Nagoya University, G30 program Spring 2015

Math tutorial II, linear algebra Instructor : Serge Richard

Quiz II Name : ____________________

Exercise 1 *We consider the real vector space $V := C([0,\pi])$ made of continuous real functions on $[0,\pi]$ and endow it with the scalar product*

$$V \times V \ni (f,g) \mapsto \langle f,g\rangle := \int_0^\pi f(x)g(x)\,\mathrm{d}x \ \in \mathbb{R}.$$

(i) Are the constant function $x \mapsto 1$ and the sine function $x \mapsto \sin(x)$ orthogonal, and why ?

$\langle 1, \sin \rangle = \int_0^\pi 1 \sin(x)\,dx = -\cos(x)\big|_1^\pi = 2.$

The 2 functions are not orthogonal.

(ii) Are the sine function $x \mapsto \sin(x)$ and the cosine function $x \mapsto \cos(x)$ orthogonal, and why ?

$\langle \sin, \cos \rangle = \int_0^\pi \cos(x)\sin(x)\,dx = \frac{1}{2}\sin^2(x)\big|_0^\pi = 0.$

The 2 functions are orthogonal.

Exercise 2 *Consider the matrices $\mathcal{A} = \begin{pmatrix} 1&0&0\\0&0&0\\0&0&1\end{pmatrix}$ and $\mathcal{B} = \begin{pmatrix} 0&0&0\\0&1&0\\0&0&2\end{pmatrix}$ in $M_3(\mathbb{R})$.*

1. *Are $\mathcal{A}$ and/or $\mathcal{B}$ projections, and why ?*

$\mathcal{A}^2 = \begin{pmatrix} 1&0&0\\0&0&0\\0&0&1\end{pmatrix}\begin{pmatrix} 1&0&0\\0&0&0\\0&0&1\end{pmatrix} = \begin{pmatrix} 1&0&0\\0&0&0\\0&0&1\end{pmatrix} = \mathcal{A} \Rightarrow \mathcal{A}$ is a projection,

$\mathcal{B}^2 = \begin{pmatrix} 0&0&0\\0&1&0\\0&0&2\end{pmatrix}\begin{pmatrix} 0&0&0\\0&1&0\\0&0&2\end{pmatrix} = \begin{pmatrix} 0&0&0\\0&1&0\\0&0&4\end{pmatrix} \neq \mathcal{B} \Rightarrow \mathcal{B}$ is not a projection.

2. *Are $\mathcal{A}$ and/or $\mathcal{B}$ symmetric, and why ?*

Yes, because ${}^t\mathcal{A} = \mathcal{A}$ and ${}^t\mathcal{B} = \mathcal{B}$.

3. *If one defines the scalar product $\langle \mathcal{A},\mathcal{B}\rangle$ by $\mathsf{Tr}(\mathcal{AB})$ with Tr the trace matrices, are $\mathcal{A}$ and $\mathcal{B}$ orthogonal ?*

$\langle \mathcal{A}, \mathcal{B}\rangle = \mathrm{Tr}(\mathcal{A}\mathcal{B}) = \mathrm{Tr}\left(\begin{pmatrix} 1&0&0\\0&0&0\\0&0&1\end{pmatrix}\begin{pmatrix} 0&0&0\\0&1&0\\0&0&2\end{pmatrix}\right) = \mathrm{Tr}\begin{pmatrix} 0&0&0\\0&0&0\\0&0&2\end{pmatrix} = 2 \neq 0$

A, B are not orthogonal.

4. *What is the norm of $\mathcal{A}$ for the norm based on the previous scalar product ?*

$\|\mathcal{A}\|^2 = \langle \mathcal{A},\mathcal{A}\rangle = \mathrm{Tr}(\mathcal{A}^2) = \mathrm{Tr}(\mathcal{A}) = 2$

$\Rightarrow \underline{\|\mathcal{A}\| = \sqrt{2}}.$