Nagoya University, G30 program Spring 2015
Math tutorial II, linear algebra Instructor : Serge Richard
Quiz IV Name : --

Exercise 1 *Compute the following expressions:*

(i) $(2-2i)+(1-i)$ (ii) $i(1-2i)$ (iii) $(i+1)^{-1}$ (iv) i^{2015} (v) $e^{2\pi i}$.

i) $3-3i$,

ii) $i+2 = 2+i$,

iii) $\frac{1}{1+i} \cdot \frac{1-i}{1-i} = \frac{1-i}{2} = \frac{1}{2} - \frac{i}{2}$,

iv) $i^{2012}\, i^3 = i^{4(503)}\, i^3 = 1^{503}\, i^3 = 1 \cdot (-i) = -i$,

v) $\cos(2\pi) + i\sin(2\pi) = 1$.

Exercise 2 *Consider the complex number* $1+i$ *and determine*

(i) *its modulus* (ii) *its argument* (iii) *its complex conjugate.*

i) $|1+i| = (1+1)^{1/2} = \sqrt{2}$,

ii) $1+i = \sqrt{2}\big(\cos(\pi/4) + i\sin(\pi/4)\big) \Rightarrow$ argument $= \pi/4$

iii) $1-i$.

Exercise 3 *Show that two similar square matrices share the same determinant.*

If $A', A \in M_n(\mathbb{R})$ and if there exits $B \in M_n(\mathbb{R})$ invertible such that $A' = BAB^{-1}$, then

$$\mathrm{Det}(A') = \mathrm{Det}(BAB^{-1}) = \mathrm{Det}(B)\,\mathrm{Det}(A)\,\mathrm{Det}(B^{-1})$$

$$= \mathrm{Det}(B)\,\frac{1}{\mathrm{Det}(B)}\,\mathrm{Det}(A) = \mathrm{Det}(A).$$

Exercise 4 *Determine the eigenvalues and the eigenspaces of the linear map* $\mathrm{L}_{\mathcal{A}}: \mathbb{R}^3 \to \mathbb{R}^3$ *associated with the matrix*

$$\mathcal{A} = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & -1 \\ 1 & 1 & 0 \end{pmatrix}.$$

$$P_A(\lambda) = \mathrm{Det}\begin{pmatrix} -\lambda & 1 & 1 \\ 1 & -\lambda & -1 \\ 1 & 1 & -\lambda \end{pmatrix} = -\lambda(\lambda^2+1) - (-\lambda-1) + (-1+\lambda)$$

$$= -\lambda^3 - \lambda + \lambda + 1 - 1 + \lambda = -\lambda(\lambda^2-1) = -\lambda(\lambda+1)(\lambda-1)$$

The eigenvalues of L_A are $0, -1, 1$.

$\underline{\lambda = 0}$: The corresponding eigenspace is given by

$$\begin{cases} y+z=0 \\ x-z=0 \\ x+y=0 \end{cases} \Leftrightarrow \begin{cases} y=-x \\ z=x \end{cases}, \quad \underline{V_0 = \left\{ x\begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} \mid x \in \mathbb{R} \right\}}$$

$\underline{\lambda = -1}$: The corresponding eigenspace is given by

$$\begin{cases} x+y+z=0 \\ x+y-z=0 \\ x+y+z=0 \end{cases} \Leftrightarrow \begin{cases} z=0 \\ y=-x \end{cases} \quad \underline{V_{-1} = \left\{ x\begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} \mid x \in \mathbb{R} \right\}}$$

$\underline{\lambda = 1}$: The corresponding eigenspace is given by

$$\begin{cases} -x+y+z=0 \\ x-y+z=0 \\ x+y-z=0 \end{cases} \Leftrightarrow \begin{cases} y=0 \\ z=x \end{cases} \quad V_1 = \left\{ x\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \mid x \in \mathbb{R} \right\}.$$

Exercise 5 *Let* $\mathcal{A} = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}$, *and consider the associated linear map* $\mathrm{L}_{\mathcal{A}} : \mathbb{C}^2 \to \mathbb{C}^2$. *Determine the eigenvalues of* $\mathrm{L}_{\mathcal{A}}$ *and the corresponding eigenspaces. Show that these eigenspaces are orthogonal.*

$$P_{\mathcal{A}}(\lambda) = \mathrm{Det}\begin{pmatrix} -\lambda & i \\ -i & -\lambda \end{pmatrix} = \lambda^2 - 1 = (\lambda+1)(\lambda-1).$$

$\underline{\lambda = 1}$: One has $\begin{cases} -x + iy = 0 \\ -ix - y = 0 \end{cases} \iff y = -ix$.

Thus $V_1 = \left\{ x\begin{pmatrix} 1 \\ -i \end{pmatrix} \mid x \in \mathbb{C} \right\}$.

$\underline{\lambda = -1}$: One has $\begin{cases} x + iy = 0 \\ -ix + y = 0 \end{cases} \iff y = ix$

Thus $V_{-1} = \left\{ x\begin{pmatrix} 1 \\ i \end{pmatrix} \mid x \in \mathbb{C} \right\}$.

Note that $\begin{pmatrix} 1 \\ -i \end{pmatrix}$ and $\begin{pmatrix} 1 \\ i \end{pmatrix}$ are orthogonal in $\mathbb{C}^2$

since $\left\langle \begin{pmatrix} 1 \\ -i \end{pmatrix}, \begin{pmatrix} 1 \\ i \end{pmatrix} \right\rangle = 1 \cdot \bar{1} + (-i)\cdot(\bar{i}) = 1 + (-i)^2 = 1 - 1 = 0.$