「Hilbert space methods for quantum mechanics

Report」

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We define the Sobolev space $H^s(\mathbb{R}^n)$ not only in the case of $s \in \mathbb{N}$ but also for $s \in \mathbb{R}$.

Def Let $s \in \mathbb{R}$.

$f \in H^s(\mathbb{R}^n)$

$\overset{\text{def}}{\Longleftrightarrow}$ f satisfies that

(i) $f \in \mathcal{S}'(\mathbb{R}^n)$

(ii) $\exists \tilde{f} \in L^1_{loc}(\mathbb{R}^n)$ s.t. $\hat{f} = T_{\tilde{f}}$ in $\mathcal{S}'(\mathbb{R}^n)$

(iii) $\int_{\mathbb{R}^n} (1+|\xi|)^{2s} |\tilde{f}(\xi)|^2 d\xi < \infty$

And for $f \in H^s(\mathbb{R}^n)$,

$$\|f\|_{H^s} \overset{\text{def}}{:=} \left(\int_{\mathbb{R}^n} (1+|\xi|)^{2s} |\tilde{f}(\xi)|^2 d\xi \right)^{\frac{1}{2}}$$

It is known that $\|\cdot\|_{H^s}$ satisfies the condition of a norm and that $H^s(\mathbb{R}^n)$ is complete with respect to $\|\cdot\|_{H^s}$.

Consider $s = m \in \mathbb{N}$. For $f \in H^m(\mathbb{R}^n)$,

$$\|f\|'_{H^m} \overset{\text{def}}{:=} \left(\sum_{|\alpha| \leq m} \int_{\mathbb{R}^n} |\partial_x^\alpha f(x)|^2 dx \right)^{\frac{1}{2}}$$

It is also known that $\|\cdot\|'_{H^m}$ satisfies the condition of a norm and that $\|\cdot\|'_{H^m}$ is equivalent to $\|\cdot\|_{H^m}$.

I'm writing this report in order to prove the next important theorem.

Here, we denote by $C_0^\infty(\mathbb{R}^n)$ the space of infinitely differentiable functions in $\mathbb{R}^n$ with compact supports.

In the following, let $s \in \mathbb{R}$.

Th (Density of $C_0^\infty(\mathbb{R}^n)$ in $H^s(\mathbb{R}^n)$)

$C_0^\infty(\mathbb{R}^n)$ is dense in $H^s(\mathbb{R}^n)$

In the first part, we prepare what we need in order to prove the theorem

In the second part, we prove the theorem.

Preparation

(References : Lectures on linear partial differential equations / Gregory Eskin / Graduate studies in mathematics ; v.123)

Def (§3.3 Change of variables for distributions)

Let $S : \mathbb{R}^n \to \mathbb{R}^n$: diffeomorphism.

$$\begin{array}{ccc} \mathbb{R}^n & \longrightarrow & \mathbb{R}^n \\ \Psi & & \Psi \\ x & \longmapsto & S(x) = (S_1(x), \ldots, S_n(x)) \end{array}$$

$$J_{S^{-1}}(x) := \det\left[\frac{\partial S_i^{-1}(x)}{\partial x_j}\right]_{i,j=1}^{n} \quad \text{for} \quad x \in \mathbb{R}^n.$$

(Jacobian matrix of S^{-1})

Let $T \in \mathcal{D}'(\mathbb{R}^n)$.

$$\langle T \circ S, \varphi \rangle \overset{\text{def}}{:=} \langle T, (\varphi \circ S^{-1}) \cdot |J_{S^{-1}}| \rangle \quad \text{for } \forall \varphi \in \mathcal{D}(\mathbb{R}^n)$$

Lem $T \circ S \in \mathcal{D}'(\mathbb{R}^n)$.

Def 7.1 For $f \in \mathcal{D}'(\mathbb{R}^n)$, $\varphi \in C_0^\infty(\mathbb{R}^n)$

$$(f * \varphi)(x) \overset{\text{def}}{:=} \langle f, \varphi(x - \bullet) \rangle \quad \text{for} \quad x \in \mathbb{R}^n$$

Lem $f * \varphi \in C^\infty(\mathbb{R}^n)$.

Def $f \in \mathcal{E}'(\mathbb{R}^n)$

$\overset{\text{def}}{\iff}$ $f \in \mathcal{D}'(\mathbb{R}^n)$ with $\operatorname{supp} f \subset \mathbb{R}^n$: compact

Def 7.2 For $f \in \mathcal{D}'(\mathbb{R}^n)$, $g \in \mathcal{E}'(\mathbb{R}^n)$

$$\langle f * g, \varphi \rangle := \langle f, (g \circ \chi) * \varphi \rangle \quad \text{for } {}^\forall \varphi \in \mathcal{D}(\mathbb{R}^n)$$

(Set $\chi(x) := -x$ for $x \in \mathbb{R}^n$)

Lem
- $(g \circ \chi) * \varphi \in \mathcal{D}(\mathbb{R}^n)$
- $f * g \in \mathcal{D}'(\mathbb{R}^n)$

Observation

$$\begin{aligned}(g \circ \chi) * \varphi(x) &= \langle g \circ \chi, \varphi(x - \bullet) \rangle \\ &= \langle g, (\varphi(x - \bullet) \circ \chi^{-1}) \cdot \underbrace{|J_{\chi^{-1}}|}_{=1} \rangle \\ &= \langle g, \varphi(x + \bullet) \rangle \quad \text{for } x \in \mathbb{R}^n\end{aligned}$$

Prop 7.2 For $f \in \mathcal{D}'(\mathbb{R}^n)$, $\varphi, \psi \in \mathcal{D}(\mathbb{R}^n)$,

$$\int_{\mathbb{R}^n} (f * \varphi)(x) \psi(x)\, dx = \left\langle f, \int_{\mathbb{R}^n} \varphi(t - \bullet) \psi(t)\, dt \right\rangle$$

Prop 12.1 $f \in \mathcal{S}'(\mathbb{R}^n)$, $g \in \mathcal{E}'(\mathbb{R}^n)$

$\Rightarrow$ (i) $\exists\, h \in C^\infty(\mathbb{R}^n)$ s.t. $\hat{g} = T_h$ in $\mathcal{S}'(\mathbb{R}^n)$

(Prop 11.1)

(ii) $f * g \in \mathcal{S}'(\mathbb{R}^n)$

(iii) $\widehat{f * g} = h \cdot \hat{f}$ in $\mathcal{S}'(\mathbb{R}^n)$

Using these lemmas and propositions without proofs,

we prove the main theorem (Density of C_0^∞ in H^s).

Proof of Th Let $f \in H^s(\mathbb{R}^n)$.

Step1 Make $f_\delta \in H^s(\mathbb{R}^n)$ with $\delta > 0$.

Take $\beta \in C_0^\infty(\mathbb{R}^n)$ with

$$\begin{cases} \circ\ \beta(x) \geqq 0 & \text{for } x \in \mathbb{R}^n \\ \circ\ \beta(x) = \begin{cases} 1 & |x| < \frac{1}{2} \\ 0 & |x| > 1 \end{cases} \\ \circ\ \int_{\mathbb{R}^n} \beta(x)\, dx = 1 \end{cases}$$

For $\delta > 0$,

set $\beta_\delta(x) := \dfrac{1}{\delta^n} \beta\left(\dfrac{x}{\delta}\right)$ for $x \in \mathbb{R}^n$

$$f_\delta := f * T_{\beta_\delta} \quad \text{in } \mathcal{S}'(\mathbb{R}^n)$$

Since $f \in \mathcal{S}'(\mathbb{R}^n)$ and $T_{\beta_\delta} \in \mathcal{E}'(\mathbb{R}^n)$, we can define the convolution, and $f * T_{\beta_\delta} \in \mathcal{S}'(\mathbb{R}^n)$ (by Prop 12.1)

Since $f \in H^s(\mathbb{R}^n)$, $\exists\, \hat{f} \in L^1_{loc}(\mathbb{R}^n)$ s.t.

$\circ$ $\widehat{f} = T_{\hat{f}}$ in $\mathcal{S}'(\mathbb{R}^n)$

$\circ$ $\displaystyle\int_{\mathbb{R}^n} (1+|\xi|)^{2s} |\hat{f}(\xi)|^2\, d\xi < \infty$

By Prop12.1, we can get that

$$\widehat{f_\delta} = \widehat{\beta_\delta} \cdot T_{\hat{f}} = T_{\widehat{\beta_\delta} \cdot \hat{f}} \quad \text{in } \mathcal{S}'(\mathbb{R}^n)$$

And $\int_{\mathbb{R}^n} (1+|\xi|)^{2s} |\widehat{\beta_\delta}(\xi) \hat{f}(\xi)|^2 d\xi \leq {}^{\exists}M_\delta \int_{\mathbb{R}^n} (1+|\xi|)^{2s} |\hat{f}(\xi)|^2 d\xi < \infty$

Therefore $f_\delta \in H^s(\mathbb{R}^n)$

and $f_\delta = T_{f * \beta_\delta}$ in $\mathcal{S}'(\mathbb{R}^n)$ for $\delta > 0$.

$\because$ For $\phi \in \mathcal{S}(\mathbb{R}^n)$, $\delta > 0$,

$$\langle f_\delta, \phi \rangle = \langle f * T_{\beta_\delta}, \phi \rangle$$

$$= \langle f \ \langle T_{\beta_\delta}(y)\ \phi(\bullet + y) \rangle \rangle$$

$$= \langle f, \int_{\mathbb{R}^n} \beta_\delta(y) \phi(\bullet + y)\, dy \rangle$$

(Transform variables as $\bullet + y = t$.)

$$= \langle f, \int_{\mathbb{R}^n} \beta_\delta(t - \bullet) \phi(t)\, dt \rangle$$

By Prop9.2 $\longrightarrow$

$$= \int_{\mathbb{R}^n} (f * \beta_\delta)(x)\, \phi(x)\, dx$$

$$= \langle T_{f * \beta_\delta}, \phi \rangle$$

$\therefore f_\delta = T_{f * \beta_\delta}$ in $\mathcal{S}'(\mathbb{R}^n)$ #

Step2 Show that $f_\delta \to f$ $(\delta \to +0)$ in $H^s(\mathbb{R}^n)$

Note that $\widehat{\beta_\delta}(\xi) = \int_{\mathbb{R}^n} e^{-ix\cdot\xi} \beta_\delta(x)\,dx$

$$= \int_{\mathbb{R}^n} e^{-ix\cdot\xi} \beta\left(\frac{x}{\delta}\right) \frac{1}{\delta^n}\,dx \quad \left(\begin{array}{c}\text{Transform variable}\\ \text{as } x=\delta y\end{array}\right)$$

$$= \int_{\mathbb{R}^n} e^{-i(\delta y)\cdot\xi} \beta(y) \frac{1}{\delta^n} \delta^n\,dy$$

$$= \int_{\mathbb{R}^n} e^{-iy\cdot(\delta\xi)} \beta(y)\,dy$$

$$= \hat{\beta}(\delta\xi) \qquad \text{for } \xi\in\mathbb{R}^n,\ \delta>0 .$$

Consider

$$\| f - f_\delta \|_{H^s}^2 = \int_{\mathbb{R}^n} |\hat{f}(\xi) - \widehat{\beta_\delta}(\xi)\hat{f}(\xi)|^2 (1+|\xi|)^{2s}\,d\xi$$

$$= \int_{\mathbb{R}^n} |\hat{f}(\xi)|^2 |1-\hat{\beta}(\delta\xi)|^2 (1+|\xi|)^{2s}\,d\xi$$

Since $\hat{\beta}$ is continuous and $\hat{\beta}(0) = \int_{\mathbb{R}^n} \beta(y)\,dy = 1$,

$$|1-\hat{\beta}(\delta\xi)| \to |1-\hat{\beta}(0)| = 0 \quad (\delta\to+0) \quad \text{for any fixed } \xi\in\mathbb{R}^n.$$

and $$|1-\hat{\beta}(\delta\xi)| \le 1 + |\hat{\beta}(\delta\xi)| \le 1 + \int_{\mathbb{R}^n} |e^{-iy\cdot(\delta\xi)}\beta(y)|\,dy$$

$$= 1 + \int_{\mathbb{R}^n} \beta(y)\,dy = 2 ,$$

$$|\hat{f}(\xi)|^2 |1-\hat{\beta}(\delta\xi)|^2 (1+|\xi|)^{2s} \leq 2\, |\hat{f}(\xi)|^2 (1+|\xi|)^{2s} .$$

Using Lebesgue's Theorem, it follows that

$$\|f - f_\delta\|_{H^s}^2 \longrightarrow 0 \quad (\delta \to +0)$$

Therefore $f_\delta \longrightarrow f$ $(\delta \to +0)$ in $H^s(\mathbb{R}^n)$.

Next, $\forall \varepsilon > 0$: fix.

$\exists\, \delta_\varepsilon \in (0,1)$ s.t. $\|f - f_{\delta_\varepsilon}\|_{H^s} < \varepsilon$.

Set $g_\eta(x) := \beta(\eta x) \cdot (f * \beta_{\delta_\varepsilon})(x)$ for $x \in \mathbb{R}^n$, $\eta > 0$.

Since $\beta \in C_0^\infty(\mathbb{R}^n)$, $g_\eta \in C_0^\infty(\mathbb{R}^n)$ when $\eta > 0$.

Next, we show that $g_\eta \in C_0^\infty$ is the approximation for $f \in H^s(\mathbb{R}^n)$.

Step3 Show that $T_{g_\eta} \to f_{\delta_\varepsilon} \quad (\eta \to +0) \quad$ in $H^s(\mathbb{R}^n)$

Take $m \in \mathbb{N}$ with $s \leq m$.

At first, we prove that $f_{\delta_\varepsilon} \in H^m(\mathbb{R}^n)$

☺ Since $\hat{\beta} \in \mathcal{S}(\mathbb{R}^n)$, For $^\forall N > 0$, $^\exists C_N > 0$ s.t.

$$|\widehat{\beta_{\delta_\varepsilon}}(\xi)| = |\hat{\beta}(\delta_\varepsilon \xi)| \leq C_N (1 + |\delta_\varepsilon \xi|)^{-N}$$

(Since $\delta_\varepsilon \in (0,1)$) $\to$

$$\leq C_N (\delta_\varepsilon + \delta_\varepsilon |\xi|)^{-N}$$

$$= C_N \delta_\varepsilon^{-N} (1 + |\xi|)^{-N} \quad \text{for } \xi \in \mathbb{R}^n$$

Consider

$$\int_{\mathbb{R}^n} (1+|\xi|)^{2m} |\widehat{\beta_{\delta_\varepsilon}}(\xi) \hat{f}(\xi)|^2 d\xi$$

$$= \int_{\mathbb{R}^n} |\hat{\beta}(\delta_\varepsilon \xi)|^2 (1+|\xi|)^{2m-2s} \cdot (1+|\xi|)^{2s} |\hat{f}(\xi)|^2 d\xi$$

$$\leq \int_{\mathbb{R}^n} C_{2m-2s}^2 \cdot \delta_\varepsilon^{-2m+2s} (1+|\xi|)^{-2m+2s} (1+|\xi|)^{2m-2s} \cdot (1+|\xi|)^{2s} |\hat{f}(\xi)|^2 d\xi$$

$$= C_{2m-2s}^2 \cdot \delta_\varepsilon^{-2m+2s} \cdot \int_{\mathbb{R}^n} (1+|\xi|)^{2s} |\hat{f}(\xi)|^2 d\xi < \infty$$

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Moreover $T_{g_\eta} \in H^m(\mathbb{R}^n)$.

Consider

$(1+|\xi|)^{2s} \le (1+|\xi|)^{2m}$

$$\| f_{\delta_\varepsilon} - T_{g_\eta} \|_{H^s}^2 \le \| f_{\delta_\varepsilon} - T_{g_\eta} \|_{H^m}^2$$

$$= \| T_{f*\beta_{\delta_\varepsilon}} - T_{\beta(\eta\bullet)\cdot(f*\beta_{\delta_\varepsilon})(\bullet)} \|_{H^m}^2$$

$$= \| T_{(1-\beta(\eta\bullet))\cdot f*\beta_{\delta_\varepsilon}} \|_{H^m}^2$$

$$\le {}^{\exists}C \left(\| T_{(1-\beta(\eta\bullet))\cdot f*\beta_{\delta_\varepsilon}} \|'_{H^m} \right)^2$$

[$\|\bullet\|_{H^m}$ is equivalent to $\|\bullet\|'_{H^m}$ (No1) when $m \in \mathbb{N}$]

(since $(1-\beta(\eta\bullet))\cdot f*\beta_{\delta_\varepsilon} \in C^\infty$)

$$= C \sum_{|\alpha|\le m} \int_{\mathbb{R}^n} \left| \partial_x^\alpha (1-\beta(\eta x))(f*\beta_{\delta_\varepsilon})(x) \right|^2 dx$$

(By the definition of β,

$|\eta x| < \frac{1}{2} \Rightarrow \beta(\eta x) = 1$)

$\Updownarrow$

$|x| < \frac{1}{2\eta}$

$$= C \sum_{|\alpha|\le m} \int_{|x|>\frac{1}{2\eta}} \left| \partial_x^\alpha (1-\beta(\eta x))(f*\beta_{\delta_\varepsilon})(x) \right|^2 dx$$

Using Leibniz rule, when $|\alpha| \leq m$,

def
For $\alpha = (\alpha_1, \dots, \alpha_n)$, $\gamma = (\gamma_1, \dots, \gamma_n) \in$

$$\binom{\alpha}{\gamma} = \binom{\alpha_1}{\gamma_1} \cdots \binom{\alpha_n}{\gamma_n}$$

$$\left| \partial_x^\alpha (1-\beta(\eta x)) (f * \beta_{\delta_\varepsilon})(x) \right|$$

$$= \left| \sum_{\gamma \leq \alpha} \binom{\alpha}{\gamma} \left[\partial_x^{\alpha-\gamma} (1-\beta(\eta x)) \right] \left[\partial_x^\gamma (f * \beta_{\delta_\varepsilon})(x) \right] \right|$$

$$\leq {}^\exists C_m \cdot \left(\sum_{|\alpha| \leq m} \left| \partial_x^\alpha (1-\beta(\eta x)) \right| \right) \times \left(\sum_{|\alpha| \leq m} \left| \partial_x^\alpha (f * \beta_{\delta_\varepsilon})(x) \right| \right)$$

$$\left(\text{When } |\alpha| \leq m, \quad \left| \partial_x^\alpha \beta(\eta x) \right| = \left| \eta^{|\alpha|} (\partial_x^\alpha \beta)(\eta x) \right| \leq \eta^{|\alpha|} \cdot \sup_{\eta x \in \mathbb{R}^n} \left| (\partial^\alpha \beta)(\eta x) \right| = \eta^{|\alpha|} \cdot \underbrace{\sup_{y \in \mathbb{R}^n} \left| (\partial^\alpha \beta)(y) \right|}_{{}^\exists C_\alpha} \right)$$

(Since $\beta \in \mathcal{S}(\mathbb{R}^n)$)

$$\leq C_m \cdot \left(\sum_{|\alpha| \leq m} {}^\exists C_\alpha \cdot \eta^{|\alpha|} \right) \cdot \left(\sum_{|\alpha| \leq m} \left| \partial_x^\alpha (f * \beta_{\delta_\varepsilon})(x) \right| \right)$$

$$\| f_{\delta_\varepsilon} - T_{g_\eta} \|_{H^s}^2 \leq C \sum_{|\alpha| \leq m} \int_{|x| > \frac{1}{2\eta}} C_m^2 \left(\sum_{|\alpha| \leq m} C_\alpha \cdot \eta^{|\alpha|} \right)^2 \left(\sum_{|\alpha| \leq m} \left| \partial_x^\alpha (f * \beta_{\delta_\varepsilon})(x) \right| \right)^2 dx$$

$$\leq {}^\exists C_m' \left(\sum_{|\alpha| \leq m} C_\alpha \cdot \eta^{|\alpha|} \right)^2 \int_{|x| > \frac{1}{2\eta}} \left(\sum_{|\alpha| \leq m} \left| \partial_x^\alpha (f * \beta_{\delta_\varepsilon})(x) \right| \right)^2 dx$$

with C_m and C_m' are independent of η.

Since $f_{\delta_\varepsilon} = T_{f * \beta_{\delta_\varepsilon}} \in H^m(\mathbb{R}^n)$,

$$\partial_x^\alpha f_\delta = \partial_x^\alpha T_{f * \beta_{\delta_\varepsilon}} = T_{\partial_x^\alpha (f * \beta_{\delta_\varepsilon})} \in L^2(\mathbb{R}^n) \quad \text{when } |\alpha| \leq m.$$

$(f * \beta_{\delta_\varepsilon} \in C^\infty)$

i.e. $\partial_x^\alpha (f * \beta_{\delta_\varepsilon}) \in L^2(\mathbb{R}^n)$ when $|\alpha| \leq m$

So, $\int_{\mathbb{R}^n} \left| \left(\sum_{|\alpha| \leq m} |\partial_x^\alpha (f * \beta_{\delta_\varepsilon})(x)| \right)^2 \right| dx < \infty$.

When $\eta \to +0$, $\left(\frac{1}{2\eta} \to +\infty\right)$

$$\int_{|x| > \frac{1}{2\eta}} \left(\sum_{|\alpha| \leq m} |\partial_x^\alpha (f * \beta_{\delta_\varepsilon})(x)| \right)^2 dx \longrightarrow 0 .$$

Therefore $\| f_{\delta_\varepsilon} - T g_\eta \|_{H^s} \to 0 \quad (\eta \to +0)$.

$\exists \eta_\varepsilon > 0$ s.t. $\| f_{\delta_\varepsilon} - T g_{\eta_\varepsilon} \|_{H^s} < \varepsilon$

Finally, we get that

$$\| f - T g_{\eta_\varepsilon} \|_{H^s} \leq \| f - f_{\delta_\varepsilon} \|_{H^s} + \| f_{\delta_\varepsilon} - T g_{\eta_\varepsilon} \|_{H^s} < 2\varepsilon$$

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(If we use this theorem (Density of C_0^∞ in H^s), we can prove "Sobolev's embedding theorem", and we can consider "Restriction to hyperplanes". So this theorem is very important to check some properties of the Sobolev space.)