Report

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Proposition 1: For every $M=(m_{ij})_{i,j} \in M_{p,q}(\mathbb{C})$,

$$Tr(M^*M) = Tr(MM^*) = \sum_{i=1}^{p} \sum_{j=1}^{q} |m_{ij}|^2$$

Proposition 2: For every idempotent element e of a $\mathbb{C}$-algebra, $\sigma(e) \subset \{0,1\}$. Furthermore if e is non-trivial, $\sigma(e) = \{0,1\}$.

Proof: For all $\lambda \in \mathbb{C} - \{0,1\}$, $(e-\lambda)^{-1} = \dfrac{e}{\lambda(1-\lambda)} - \dfrac{1}{\lambda}$, thus $\sigma(e) \subset \{0,1\}$. If e is non-trivial, the relations $e \neq 0$, $e \neq 1$, $e(1-e) = 0$ imply $\{0,1\} \subset \sigma(e)$, thus $\sigma(e) = \{0,1\}$.

Proposition 3: Let P be a projection of $M_n(\mathbb{C})$. If $Tr(P) = r$, then $P \sim_0 I_r$.

Proof: Since P is idempotent, $\sigma(P) \subset \{0,1\}$ by Proposition 2, and as P is normal, there exists a unitary matrix $U \in U(n)$ and an integer $r \in \mathbb{N}$ such that $U^*PU = \begin{pmatrix} I_r & 0_{r,n-r} \\ 0_{n-r,r} & 0_{n-r,n-r} \end{pmatrix}$. Then $Tr(P) = Tr(U^*PU) = r$ and for the matrix $E_r(n) \begin{pmatrix} I_r \\ 0_{n-r,r} \end{pmatrix} \in M_{n,r}(\mathbb{C})$

$$\begin{cases} (U E_r(n))^*(U E_r(n)) = I_r \\ (U E_r(n))(U E_r(n))^* = P \end{cases}$$

Corollary 1: Let $P \in M_n(\mathbb{C})$, $Q \in M_m(\mathbb{C})$ be two projections. Then $Tr(P) = Tr(Q)$ iff $P \sim_0 Q$.

Proof: If $Tr(P) = Tr(Q) = r$, then $P \sim_0 I(r) \sim_0 Q$ by Proposition 3. If $P \sim_0 Q$, then there exists a matrix $R \in M_{m,n}(\mathbb{C})$ such that $P = R^*R$, $Q = RR^*$. By Proposition 1, $Tr(P) = Tr(R^*R) = Tr(RR^*) = Tr(Q)$.

Proposition 4 $Tr : D(\mathbb{C}) \to \mathbb{N}$ is isomorphism of monoid.

Proof: The mapping Tr is well-defined and injective by Corollary 1.

As $Tr(0) = 0$, $Tr(I_n) = n$, Tr is surjective.

And $Tr(P \oplus Q) = Tr(P) + Tr(Q)$ for all $P \in P_n(\mathbb{C})$, $Q \in P_m(\mathbb{C})$, and $Tr(0) = 0$, thus Tr is monoide morphism.

Proposition 5 : $K_0(\mathbb{C}) \simeq \mathbb{Z}$

Proof : By Proposition 4, $K_0(\mathbb{C}) \simeq G(D(\mathbb{C})) \simeq G(\mathbb{N}) \simeq \mathbb{Z}$.

Corollary 2 : $K_0(M_n(\mathbb{C})) \simeq K_0(\mathbb{C}) \simeq \mathbb{Z}$

Proof : By definition of K_0-group, $K_0(M_n(\mathbb{C})) \simeq K_0(\mathbb{C}) \simeq \mathbb{Z}$.

Proposition 6 : $U(n)$ is path-connected.

Proof : A unitary matrix $M \in U(n)$ is normal and $\det(M) \in S^1$, thus there exists $U \in U(n)$ such that

$$U^* M U = \begin{pmatrix} \exp(i a_1) & & 0 \\ & \ddots & \\ 0 & & \exp(i a_n) \end{pmatrix} = \exp\left(i \begin{pmatrix} a_1 & & 0 \\ & \ddots & \\ 0 & & a_n \end{pmatrix}\right) \quad (a_1, \dots, a_n \in \mathbb{R}),$$

Put $D = \begin{pmatrix} a_1 & & 0 \\ & \ddots & \\ 0 & & a_n \end{pmatrix}$, then the mapping $t \longmapsto U \exp(itD) U^*$ from $[0,1]$ to $U(n)$ is a path from I_n to M.

Corollary 7 : $K_1(\mathbb{C}) \simeq 0$.

Proof : By a Proposition 6, $U \sim_h I_n$ for all $U \in U(n)$, thus $U \sim_1 I_n$.

And $I_p \sim_1 I_q$ for all integer $p, q \geq 1$.

Hence $K_1(\mathbb{C}) \simeq 0$.