
Homework 11

Exercise 1 Use Green's theorem to compute the integral $\int_c f$ with $f(x, y) = (y^2, x)$ when c corresponds to the following curves, taken counterclockwise:

- (i) The square of vertices $(0, 0)$, $(2, 0)$, $(2, 2)$, $(0, 2)$,
- (ii) The square of vertices $(\pm 1, \pm 1)$,
- (iii) The circle of radius 1 and centered at $(0, 0)$,
- (iv) The ellipse of equation $(x/a)^2 + (y/b)^2 = 1$ for some $a, b > 0$.

Exercise 2 Check the validity of Green's theorem for the function $f : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ defined for $(x, y) \in \mathbb{R}^2$ by $f(x, y) = (2xy, x^2)$ on the domain $\Omega = [-1, 2] \times [-1, 3] \subset \mathbb{R}^2$.

Exercise 3 Consider the function $f : \mathbb{R}^2 \setminus \{0, 0\} \rightarrow \mathbb{R}^2$ defined for $(x, y) \in \mathbb{R}^2$ by

$$f(x, y) = \left(\frac{-y}{x^2 + y^2}, \frac{x}{x^2 + y^2} \right).$$

Let $c : [a, b] \rightarrow \mathbb{R}^2 \setminus \{(0, 0)\}$ be a parametric curve of class C^1 and non-intersecting such that its interior Ω is located on the left of the curve. We also assume that $(0, 0) \in \Omega$. Compute $\int_c f$, and explain your result.

Exercise 4 Let $f : \mathbb{R}^2 \rightarrow \mathbb{R}$ be sufficiently many times differentiable and satisfying the Laplace equation

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = 0.$$

- (i) Let c be a closed parametric curve oriented counterclockwise and non-intersecting. Show that

$$\int_c \begin{pmatrix} \partial_y f \\ -\partial_x f \end{pmatrix} = 0,$$

- (ii) Show that $f(0, 0) = \frac{1}{2\pi} \int_0^{2\pi} f(r \cos(\theta), r \sin(\theta)) d\theta$ for any $r > 0$.

Exercise 5 Compute the area of the domain defined in $\mathbb{R}_+ \times \mathbb{R}_+$ by the four curves of equation

$$y = ax, \quad y = x/a, \quad y = b/x, \quad y = 1/bx \quad \text{for } a > 1, b > 1.$$