

Chapter 1

Geometric setting

In this Chapter we recall some basic notions on points or vectors in \mathbb{R}^n . The norm of a vector and the scalar product between two vectors are also introduced.

1.1 The Euclidean space \mathbb{R}^n

We set $\mathbb{N} := \{1, 2, 3, \dots\}$ for the set of *natural numbers*, also called *positive integers*, and let \mathbb{R} be the set of all real numbers.

Definition 1. *One sets*

$$\mathbb{R}^n = \{(x_1, x_2, \dots, x_n) \mid x_j \in \mathbb{R} \text{ for all } j \in \{1, 2, \dots, n\}\}^1.$$

Alternatively, an element of \mathbb{R}^n , also called a n -tuple or a vector, is a collection of n numbers (x_1, x_2, \dots, x_n) with $x_j \in \mathbb{R}$ for any $j \in \{1, 2, \dots, n\}$. The number n is called the dimension of \mathbb{R}^n .

In the sequel, we shall often write $X \in \mathbb{R}^n$ for the vector $X = (x_1, x_2, \dots, x_n)$. With this notation, the values x_1, x_2, \dots, x_n are called *the components* or *the coordinates* of X . Note that one often writes (x, y) for elements of \mathbb{R}^2 and (x, y, z) for elements of \mathbb{R}^3 . However this notation is not really convenient in higher dimensions.

The set \mathbb{R}^n can be endowed with two operations, *the addition* and *the multiplication by a scalar*.

Definition 2. *For any $X, Y \in \mathbb{R}^n$ with $X = (x_1, x_2, \dots, x_n)$ and $Y = (y_1, y_2, \dots, y_n)$ and for any $\lambda \in \mathbb{R}$ one defines the addition of X and Y by*

$$X + Y := (x_1 + y_1, x_2 + y_2, \dots, x_n + y_n) \in \mathbb{R}^n$$

and the multiplication of X by the scalar λ by

$$\lambda X := (\lambda x_1, \lambda x_2, \dots, \lambda x_n) \in \mathbb{R}^n.$$

¹The vertical line \mid has to be read “such that”.

Examples 3. (i) $(1, 3) + (2, 4) = (3, 7) \in \mathbb{R}^2$,

(ii) $(1, 2, 3, 4, 5) + (5, 4, 3, 2, 1) = (6, 6, 6, 6, 6) \in \mathbb{R}^5$,

(iii) $3(1, 2) = (3, 6) \in \mathbb{R}^2$,

(iv) $\pi(0, 0, 1) = (0, 0, \pi) \in \mathbb{R}^3$.

One usually sets

$$\mathbf{0} = (0, 0, \dots, 0) \in \mathbb{R}^n$$

and this element satisfies $X + \mathbf{0} = \mathbf{0} + X = X$ for any $X \in \mathbb{R}^n$. If $X = (x_1, x_2, \dots, x_n)$ one also writes $-X$ for the element $-1X = (-x_1, -x_2, \dots, -x_n)$. Then, by an abuse of notation, one writes $X - Y$ for $X + (-Y)$ if $X, Y \in \mathbb{R}^n$, and obviously one has $X - X = \mathbf{0}$. Note that $X + Y$ is defined if and only if X and Y belong to \mathbb{R}^n , but has no meaning if $X \in \mathbb{R}^n$ and $Y \in \mathbb{R}^m$ with $n \neq m$.

Properties 4. If $X, Y, Z \in \mathbb{R}^n$ and $\lambda, \mu \in \mathbb{R}$ then one has

(i) $X + Y = Y + X$, (commutativity)

(ii) $(X + Y) + Z = X + (Y + Z)$, (associativity)

(iii) $\lambda(X + Y) = \lambda X + \lambda Y$, (distributivity)

(iv) $(\lambda + \mu)X = \lambda X + \mu X$,

(v) $(\lambda\mu)X = \lambda(\mu X)$.

1.2 Scalar product and norm

Definition 5. For any $X, Y \in \mathbb{R}^n$ with $X = (x_1, x_2, \dots, x_n)$ and $Y = (y_1, y_2, \dots, y_n)$ one sets

$$X \cdot Y := x_1 y_1 + x_2 y_2 + \dots + x_n y_n = \sum_{j=1}^n x_j y_j$$

and calls this number the scalar product between X and Y .

For example, if $X = (1, 2)$ and $Y = (3, 4)$, then $X \cdot Y = 1 \cdot 3 + 2 \cdot 4 = 3 + 8 = 11$, but if $X = (1, 3)$ and $Y = (6, -2)$, then $X \cdot Y = 6 - 6 = 0$. Be aware that the previous notation is slightly misleading since the dot \cdot between X and Y corresponds to the scalar product while the dot between numbers just corresponds to the usual multiplication of numbers.

Properties 6. For any $X, Y, Z \in \mathbb{R}^n$ and $\lambda \in \mathbb{R}$ one has

(i) $X \cdot Y = Y \cdot X$,

$$(ii) X \cdot (Y + Z) = X \cdot Y + X \cdot Z,$$

$$(iii) (\lambda X) \cdot Y = X \cdot (\lambda Y) = \lambda(X \cdot Y),$$

$$(iv) X \cdot X \geq 0, \text{ and } X \cdot X = 0 \text{ if and only if } X = \mathbf{0}.$$

Definition 7. The Euclidean norm or simply norm of a vector $X \in \mathbb{R}^n$ is defined by

$$\|X\| := \sqrt{X^2} \equiv \sqrt{X \cdot X} = \sqrt{x_1^2 + x_2^2 + \cdots + x_n^2}.$$

The positive number $\|X\|$ is also referred to as the magnitude of X . A vector of norm 1 is called a unit vector.

Example 8. If $X = (-1, 2, 3) \in \mathbb{R}^3$, then $X \cdot X = (-1)^2 + 2^2 + 3^2 = 14$ and therefore $\|X\| = \sqrt{14}$.

Remark 9. If $n = 2$ or $n = 3$ this norm corresponds to our geometric intuition.

Properties 10. For any $X \in \mathbb{R}^n$ and $\lambda \in \mathbb{R}$ one has

$$(i) \|X\| = 0 \text{ if and only if } X = \mathbf{0},$$

$$(ii) \|\lambda X\| = |\lambda| \|X\|,$$

$$(iii) \|-X\| = \|X\|.$$

Note that the third point is a special case of the second point.

1.3 Vectors and located vectors

Definition 11. For any $X, Y \in \mathbb{R}^n$ we set \overrightarrow{XY} for the arrow starting at X and ending at Y , and call it the located vector \overrightarrow{XY} .

With this definition and for any $X \in \mathbb{R}^n$ the located vector $\overrightarrow{\mathbf{0}X}$ corresponds to the arrow starting at $\mathbf{0}$ and ending at X . This located vector is simply called a vector and is often identified with the element X of \mathbb{R}^n . This identification should not lead to any confusion in the sequel.

Definition 12. Let $A, B, C, D \in \mathbb{R}^n$ and consider the located vectors \overrightarrow{AB} and \overrightarrow{CD} .

$$(i) \overrightarrow{AB} \text{ is equivalent to } \overrightarrow{CD} \text{ if } B - A = D - C,$$

$$(ii) \overrightarrow{AB} \text{ is parallel to } \overrightarrow{CD} \text{ if } B - A = \lambda(D - C) \text{ for some } \lambda \in \mathbb{R},$$

(iii) \overrightarrow{AB} and \overrightarrow{CD} have the same direction if $B - A = \lambda(D - C)$ for some $\lambda > 0$.

Note that the located vector \overrightarrow{XY} is always equivalent to the located vector $\overrightarrow{\mathbf{0}(Y - X)}$ which is located at *the origin* $\mathbf{0}$. This fact follows from the equality

$$(Y - X) - \mathbf{0} = (Y - X) = Y - X.$$

Definition 13. The norm of a located vector \overrightarrow{XY} is defined by $\|\overrightarrow{XY}\| := \|Y - X\|$.

Note that this definition corresponds to our intuition in dimension $n = 2$ or $n = 3$ for the length of an arrow. It should also be observed that two equivalent located vectors have the same norm.

Definition 14. Let $A, B, C, D \in \mathbb{R}^n$.

- (i) The vectors $\overrightarrow{\mathbf{0}A}$ and $\overrightarrow{\mathbf{0}B}$ are orthogonal (or perpendicular) if $A \cdot B = 0$. In this case one writes $\overrightarrow{\mathbf{0}A} \perp \overrightarrow{\mathbf{0}B}$ or simply $A \perp B$,
- (ii) The located vectors \overrightarrow{AB} and \overrightarrow{CD} are orthogonal if $\overrightarrow{\mathbf{0}(B - A)}$ and $\overrightarrow{\mathbf{0}(D - C)}$ are orthogonal, namely if $(B - A) \cdot (D - C) = 0$. In this case one writes $\overrightarrow{AB} \perp \overrightarrow{CD}$.

Let us emphasize again that this notion corresponds to our intuition in dimension $n = 2$ or $n = 3$.

Example 15. In \mathbb{R}^n let us set $E_1 = (1, 0, \dots, 0)$, $E_2 = (0, 1, 0, \dots, 0)$, \dots , $E_n = (0, \dots, 0, 1)$ the n different vectors obtained by assigning a 1 at the coordinate j of E_j and 0 for all its other coordinates. Then, one easily checks that

$$E_j \cdot E_k = 0 \text{ whenever } j \neq k \quad \text{and} \quad E_j \cdot E_j = 1 \text{ for any } j \in \{1, 2, \dots, n\}.$$

These n vectors are said to be mutually orthogonal.

Theorem 16 (General Pythagoras theorem). *For any $X, Y \in \mathbb{R}^n$ one has*

$$\|X + Y\|^2 = \|X\|^2 + \|Y\|^2 \iff X \cdot Y = 0.$$

In other words, the equality on the left-hand side holds if and only if X and Y are orthogonal.

Let us add some other useful properties:

Lemma 17. *For any $X, Y \in \mathbb{R}^n$:*

$$(i) \quad |X \cdot Y| \leq \|X\| \|Y\|,$$

$$(ii) \quad \|X + Y\| \leq \|X\| + \|Y\|,$$

(iii) *Let us recall from plane geometry that if one considers the triangle with vertices the points $\mathbf{0}$, X and Y , then the angle θ at the vertex $\mathbf{0}$ satisfies*

$$\cos(\theta) = \frac{X \cdot Y}{\|X\| \|Y\|}.$$

1.4 Lines and hyperplanes

Let us consider $P, N \in \mathbb{R}^n$ with $N \neq \mathbf{0}$.

The set

$$L_{P,N} := \{P + tN \mid t \in \mathbb{R}\}$$

defines the line in \mathbb{R}^n passing through P and having the direction parallel to $\overrightarrow{\mathbf{0}N}$.

The set

$$H_{P,N} := \{X \in \mathbb{R}^n \mid X \cdot N = P \cdot N\}$$

defines a *hyperplane* passing through P and normal to $\overrightarrow{\mathbf{0}N}$. Note that if $P = (p_1, p_2, \dots, p_n)$ and if $N = (n_1, n_2, \dots, n_n)$, then

$$H_{P,N} = \left\{ (x_1, x_2, \dots, x_n) \in \mathbb{R}^n \mid n_1x_1 + n_2x_2 + \dots + n_nx_n = \sum_{j=1}^n p_j n_j \right\}.$$