bounded φ to bounded and measurable functions φ. This extension can be realized by considering the Lebesgue-Stieltjes integrals in the weak form. In particular, this extension is necessary for defining $\varphi(A)$ whenever φ is the characteristic function on some Borel set V.

The second observation is going to provide an alternative formula for $\varphi(A)$ in terms of the unitary group $\{\mathrm{e}^{-itA}\}_{t\in\mathbb{R}}$. Indeed, assume that the inverse Fourier transform $\check{\varphi}$ of φ belongs to $L^1(\mathbb{R})$, then the following equality holds

$$\varphi(A) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \check{\varphi}(t)\, \mathrm{e}^{-itA}\, \mathrm{d}t. \tag{4.13}$$

Indeed, observe that

$$\langle f, \varphi(A)f \rangle = \int_{\mathbb{R}} \varphi(\lambda) m_f(\mathrm{d}\lambda) = \int_{\mathbb{R}} m_f(\mathrm{d}\lambda) \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \mathrm{e}^{-i\lambda t}\, \check{\varphi}(t) \mathrm{d}t.$$

By application of Fubini's theorem one can interchange the order of integrations and obtain

$$\begin{aligned}\langle f, \varphi(A)f\rangle =&\frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} \mathrm{d}t\check{\varphi}(t)\int_{\mathbb{R}} \mathrm{e}^{-i\lambda t}\, m_f(\mathrm{d}\lambda)\\ =&\frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} \mathrm{d}t\,\check{\varphi}(t)\langle f, \mathrm{e}^{-itA} f\rangle = \Big\langle f, \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} \mathrm{d}t\,\check{\varphi}(t)\,\mathrm{e}^{-itA} f\Big\rangle,\end{aligned}$$

and one gets (4.13) by applying the polarisation identity (1.1).

4.3 Spectral parts of a self-adjoint operator

In this section, we consider a fixed self-adjoint operator A (and its associated spectral family $\{E_\lambda\}$), and show that there exists a natural decomposition of the Hilbert space $\mathcal{H}$ with respect to this operator. First of all, recall from Lemma 2.2.6 that the spectrum of any self-adjoint operator is real. In addition, let us recall that for any $\mu \in \mathbb{R}$, one has

$$\mathsf{Ran}\,\big(E(\{\mu\})\big) = \{f \in \mathcal{H} \mid E(\{\mu\})f = f\}.$$

Then, one observes that the following equivalence holds:

$$f \in \mathsf{Ran}\,\big(E(\{\mu\})\big) \Longleftrightarrow f \in \mathsf{D}(A) \ \text{ with } Af = \mu f.$$

Indeed, this can be inferred from the equality

$$\|Af - \mu f\|^2 = \int_{-\infty}^{\infty} |\lambda - \mu|^2\, m_f(\mathrm{d}\lambda)$$

which itself can be deduced from the point (iii) of Proposition 4.2.3. In fact, since the integrand is strictly positive for each $\lambda \neq \mu$, one has $\|Af - \mu f\| = 0$ if and only if $m_f(V) = 0$ for any Borel set V on $\mathbb{R}$ with $\mu \notin V$. In other words, the measure m_f is supported only on $\{\mu\}$.

Definition 4.3.1. *The set of all* $\mu \in \mathbb{R}$ *such that* $\mathsf{Ran}\,\big(E(\{\mu\})\big) \neq 0$ *is called* the point spectrum of A *or* the set of eigenvalues of A. *One then sets*

$$\mathcal{H}_p(A) := \bigoplus \mathsf{Ran}\,\big(E(\{\mu\})\big)$$

where the sum extends over all eigenvalues of A.

In accordance with what has been presented in Theorem 4.1.3, we define two additional subspaces of $\mathcal{H}$.

Definition 4.3.2.

$$\begin{aligned}\mathcal{H}_{ac}(A) \ &:= \ \big\{f \in \mathcal{H} \mid m_f \textit{ is an absolutely continuous measure}\big\}\\ &= \ \big\{f \in \mathcal{H} \mid \ \textit{the function } \lambda \mapsto \|E_\lambda f\|^2 \textit{ is absolutely continuous}\big\},\end{aligned}$$

$$\begin{aligned}\mathcal{H}_{sc}(A) &:= \big\{f \in \mathcal{H} \mid m_f \textit{ is a singular continuous measure}\big\} \\ &= \big\{f \in \mathcal{H} \mid \textit{ the function } \lambda \mapsto \|E_\lambda f\|^2 \textit{ is singular continuous}\big\},\end{aligned}$$

for which the comparison measure is always the Lebesgue measure on $\mathbb{R}$*.*

Note that one also uses the notation $\mathcal{H}_c(A)$ for the set of $f \in \mathcal{H}$ such that m_f is continuous, *i.e.* $m_f(\{x\}) = 0$ for any $x \in \mathbb{R}$. One also speaks then about the continuous subspace of $\mathcal{H}$ with respect to A.

The following statement provides the decomposition of any self-adjoint operator into three distinct parts. Note that depending on the operators, some parts of the following decomposition can be trivial. The proof of the statement is not difficult and consists in some routine computations.

Theorem 4.3.3. *Let* A *be a self-adjoint operator in a Hilbert space* $\mathcal{H}$*.*

a) This Hilbert space can be decomposed as follows

$$\mathcal{H} = \mathcal{H}_p(A) \oplus \mathcal{H}_{ac}(A) \oplus \mathcal{H}_{sc}(A),$$

and the restriction of the operator A *to each of these subspaces defines a self-adjoint operator denoted respectively by* A_p*,* A_{ac} *and* A_{sc}*.*

b) For any $\varphi \in C_b(\mathbb{R})$*, one has the decomposition*

$$\varphi(A) = \varphi(A_p) \oplus \varphi(A_{ac}) \oplus \varphi(A_{sc}).$$

Moreover, the following equality holds

$$\sigma(A) = \sigma(A_p) \cup \sigma(A_{ac}) \cup \sigma(A_{sc}).$$

Exercise 4.3.4. *Provide a full proof of the above statement.*

Note that one often writes $E_p(A)$, $E_{ac}(A)$ and $E_{sc}(A)$ for the orthogonal projection on $\mathcal{H}_p(A)$, $\mathcal{H}_{ac}(A)$ and $\mathcal{H}_{sc}(A)$, respectively, and with these notations one has $A_p = AE_p(A)$, $A_{ac} = AE_{ac}(A)$ and $A_{sc} = AE_{sc}(A)$. In addition, note that the relation between the set of eigenvalues $\sigma_p(A)$ introduced in Definition 2.2.4 and the set $\sigma(A_p)$ is

$$\sigma(A_p) = \overline{\sigma_p(A)}.$$

Two additional sets are often introduced in relation with the spectrum of A, namely $\sigma_d(A)$ and $\sigma_{ess}(A)$.

Definition 4.3.5. *An eigenvalue* λ *belongs to* the discrete spectrum $\sigma_d(A)$ of A *if and only if* $\mathsf{Ran}\big(E(\{\lambda\})\big)$ *is of finite dimension, and* λ *is isolated from the rest of the spectrum of* A*.* The essential spectrum $\sigma_{ess}(A)$ of A *is the complementary set of* $\sigma_d(A)$ *in* $\sigma(A)$*, or more precisely*

$$\sigma_{ess}(A) = \sigma(A) \setminus \sigma_d(A).$$

Since we have now all type of spectra at our disposal, let us come back to the examples of Chapter 3. As already mentioned in Exercise 3.1.2 the spectrum of any self-adjoint multiplication operator $\varphi(X)$ in $L^2(\mathbb{R}^d)$ is given by the closure of $\varphi(\mathbb{R}^d)$. Now, it is easily observed that $\lambda \in \sigma_p\big(\varphi(X)\big)$ if and only if there exists a Borel set V with strictly positive Lebesgue measure such that $\varphi(x) = \lambda$ for any $x \in V$. In this case, the multiplicity of the eigenvalue is infinite, since an infinite family of orthogonal eigenfunctions corresponding to the eigenvalue λ can easily be constructed. Obviously, the previous requirement is not necessary for $\lambda \in \sigma_{ess}\big(\varphi(X)\big)$. Let us also mention that if the function φ is continuously differentiable and if $\nabla\varphi(x) \neq 0$ for any $x \in \mathbb{R}^d$, then the operator $\varphi(X)$ has only absolutely continuous spectrum. Such a statement will be a consequence of the conjugate operator method introduced in subsequent chapters. Note also that for a convolution operator $\varphi(D)$ the situation is rather similar, and this can be deduced easily from Remark 4.3.6.

For the harmonic oscillator of Section 3.1.1, the corresponding operator has only discrete eigenvalues and no continuous spectrum, *i.e.* $\mathcal{H}_c(A) = \{0\}$. For Schrödinger operators, one has typically a mixture of continuous spectrum and of point spectrum. Note that the eigenvalues can be embedded in the continuous spectrum, but that the situation of eigenvalues below the continuous spectrum often appears for such operators. For the hydrogen atom of Section 3.2.1 this situation takes place, since the continuous spectrum corresponds to $[0, \infty)$ while the point spectrum consists in the eigenvalues which are all located below 0 and are converging to 0. For more general Schrödinger operators, it is also often expected that $\mathcal{H}_{sc}(A) = \{0\}$, but proving such a statement can be a difficult task. We shall come back to this question in the following chapters.

Let us still consider any compact self-adjoint operator A in a Hilbert space $\mathcal{H}$. It is easily observed that for any $\varepsilon > 0$ the subspace $E\big((-\infty, -\varepsilon]\big)\mathcal{H} \cup E\big([\varepsilon, \infty)\big)\mathcal{H}$ is of finite dimension, where $E(\cdot)$ corresponds to the spectral measure associated with A. In other words, away from 0 the spectrum of A consists of eigenvalues of finite multiplicity, and these eigenvalues can only converge to 0. On the other hand, 0 can be either an eigenvalue with finite or infinite multiplicity, or a point of accumulation of the spectrum without being itself a eigenvalue.

Remark 4.3.6. *If A is a self-adjoint operator in a Hilbert space $\mathcal{H}$ and if U is a unitary operator in $\mathcal{H}$, conjugating A by U does not change its spectral properties. More precisely, in one considers $A_U := UAU^*$ with domain $\mathsf{D}(A_U) = U\mathsf{D}(A)$, then this operator is self-adjoint and the following equalities hold: $\sigma(A_U) = \sigma(A)$, $\sigma_p(A_U) = \sigma_p(A)$, ... These facts are a consequence of the following observations: $\{UE_\lambda U^*\}$ corresponds to the spectral family for the operator A_U, and then for any Borel set V*

$$m_{Uf}(V) = \big\langle \big(UE(V)U^*\big)Uf, Uf\big\rangle = \langle E(V)f, f\rangle = m_f(V).$$

We end this section with a few results which are related to the essential spectrum of a self-adjoint operator. The first one provides another characterization of the spectrum or of the essential spectrum of a self-adjoint operator A.

Proposition 4.3.7 (Weyl's criterion). *Let A be a self-adjoint operator in a Hilbert space $\mathcal{H}$.*

a) A real number λ belongs to $\sigma(A)$ if and only if there exists a sequence $\{f_n\}_{n\in\mathbb{N}} \subset D(A)$ such that $\|f_n\| = 1$ and $s-\lim_{n\to\infty}(A-\lambda)f_n = 0$.

b) A real number λ belongs to $\sigma_{ess}(A)$ if and only if there exists a sequence $\{f_n\}_{n\in\mathbb{N}} \subset D(A)$ such that $\|f_n\| = 1$, $w-\lim_{n\to\infty} f_n = 0$ and $s-\lim_{n\to\infty}(A-\lambda)f_n = 0$.

Exercise 4.3.8. *Provide a proof of the above statement. For convenience, you can first provide such a proof in the special case of a multiplication operator in the Hilbert space $L^2(\mathbb{R})$.*

The second result deals with the conservation of the essential spectrum under a relatively compact perturbation. Before its statement, recall that the addition of a relatively compact perturbation does not change the self-adjointness property, see Proposition 2.3.5 in conjunction with Rellich-Kato theorem.

Proposition 4.3.9. *Let A be a self-adjoint operator in a Hilbert space $\mathcal{H}$, and let B be a symmetric operator in $\mathcal{H}$ which is A-compact. Then the following equality holds:*

$$\sigma_{ess}(A+B) = \sigma_{ess}(A). \tag{4.14}$$

Proof. Let us consider $\lambda \in \sigma_{ess}(A)$, and choose a sequence $\{f_n\}_{n\in\mathbb{N}} \subset D(A)$ such that $\|f_n\| = 1$, $w-\lim_{n\to\infty} f_n = 0$ and $s-\lim_{n\to\infty}(A-\lambda)f_n = 0$. Note that the existence of such a sequence is provided by Proposition 4.3.7. By the same proposition, one would get $\lambda \in \sigma_{ess}(A+B)$ if one shows that $s-\lim_{n\to\infty}(A+B-\lambda)f_n = 0$, which is itself implied by $s-\lim_{n\to\infty} Bf_n = 0$.

For that purpose, let us fix $z \in \rho(A)$ such that $B(A-z)^{-1} \in \mathscr{K}(\mathcal{H})$ and write

$$\begin{aligned} Bf_n &= [B(A-z)^{-1}](A-z)f_n \\ &= [B(A-z)^{-1}](A-\lambda)f_n + (\lambda - z)[B(A-z)^{-1}]f_n. \end{aligned}$$

Observe now that both terms converge to 0 as $n \to \infty$. For the first term, this follows directly from the assumptions. For the second one, recall that $w-\lim_{n\to\infty} f_n = 0$ and that a compact operator transform a weak convergence into a strong convergence, see Proposition 1.4.12, which means that $s-\lim_{n\to\infty}[B(A-z)^{-1}]f_n = 0$. As a consequence one infers that $\sigma_{ess}(A) \subset \sigma_{ess}(A+B)$.

The converse statement $\sigma_{ess}(A+B) \subset \sigma_{ess}(A)$ can be obtained similarly by considering first $A+B$ and by perturbing this operator with the relatively compact operator $-B$. Note that the relative compactness of $-B$ with respect to $A+B$ is a direct consequence of the point (iii) of Proposition 2.3.5. □

In the previous statement we have obtained the stability of the essential spectrum under relatively compact perturbation. However, this stability does not imply anything about the conservation of the nature of the spectrum. In that respect the following statement shows that the nature of the spectrum can drastically change even under a small perturbation.

Proposition 4.3.10 (Weyl-von Neumann). *Let A be an arbitrary self-adjoint operator in a Hilbert space. Then, for any $\varepsilon > 0$ there exists a self-adjoint Hilbert-Schmidt operator B with its Hilbert-Schmidt norm $\|B\|_{HS}$ satisfying $\|B\|_{HS} < \varepsilon$ such that $A+B$ has only pure point spectrum.*

Exercise 4.3.11. *Study the proof of this proposition, as presented for example in [Kat, Sec. X.2].*

4.4 The resolvent near the spectrum

In this section we study the resolvent $(A - z)^{-1}$ of any self-adjoint operator A when z approaches a value in $\sigma(A)$. Such investigations lead quite naturally to the spectral theorem, but also allow us to deduce useful information on the spectrum of the operator A. The spectral family $\{E_\lambda\}$ associated with the operator A can also be deduced from such investigations.

As a motivation, consider the following function defined for any $\lambda \in \mathbb{R}$ and any $\varepsilon > 0$:

$$\mathbb{R} \ni x \mapsto \frac{1}{x-\lambda-i\varepsilon} - \frac{1}{x-\lambda+i\varepsilon} = \frac{2i\varepsilon}{(x-\lambda)^2+\varepsilon^2} \in \mathbb{C}.$$

It is known that this function converges as $\varepsilon \to 0$ and in the sense of distributions to $2\pi i\delta_0(x - \lambda)$. Thus, if we replace x by the self-adjoint operator A one formally infers that

$$\begin{aligned}\frac{1}{2\pi i}\int_a^b \big[(A-\lambda-i\varepsilon)^{-1} - (A-\lambda+i\varepsilon)^{-1}\big]\mathrm{d}\lambda \to \int_a^b \delta_0(A-\lambda)\mathrm{d}\lambda &= \chi_{(a,b)}(A) \\ &= E\big((a,b)\big).\end{aligned}$$

The next statement shows that this argument is almost correct, once the behavior of the operator A at the endpoints a and b is taken into account.

Proposition 4.4.1 (Stone's formula). *Let A be a self-adjoint operator with associated spectral family $\{E_\lambda\}$. Then for any $-\infty < a < b < \infty$ the following formulas hold:*

$$\begin{aligned}\frac{1}{2\pi i} s-\lim_{\varepsilon\searrow 0}\int_a^b &\big[(A-\lambda-i\varepsilon)^{-1} - (A-\lambda+i\varepsilon)^{-1}\big]\mathrm{d}\lambda \\ &= E\big((a,b)\big) + \frac{1}{2}E(\{a\}) + \frac{1}{2}E(\{b\}) \qquad (4.15)\end{aligned}$$

and

$$E\big((a,b]\big) = \frac{1}{2\pi i}\ s-\lim_{\delta\searrow 0}\ s-\lim_{\varepsilon\searrow 0}\int_{a+\delta}^{b+\delta} \big[(A-\lambda-i\varepsilon)^{-1} - (A-\lambda+i\varepsilon)^{-1}\big]\mathrm{d}\lambda\ . \qquad (4.16)$$

Note that in the last formula the order of the two limits is important.

Proof. i) For any $\varepsilon > 0$ and $\lambda \in \mathbb{R}$ let us set

$$\begin{aligned}\Psi_\varepsilon(\lambda) =& \frac{1}{2\pi i}\int_a^b \Big(\frac{1}{x-\lambda-i\varepsilon} - \frac{1}{x-\lambda+i\varepsilon}\Big)\mathrm{d}x \\ =& \frac{1}{2\pi i}\int_a^b \frac{2i\varepsilon}{(x-\lambda)^2+\varepsilon^2}\mathrm{d}x \\ =& \frac{1}{\pi}\Big[\arctan\Big(\frac{b-\lambda}{\varepsilon}\Big) - \arctan\Big(\frac{a-\lambda}{\varepsilon}\Big)\Big].\end{aligned}$$

Clearly, Ψ_ε is continuous in λ and satisfies $|\Psi_\varepsilon(\lambda)| \leq 1$ for any $\lambda \in \mathbb{R}$. In addition, since $\lim_{\varepsilon\searrow 0}\arctan\big(\frac{x-\lambda}{\varepsilon}\big) = \frac{\pi}{2}$ if $\lambda < x$ and $\lim_{\varepsilon\searrow 0}\arctan\big(\frac{x-\lambda}{\varepsilon}\big) = -\frac{\pi}{2}$ if $\lambda > x$ one infers that

$$\Psi_0(\lambda) := \lim_{\varepsilon\searrow 0}\Psi_\varepsilon(\lambda) = \frac{1}{2}\Big(\chi_{(a,b)}(\lambda) + \chi_{[a,b]}(\lambda)\Big). \tag{4.17}$$

ii) For any $\varepsilon > 0$ the operator $\Psi_\varepsilon(A) := \int_{\mathbb{R}}\Psi_\varepsilon(\lambda)E(\mathrm{d}\lambda)$ is well-defined and belongs to $\mathscr{B}(\mathcal{H})$. Our aim is to show that this operator is strongly continuous in ε and strongly convergent for $\varepsilon \searrow 0$ to a limit corresponding to what is suggested by (4.17). For any $f \in \mathcal{H}$ one can write $f = E(\{a\})f + E(\{b\})f + f_0$ with $f_0 \in \big(E(\{a\})\mathcal{H} + E(\{b\})\mathcal{H}\big)^\perp$. One then has

$$\Psi_\varepsilon(A)E(\{a\})f = \Psi_\varepsilon(a)E(\{a\})f = \frac{1}{\pi}\arctan\Big(\frac{b-a}{\varepsilon}\Big)E(\{a\})f,$$

which converges strongly to $\frac{1}{2}E(\{a\})f$ as $\varepsilon \searrow 0$. Similarly,

$$s - \lim_{\varepsilon\searrow 0}\Psi_\varepsilon(A)E(\{b\})f = \frac{1}{2}E(\{b\})f.$$

On the other hand, since $E\big((a,b)\big)f_0 = E\big([a,b]\big)f_0 = \int_a^b E(\mathrm{d}\lambda)f_0$ one has

$$\begin{aligned}&E\big((a,b)\big)f_0 - \Psi_\varepsilon(A)f_0 \\ &= \int_a^b \big(1-\Psi_\varepsilon(\lambda)\big)E(\mathrm{d}\lambda)f_0 - \int_{-\infty}^a \Psi_\varepsilon(\lambda)E(\mathrm{d}\lambda)f_0 - \int_b^\infty \Psi_\varepsilon(\lambda)E(\mathrm{d}\lambda)f_0.\end{aligned}$$

By the dominated convergence theorem, one finds that each term on the right-hand side converges strongly to zero. Thus we have obtained that

$$\Psi_0(A) := s - \lim_{\varepsilon\searrow 0}\Psi_\varepsilon(A) = E\big((a,b)\big) + \frac{1}{2}E(\{a\}) + \frac{1}{2}E(\{b\}).$$

iii) To obtain the validity of (4.15) it is now sufficient to verify that for $\varepsilon > 0$ one has

$$\Psi_\varepsilon(A) = \frac{1}{2\pi i}\int_a^b \big[(A-\lambda-i\varepsilon)^{-1} - (A-\lambda+i\varepsilon)^{-1}\big]\mathrm{d}\lambda.$$

By the polarization identity, such an equality holds if one knows that for any $f \in \mathcal{H}$

$$\langle f, \Psi_\varepsilon(A) f\rangle = \frac{1}{2\pi i}\int_a^b \langle f, \big[(A-\lambda-i\varepsilon)^{-1} - (A-\lambda+i\varepsilon)^{-1}\big] f\rangle \mathrm{d}\lambda.$$

To prove this equality, one uses the first resolvent equation for the equality

$$\begin{aligned}(A-\lambda-i\varepsilon)^{-1} - (A-\lambda+i\varepsilon)^{-1} &= 2i\varepsilon(A-\lambda-i\varepsilon)^{-1}(A-\lambda+i\varepsilon)^{-1}\\ &= 2i\varepsilon\int_{\mathbb{R}} \frac{1}{(\mu-\lambda)^2+\varepsilon^2} E(\mathrm{d}\mu).\end{aligned}$$

It then follows that

$$\frac{1}{2\pi i}\int_a^b \big[(A-\lambda-i\varepsilon)^{-1} - (A-\lambda+i\varepsilon)^{-1}\big]\mathrm{d}\lambda = \frac{\varepsilon}{\pi}\int_a^b \mathrm{d}\lambda \int_{\mathbb{R}} \frac{1}{(\mu-\lambda)^2+\varepsilon^2} E(\mathrm{d}\mu)$$

and as a consequence

$$\frac{1}{2\pi i}\int_a^b \langle f, \big[(A-\lambda-i\varepsilon)^{-1} - (A-\lambda+i\varepsilon)^{-1}\big] f\rangle \mathrm{d}\lambda = \frac{\varepsilon}{\pi}\int_a^b \mathrm{d}\lambda \int_{\mathbb{R}} \frac{m_f(\mathrm{d}\mu)}{(\mu-\lambda)^2+\varepsilon^2}.$$

By an application of Fubini's theorem one deduces that

$$\begin{aligned}&\frac{1}{2\pi i}\int_a^b \langle f, \big[(A-\lambda-i\varepsilon)^{-1} - (A-\lambda+i\varepsilon)^{-1}\big] f\rangle \mathrm{d}\lambda\\ &= \int_{\mathbb{R}} m_f(\mathrm{d}\mu)\frac{\varepsilon}{\pi}\int_a^b \mathrm{d}\lambda \frac{1}{(\mu-\lambda)^2+\varepsilon^2}\\ &= \langle f, \Psi_\varepsilon(A) f\rangle,\end{aligned}$$

as expected.

iv) Let us finally deduce (4.16) from (4.15). As a consequence of the latter formula one has

$$\begin{aligned}&\frac{1}{2\pi i} s-\lim_{\delta\searrow 0}\int_{a+\delta}^{b+\delta} \big[(A-\lambda-i\varepsilon)^{-1} - (A-\lambda+i\varepsilon)^{-1}\big]\mathrm{d}\lambda\\ &= E\big((a+\delta, b+\delta)\big) + \frac{1}{2}E(\{a+\delta\}) + \frac{1}{2}E(\{b+\delta\})\\ &= E_{b+\delta} - E_{a+\delta} + \frac{1}{2}E(\{a+\delta\}) - \frac{1}{2}E(\{b+\delta\}).\end{aligned} \tag{4.18}$$

Observe also that if $\mu \in \mathbb{R}$ and $\delta > 0$, then by the right continuity of the spectral family one has

$$\|E(\{\mu+\delta\})f\|^2 \le \big\|E\big((\mu,\mu+\delta]\big)f\big\|^2 = \|(E_{\mu+\delta}-E_\mu)f\|^2 \ \to\ 0 \ \text{ as } \ \delta\to 0,$$

or in other words $s-\lim_{\delta\searrow 0} E(\{\mu+\delta\}) = \mathbf{0}$. Thus, by taking the right continuity of $\{E_\lambda\}$ again into account, one infers that the strong limit as $\delta \searrow 0$ of (4.18) is $E_b - E_a \equiv E\big((a,b]\big)$, which proves (4.16). □

Let us now comment on the relation between the previous proposition and the proof of the spectral theorem, as stated in Theorem 4.2.6. First of all, observe that the uniqueness of the spectral family $\{E_\lambda\}$ is a consequence of Stone's formula. Indeed, if $\{E_\lambda\}$ and $\{E'_\lambda\}$ are two spectral families satisfying $A = \int_{\mathbb{R}} \lambda E(\mathrm{d}\lambda) = \int_{\mathbb{R}} \lambda E'(\mathrm{d}\lambda)$, then it would follow from the equality (4.16) that $E\big((a,b]\big) = E'\big((a,b]\big)$ (the l.h.s. of (4.16) depends only A). Hence,

$$E_\lambda = s - \lim_{a\to-\infty} E\big((a,\lambda]\big) = s - \lim_{a\to-\infty} E'\big((a,\lambda]\big) = E'_\lambda.$$

For the existence of the spectral family, the proof is much longer, and can be found in several textbooks. However, let us just sketch it. The starting point for such a proof is (almost) always the r.h.s. of (4.16), and one has to prove the existence of its r.h.s. at least in the weak sense, and to show that the corresponding operators have the properties of a spectral measure. For the existence of the limit, one has to consider

$$\begin{aligned}&\frac{1}{2\pi i} \lim_{\delta\searrow 0} \lim_{\varepsilon\searrow 0} \Big(\int_{a+\delta}^{b+\delta} \langle f, (A-\lambda-i\varepsilon)^{-1} f\rangle \mathrm{d}\lambda - \int_{a+\delta}^{b+\delta} \langle f, (A-\lambda+i\varepsilon)^{-1} f\rangle \mathrm{d}\lambda \Big) \\ &= \lim_{\delta\searrow 0} \lim_{\varepsilon\searrow 0} \frac{1}{\pi} \int_{a+\delta}^{b+\delta} \Im\langle f, (A-\lambda-i\varepsilon)^{-1} f\rangle \mathrm{d}\lambda.\end{aligned}$$

Thus, let us set $\Phi(z) := \langle f, (A-z)^{-1} f\rangle$ and observe that this $\mathbb{C}$-valued function is holomorphic in the upper half complex plane. Observe also that this function has a non-negative imaginary part and satisfies the estimate $|\Phi(z)| \leq c/\Im(z)$ for some finite constant c (which means that $\Phi(z)$ is a Nevanlinna function). Then one can use a theorem on analytic functions saying that in such a case there exists a finite Stieltjes measure m on $\mathbb{R}$ satisfying $\Phi(z) = \int_{\mathbb{R}} (\lambda - z)^{-1} m(\mathrm{d}\lambda)$. In addition, this measure satisfies

$$m\big((a,b]\big) = \lim_{\delta\searrow 0} \lim_{\varepsilon\searrow 0} \frac{1}{\pi} \int_{a+\delta}^{b+\delta} \Im\Phi(\lambda+i\varepsilon)\mathrm{d}\lambda.$$

As a consequence, the application of this theorem provides us for any $f \in \mathcal{H}$ the measure $m \equiv m_f$. The rest of the proof consists in routine computations.

Let us add another comment in relation with the previous proposition. As already mentioned in Exercise 4.2.9, and more precisely in (4.10), for any $\lambda \in \sigma(A)$ one has

$$\|(A-\lambda \mp i\varepsilon)^{-1}\| = \frac{1}{|\varepsilon|}.$$

However, for some particular $f \in \mathcal{H}$ the expressions $(A-\lambda\mp i\varepsilon)^{-1} f$ could be convergent as $\varepsilon \searrow 0$. In particular, this is the case if the associated measure m_f is supported away from the value λ, or equivalently if there exists $\kappa > 0$ such that $E([\lambda-\kappa, \lambda+\kappa])f = 0$. Indeed, in this situation one has

$$\begin{aligned}\|(A-\lambda\mp i\varepsilon)^{-1} f\|^2 &= \int_{-\infty}^{\lambda-\kappa} |\mu-\lambda\mp i\varepsilon|^{-2} m_f(\mathrm{d}\mu) + \int_{\lambda+\kappa}^{\infty} |\mu-\lambda\mp i\varepsilon|^{-2} m_f(\mathrm{d}\mu) \\ &\leq \frac{1}{\kappa^2} \int_{\mathbb{R}} m_f(\mathrm{d}\mu) = \frac{1}{\kappa^2}\|f\|^2.\end{aligned}$$

Since this estimate holds for any $\varepsilon > 0$, and even for $\varepsilon = 0$, one easily obtains from the dominated convergence theorem that $\lim_{\varepsilon\searrow 0} \|(A-\lambda\mp i\varepsilon)^{-1}f - (A-\lambda)^{-1}f\| = 0$.

Let us also observe that if $\lambda \notin \sigma_p(A)$, then the previous argument holds for a dense set of elements of $\mathcal{H}$. Indeed, in such a case one has $s-\lim_{\kappa\searrow 0} E([\lambda-\kappa,\lambda+\kappa]) = \mathbf{0}$. However, if $E(\{\lambda\}) \neq \mathbf{0}$ then the above set of vectors can not be dense since it is orthogonal to $E(\{\lambda\})\mathcal{H}$.

Let us still assume that $\lambda \notin \sigma_p(A)$. In the previous paragraphs, it was shown that the set of vectors such that the limit $s-\lim_{\varepsilon\searrow 0}(A-\lambda\mp i\varepsilon)^{-1}f$ exists is dense in $\mathcal{H}$, but the choice of this dense set was depending on λ. A more interesting situation would be when this set can be chosen independently of λ, or at least for any λ in some interval (a,b). In the next statement, we show that if this situation takes place, then the spectrum of A in (a,b) is absolutely continuous. In order to get a better understanding of the subsequent result, let us recall that if $\phi:\mathbb{R}\to\mathbb{C}$ is a smooth function, then

$$\lim_{\varepsilon\searrow 0}\int_{\mathbb{R}}\frac{\phi(\mu)}{\mu-\lambda-i\varepsilon}\mathrm{d}\mu = i\pi\phi(\lambda) + \mathrm{Pv}\int_{\mathbb{R}}\frac{\phi(\mu)}{\mu-\lambda}\mathrm{d}\mu, \tag{4.19}$$

where Pv denotes the principal value integral. In particular, let us now choose $f \in \mathcal{H}_{ac}(A)$, which implies that there exists a non-negative measurable function $\theta_f:\mathbb{R}\to\mathbb{R}_+$ such that $m_f(V) = \int_V \theta_f(\mu)\mathrm{d}\mu$. If we assume in addition some regularity on θ_f, as for example $\theta_f \in C^1$, then one infers from (4.19) that

$$\langle f,(A-\lambda-i\varepsilon)^{-1}f\rangle = \int_{\mathbb{R}}\frac{\theta_f(\mu)}{\mu-\lambda-i\varepsilon}\mathrm{d}\mu \to i\pi\theta_f(\lambda) + \mathrm{Pv}\int_R\frac{\theta_f(\mu)}{\mu-\lambda}\mathrm{d}\mu \quad \text{as } \varepsilon\searrow 0.$$

The next statement clarifies the link between the existence of a limit for $(A-\lambda-i\varepsilon)^{-1}$ as $\varepsilon\searrow 0$ and the existence of absolutely continuous spectrum for A.

Proposition 4.4.2. *Let A be a self-adjoint operator and let $J := (\alpha,\beta)\subset\mathbb{R}$.*

(i) Let $f\in\mathcal{H}$ such that for each $\lambda\in J$ the expression $\Im\langle f,(A-\lambda-i\varepsilon)^{-1}f\rangle$ admits a limit as $\varepsilon\searrow 0$ and that this convergence holds uniformly in λ on any compact subset of J. Then $E(J)f\in\mathcal{H}_{ac}(A)$,

(ii) Assume that there exists a dense set $\mathcal{D}\subset\mathcal{H}$ such that the assumptions of (i) hold for any $f\in\mathcal{D}$, then $E(J)\mathcal{H}\subset\mathcal{H}_{ac}(A)$. In particular, it follows that

$$\sigma_p(A)\cap J = \emptyset = \sigma_{sc}(A)\cap J.$$

Proof. i) First of all, for z in the upper half complex plane let us set

$$\phi(z) := \Im\langle f,(A-z)^{-1}f\rangle.$$

By assumption $\phi(\lambda) := \lim_{\varepsilon\searrow 0}\phi(\lambda+i\varepsilon)$ exists for any $\lambda\in J$ and defines a bounded and uniformly continuous function on any compact subset of J.

Let us also consider $\lambda \in [a,b] \subset J$ and deduce from Stone's formula that

$$\big\langle f, E\big((a,\lambda]\big) f\big\rangle = \frac{1}{\pi} \lim_{\delta \searrow 0} \lim_{\varepsilon \searrow 0} \int_{a+\delta}^{\lambda+\delta} \Im \big\langle f, (A-\mu-i\varepsilon)^{-1} f\big\rangle \mathrm{d}\mu$$

where we can choose δ small enough such that $[a+\delta, \lambda+\delta] \subset J$. Then, by the observation made in the previous paragraph and by an application of the dominated convergence theorem one infers that

$$\big\langle f, E\big((a,\lambda]\big) f\big\rangle = \frac{1}{\pi} \lim_{\delta \searrow 0} \int_{a+\delta}^{\lambda+\delta} \phi(\mu)\mathrm{d}\mu = \frac{1}{\pi}\int_a^\lambda \phi(\mu)\mathrm{d}\mu.$$

Since ϕ is continuous on $[a,b]$ if follows that $\phi \in L^1([a,b])$. Finally, since $E\big((a,\lambda]\big)f = E\big((a,\lambda]\big)E(J)f$ one infers that the map $\lambda \mapsto \big\langle f, E\big((a,\lambda]\big)f\big\rangle$ defines an absolutely continuous measure on $(a,b]$.

ii) One has $E(J)f \in \mathcal{H}_{ac}(A)$ for each $f \in \mathcal{D}$. Since $\mathcal{D}$ is assumed to be dense in $\mathcal{H}$, it easily follows that $\{E(J)f \mid f \in \mathcal{D}\}$ is dense in $E(J)\mathcal{H}$, and hence $E(J)\mathcal{H} \subset \mathcal{H}_{ac}(A)$. □

The following result will be at the root of the commutator methods introduced later on.

Theorem 4.4.3 (Putnam's theorem)**.** *Let H and A be bounded self-adjoint operators satisfying $[iH, A] \geq CC^*$ for some $C \in \mathscr{B}(\mathcal{H})$. Then for all $\lambda \in \mathbb{R}$, any $\varepsilon > 0$ and each $f \in \mathcal{H}$ one has*

$$\Im\langle Cf, (H-\lambda-i\varepsilon)^{-1}Cf\rangle \leq 4\|A\|\,\|f\|^2, \tag{4.20}$$

and $\mathsf{Ran}(C) \subset \mathcal{H}_{ac}(H)$. In particular, if $\mathsf{Ker}(C^*) = \{0\}$, then the spectrum of H is purely absolutely continuous.

Proof. In this proof, we use the notation $R(z)$ for $(H-z)^{-1}$ when $z \in \rho(H)$.

i) For any $f \in \mathcal{H}$ one has

$$\begin{aligned}
\Im\langle Cf, R(\lambda+i\varepsilon)Cf\rangle &= \frac{1}{2i}\big\langle Cf, [R(\lambda+i\varepsilon) - R(\lambda-i\varepsilon)]Cf\big\rangle \\
&= \varepsilon\big\langle Cf, R(\lambda+i\varepsilon)R(\lambda-i\varepsilon)Cf\big\rangle \\
&= \varepsilon\|R(\lambda-i\varepsilon)Cf\|^2 \\
&\leq \varepsilon\|R(\lambda-i\varepsilon)C\|\,\|f\|^2.
\end{aligned} \tag{4.21}$$

Since for any bounded operator T one has $\|T\|^2 = \|TT^*\|$ one infers that

$$\begin{aligned}
\|R(\lambda-i\varepsilon)C\|^2 &= \|R(\lambda-i\varepsilon)CC^*R(\lambda+i\varepsilon)\| \\
&= \sup_{f\in\mathcal{H}, \|f\|=1} \big\langle R(\lambda+i\varepsilon)f, CC^*R(\lambda+i\varepsilon)f\big\rangle \\
&\leq \sup_{f\in\mathcal{H}, \|f\|=1} \big\langle R(\lambda+i\varepsilon)f, [iH,A]R(\lambda+i\varepsilon)f\big\rangle \\
&= \big\|R(\lambda-i\varepsilon)[iH,A]R(\lambda+i\varepsilon)\big\| \\
&= \big\|R(\lambda-i\varepsilon)[i(H-\lambda+i\varepsilon),A]R(\lambda+i\varepsilon)\big\| \\
&\leq \|AR(\lambda+i\varepsilon)\| + \|R(\lambda-i\varepsilon)A\| + \|2i\varepsilon R(\lambda-i\varepsilon)AR(\lambda+i\varepsilon)\|.
\end{aligned}$$

Finally, since $\|R(\lambda \pm i\varepsilon)\| \leq \varepsilon^{-1}$ one gets $\|R(\lambda - i\varepsilon)C\|^2 \leq 4\varepsilon^{-1}\|A\|$. By inserting this estimate into (4.21) one directly deduces the inequality (4.20).

ii) Let $\{E_\lambda\}$ be the spectral family of H. For any $J = (a,b]$ one has by Stone's formula

$$\big\langle Cf, E\big((a,b]\big)Cf\big\rangle = \frac{1}{\pi} \lim_{\delta\searrow 0} \lim_{\varepsilon\searrow 0} \int_{a+\delta}^{b+\delta} \Im\big\langle Cf, (H-\lambda-i\varepsilon)^{-1}Cf\big\rangle \mathrm{d}\lambda.$$

Now since one has

$$\int_{a+\delta}^{b+\delta} \Im\big\langle Cf, (H-\lambda-i\varepsilon)^{-1}Cf\big\rangle \mathrm{d}\lambda \leq 4\|A\|\,\|f\|^2\,(b-a)$$

one infers that

$$\langle Cf, E(J)Cf\rangle \equiv m_{Cf}(J) \leq 4\|A\|\,\|f\|^2\,|J|,$$

where $|J|$ means the Lebesgue measure of J. Such an inequality implies that $m_{Cf}(V) \leq 4\|A\|\ \|f\|^2\ |V|$ for any Borel set V of $\mathbb{R}$, and consequently that the measure m_{Cf} is absolutely continuous with respect to the Lebesgue measure. It thus follows that $Cf \in \mathcal{H}_{ac}(H)$. Finally, if $\mathsf{Ker}(C^*) = \{0\}$, then $\mathsf{Ran}(C)$ is dense in $\mathcal{H}$, as proved in (ii) of Lemma 2.1.10. As a consequence, one obtains that $\mathcal{H}_{ac}(H) = \mathcal{H}$. $\square$