Reflexive space and relation between norm and inner product
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Theorem 1 *Let X be a Banach Space. Then X is reflexive if and only if X^* is reflect.*

(proof) ($\Rightarrow$) Let X be reflexive. There is a natural bijection J which is defined as follow,

$$J : x \in X \mapsto J(x) \in X^{**}$$

where we define $J(x)$ as $J(x)(f) = f(x), \forall f \in X^*$.Thus, since X is reflect, J gives a one-to-one correspondence between X and X^{**}. So, for any $x^{**} \in X^{**}$ there exists a unique $x \in X$ such that $x^{**} = J(x)$ and we can define $K : X^* \to X^{***}$ as follow,

$$K : f \in X^* \mapsto K(f) \in X^{***}$$
$$K(f)(x^{**}) = f(x) \ \ (x \in X \ \ satisfies \ \ x^{**} = J(x)).$$

K is a natural injection. Thus, for any $h \in X^{***}$, set

$$f(x) = h(J(x))$$

then $f \in X^*$ and $h = K(f)$.

Hence, K is natural bijection and $X^* \cong X^{***}$. X^* is reflexive as desired.

($\Leftarrow$) Let X^* be reflexive and assume $X \ncong X^{**}$. One sets

$$||x||_X = ||J(x)||_{X^{**}}$$
$$(\because ||x||_X = \sup_{||f|| \leqq 1} |f(x)| = ||x||_{X^{**}}).$$

Since X is a banach space and from $||x||_X = ||J(x)||_{X^{**}}$,we have $X^{**} \supsetneqq JX (\cong X)$ is closed subspace.

Then, from Hahn-Banach Theorem there exists $h \in X^{***}$ such that $h(x^{**}) = 0 (\forall x^{**} \in JX)$, $||h||_{X^{***}} > 0$. Since $X^* \cong X^{***}$, there exists a unique $f \in X^*$ such that $h = K(f)$.

Then, for any $x \in X$,

$$0 = h(f(x)) = K(f)(J(x)) = f(x)$$
$$\therefore \ \ f = 0.$$

But $h \neq 0$ by assumption, and we get the contradiction,

$$0 \neq h = K(f) = K(0) = 0.$$

Therefore we have proved that X is reflexive if and only if X^* is reflexive.

Proposition 1 *Let H be a Hilbert space. Then H is reflexive.*

(proof) Let us consider J such that

$$J : u \in H \mapsto J_u \in H^{**} \ with \ \ J_u(f) = f(u), \forall f \in H^*.$$

Thus, from Riesz Theorem, for any $f \in H^*$ there exists a unique $v \in H$ such that

$$f(u) = \langle u,v \rangle_H \ (\forall u \in H)$$
$$||f||_{H^*} = ||v||_H$$

Hence, there exists unique $v, w \in H$ corresponding to $f, g \in H^*$, and we define

$$\langle f, g\rangle_{H^*} = \langle f_v, g_w\rangle_{H^*} = \langle w, v\rangle_H$$

then H^* is a Hilbert space and we have $H \cong H^*$ because $J_1 : u \in H \mapsto f_u \in H^*$ is natural bijection. By the same argument we obtain $H^* \cong H^{**}$. So, for any $h \in H^{**}$ there exists $g \in H^*$ such that

$$h(f) = \langle f, g\rangle_{H^*}(\forall f \in H^*)$$

Then, there exist $v, w \in H$ corresponding to f, g and $\langle f, g\rangle_{H^*} = \langle w, v\rangle_{H^*} = f(v)(\because$Riesz Theorem). So, we have $h = J(v)$ and $J : H \mapsto H^{**}$ is proved to be a natural bijection, hence H is reflexive.

Proposition 2 *Let X be a norm space and A be a subset in X. Then $A^* \cong (\bar{A})^*$*

(proof) We choose any $f \in A^*$ and $x \in \bar{A}$. Then,there exists $x_n \subseteqq A$ such that $\lim_{n\to\infty} ||x - x_n||_X = 0$. Since $f \in A^*$, for $n, m \in N$,

$$\lim_{m,n\to\infty} |f(x_m - x_n)| \leqq \lim_{m,n\to\infty} ||f||_A^* ||x_m - x_n||_X = 0$$

Hence $\{f(x_n)\}$ is Cauchy sequence. Since $\mathbb{C}$ is a banach space, there exists α_∞ in $\mathbb{C}$ such that $\lim_{n\to\infty} |fx_n - \alpha_\infty| = 0$. Then we define $\bar{f}(x)$ to be equal to α_∞,and $\bar{f}$ is a liner operator and

$$|\bar{f}(x)| = \lim_n |f(x_n)| \leqq \lim_n ||f||_{A^*} ||x_n||_X = ||f||_{A^*} ||x||_X$$
$$\therefore ||\bar{f}||_{\bar{A}^*} \leqq ||f||_{A^*}$$

and

$$||f||_{A^*} = \sup_{||x||=1, x\in A} |f(x)| \leqq \sup_{||x||=1, x\in \bar{A}} |\bar{f}(x)| = ||\bar{f}||_{\bar{A}^*}$$

then,

$$||\bar{f}||_{\bar{A}^*} = ||f||_A \tag{1}$$

Therefore, for any $f \in A^*, \bar{f}$ belongs to $(\bar{A})^*$, so we can define a natural operator $T : A^* \to (\bar{A})^*$ as $Tf = \bar{f}$. Thus, we can find $g|_A \in A^*$ for any g $\in (\bar{A})^*$. Hence, from (1), we have proved T is natural isometry bijection, then $A^* \cong (\bar{A})^*$.

Von Neumann Theorem 1 *Let $(X, ||\ ||)$ be a normed space on $\mathbb{C}$ which satisfies*

$$||x + y||^2 + ||x - y||^2 = 2(||x||^2 + ||y||^2),\ \forall x, y \in X \tag{2}$$

Then,

$$(x, y) = \frac{1}{2}(||x + y||^2 - ||x||^2 - ||y||^2) + \frac{i}{2}(||x + iy||^2 - ||x||^2 - ||y||^2) \tag{3}$$

defines an inner product on X.

(proof) (The proof is written based on Reference(2))
We shall prove following formula

$$(x, y) = \frac{1}{4}(||x + y||^2 - ||x - y||^2) + \frac{i}{4}(||x + iy||^2 - ||x - iy||^2) \tag{4}$$

In fact, from formula (2),

$$-||x - y||^2 = ||x + y||^2 - 2(||x||^2 + ||y||^2)$$

replacing y with iy implies that,

$$-||x-iy||^2 = ||x+iy||^2 - 2(||x||^2 + ||y||^2).$$

Substituting these two formulas in $\frac{1}{4}(||x+y||^2 - ||x-y||^2) + \frac{i}{4}(||x+iy||^2 - ||x-iy||^2)$, we find that formula (3) and (4) are equivalent.
Then, from formula (4) and the norm property, the follwing formula holds.

$$\begin{aligned}(x,0) &= \frac{1}{4}(||x||^2 - ||x||^2) + \frac{i}{4}(||x||^2 - ||x||^2) = 0\\ (0,y) &= \frac{1}{4}(||y||^2 - ||-y||^2) + \frac{i}{4}(||iy||^2 - ||-iy||^2) = 0\end{aligned}$$

and,

$$\begin{aligned}(x,x) &= \frac{1}{4}(||2x||^2 - ||x-x||^2) + \frac{i}{4}(||x+ix||^2 - ||x-ix||^2)\\ &= \frac{1}{4}\cdot 4\cdot ||x||^2 + \frac{i}{4}\cdot 4\cdot(||x||^2 - ||x||^2) \quad (\because |1+i| = |1-i| = 2)\\ &= ||x||^2.\end{aligned}$$

Hence, (x,x) is a positive real number or 0 from the norm property and $(x,x) = 0$ if and only if $x = 0$. Then, we shall prove $(y,x) = \overline{(x,y)}$. The formula $||y+ix|| = ||i(x-iy)|| = ||x-iy||$ holds. Replacing y with iy in formula (2) implies that $||x-iy||^2 = 2(||x||^2 + ||y||^2) - ||x+iy||^2$ holds. Change x with y in formula (3), then

$$\begin{aligned}(y,x) &= \frac{1}{2}(||x+y||^2 - ||x||^2 - ||y||^2) + \frac{i}{2}(||x-iy||^2 - ||x||^2 - ||y||^2)\\ &= \frac{1}{2}(||x+y||^2 - ||x||^2 - ||y||^2) + \frac{i}{2}(||x||^2 + ||y||^2 - ||x+iy||^2)\\ &= \frac{1}{2}(||x+y||^2 - ||x||^2 - ||y||^2) - \frac{i}{2}(||x+iy||^2 - ||x||^2 - ||y||^2)\\ &= \overline{(x,y)}.\end{aligned}$$

Next we shall prove following formula

$$(x+y,z) = (x,z) + (y,z)$$

Replace x with $x+z$ and y with $y+z$ in formula (2), then,

$$\begin{aligned}||(x+z)+(y+z)||^2 + ||(x+z)-(y+z)||^2 &= 2(||x+z||^2 + ||y+z||^2)\\ \therefore \quad ||x+y+2z||^2 + ||x-y||^2 &= 2(||x+z||^2 + ||y+z||^2)\end{aligned}$$

and, replacing x with $x-z$ and y with $y-z$ implies that,

$$\begin{aligned}||(x-z)+(y-z)||^2 + ||(x-z)-(y-z)||^2 &= 2(||x-z||^2 + ||y-z||^2)\\ \therefore \quad ||x+y-2z||^2 + ||x-y||^2 &= 2(||x-z||^2 + ||y-z||^2)\end{aligned}$$

using these formulas show that,

$$\begin{aligned}&2||\frac{x+y}{2} + z||^2 - 2||\frac{x+y}{2} - z||^2\\ &= \frac{1}{2}||x+y+2z||^2 - \frac{1}{2}||x+y-2z||^2\\ &= \left(||x+y||^2 + ||y+z||^2 - \frac{1}{2}||x-y||^2\right) - \left(||x-z||^2 + ||y-z||^2 - \frac{1}{2}||x-y||^2\right)\\ &= ||x+z||^2 - ||x-z||^2 + ||y+z||^2 - ||y-z||^2\end{aligned}$$

By replacing z with iz we find that

$$2||\frac{x+y}{2}+iz||^2-2||\frac{x+y}{2}-iz||^2=||x+iz||^2-||x-iz||^2+||y+iz||^2-||y-iz||^2$$

Now,from formula (4), one sets

$$\begin{aligned}(x,z)+(y,z)&=\frac{1}{4}(||x+z||^2-||x-z||^2)+\frac{i}{4}(||x+iz||^2-||x-iz||^2)\\&\quad+\frac{1}{4}(||y+z||^2-||y-z||^2)+\frac{i}{4}(||y+iz||^2-||y-iz||^2)\\&=\frac{1}{4}(||x+z||^2-||x-z||^2+||y+z||^2-||y-z||^2)\\&\quad+\frac{i}{4}(||x+iz||^2-||x-iz||^2+||y+iz||^2-||y-iz||^2)\\&=\frac{1}{2}\left(||\frac{x+y}{2}+z||^2-||\frac{x+y}{2}-z||^2\right)\\&\quad+\frac{i}{2}\left(||\frac{x+y}{2}+iz||^2-||\frac{x+y}{2}-iz||^2\right)\\&=2\left(\frac{x+y}{2},z\right)(\because (2)\ ).\end{aligned}$$

Setting $y=0$ in this formula implies that $(x,z)=2\left(\frac{x}{2},z\right)$ $(\because (0,z)=0)$. Now,replacing x with $x+y$ implies that $(x+y,z)=2\left(\frac{x+y}{2},z\right)$.Therefore we find that,

$$(x,z)+(y,z)=2\left(\frac{x+y}{2},z\right)=(x+y,z).$$

Then, we shall prove $(-x,y)=-(x,y)$ and $(ix,y)=i(x,y)$. In fact, using formula (4), we find that,

$$\begin{aligned}(-x,y)&=\frac{1}{4}\left(||-x+y||^2-||-x-y||^2\right)+\frac{i}{4}\left(||-x+iy||^2-||-x-iy||^2\right)\\&=\frac{1}{4}\left(||x-y||^2-||x+y||^2\right)+\frac{i}{4}\left(||x-iy||^2-||x+iy||^2\right)\\&=-\left(\frac{1}{4}\left(||x+y||^2-||x-y||^2\right)+\frac{i}{4}\left(||x+iy||^2-||x-iy||^2\right)\right)\\&=-(x,y)\end{aligned}$$

and,

$$\begin{aligned}(ix,y)&=\frac{1}{4}\left(||ix+y||^2-||ix-y||^2\right)+\frac{i}{4}\left(||ix+iy||^2-||ix-iy||^2\right)\\&=\frac{1}{4}\left(||i(x-iy)||^2-||i(x+iy)||^2\right)+\frac{i}{4}\left(||i(x+y)||^2-||i(x-y)||^2\right)\\&=\frac{1}{4}\left(||x-iy||^2-||x+iy||^2\right)+\frac{i}{4}\left(||x+y||^2-||x-y||^2\right)\\&=i\left(\frac{1}{4}\left(||x+y||^2-||x-y||^2\right)+\frac{i}{4}\left(||x+iy||^2-||x-iy||^2\right)\right)\\&=i(x,y).\end{aligned}$$

Now we shall prove $(\alpha x, y) = \alpha(x, y)$, for all α in $\mathbb{Q}$. At first, for any positive integer m we find that,

$$(mx, y) = (x + (m-1)x, y) = (x, y) + ((m-1)x, y) = \cdots = m(x, y).$$

Next, for any negative integer m', there exists a positive integer m such that $m' = -m$. Now, using $(-x, y) = -(x, y)$, one gets

$$(m'x, y) = (-mx, y) = -(mx, y) = -m(x, y) = m'(x, y).$$

Thus, from $(0, y) = 0$, $(mx, y) = m(x, y)$ holds for all integer m.
Next, let n be another integer $(n \neq 0)$. Replacing x with $\frac{x}{n}$ in $(nx, y) = n(x, y)$ implies that $(x, y) = n(\frac{1}{n}x, y)$, i.e $(\frac{1}{n}x, y) = \frac{1}{n}(x, y)$. Therefore

$$(\frac{m}{n}x, y) = \frac{1}{n}(mx, y) = \frac{m}{n}(x, y).$$

Now, we shall prove $(\alpha x, y) = \alpha(x, y)(\forall \alpha \in R)$. There exists a sequence of rational number $\{\alpha_n\}$ which converges to α. The sequence $\{\alpha_n\}$ satisfies $(\alpha_n x, y) = \alpha_n(x, y)$. Hence,

$$(\alpha x, y) = (\lim_n \alpha_n x, y) = \lim_n(\alpha_n x, y) = \lim_n \alpha_n(x, y) = \alpha(x, y).$$

Finally, if α is a complex number, setting $\alpha = a + ib$ implies that $(\alpha x, y) = \alpha(x, y)$
From the above, we prove that (x, y) which is defined by (3) formula satisfies all condition of an inner product conditions.

Proposition 3 *Let A be a norm space. If there is a scalar product $\langle \cdot, \cdot \rangle_A$, then the norm $|| \cdot ||_A$ associated with it satisfies*

$$||x + y||_A^2 + ||x - y||_A^2 = 2(||x||_A^2 + ||y||_A^2), \ \forall x, y \in A$$

and vise versa.

(proof) At first, we shall prove $(\Rightarrow)$. Let assume there exists an inner product $\langle \cdot, \cdot \rangle_A$. Then,

$$\begin{aligned}
||x + y||_A^2 + ||x - y||_A^2 &= \langle x + y, x + y \rangle_A + \langle x - y, x - y \rangle_A \\
&= \langle x, x \rangle_A + \langle x, y \rangle_A + \langle y, x \rangle_A + \langle y, y \rangle_A \\
&\qquad \{\langle x, x \rangle_A + \langle x, -y \rangle_A + \langle -y, x \rangle_A + \langle -y, -y \rangle_A\} \\
&= 2(\langle x, x \rangle_A + \langle y, y \rangle_A) + (\langle x, y - y \rangle_A + \langle y - y, x \rangle_A) \\
&= 2(||x||_A^2 + ||y||_A^2)
\end{aligned}$$

Thus $(\Rightarrow)$ hold.
Then, $(\Leftarrow)$ follows from Von Neumann Theorem. Hence, the Proposition is proved.

Proposition 4 *Let X be a reflexive space and Y be a banach space. If $Y \cong X$, then X is reflexive.*

We shall only write statement of Proposition 4 here.

Proposition 5 *Let H be a Hilbert space and A be a subspace in H. Then A is a Hilbert space.*

We shall only write the statement of Proposition 5 here.

Theorem 2 *Let A be a partial norm space in another norm space X whose norm satisfies*

$$||x+y||_A^2 + ||x-y||_A^2 = 2(||x||_A^2 + ||y||_A^2),\ \forall x,y \in A$$

Then A^* is reflexive.

(proof) There exists an inner product $\langle\cdot,\cdot\rangle_A$ from proposition 3. Then $\bar{A}$ is a Hilbert space. So, $\bar{A}^*$ is reflexive from Theorem 1 and Proposition 1. Now we find that A^* is reflexive from Proposition 2 and Proposition 4 as desired.

Theorem 3 *Let A be a norm space which is embedded in a Hilbert space. Then A^* is reflexive.*

(proof) We can prove in the same way as the proof of theorem 2 by using Proposition 5.

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(2) ノルムと内積
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(3) P. Jordan and J. von Neumann, "On Inner Product in Linear Metric Spaces"