Nagoya University, G30 program Spring 2015

Math tutorial II, linear algebra Instructor : Serge Richard

Quiz I Name : ____________________

Exercise 1 *Let V be a real vector space and let $T : V \to V$ be a linear map satisfying $T^2 = \mathbf{1}$. Show that $P := \frac{1}{2}(\mathbf{1} + T)$ is a projection.*

One has $P^2 = \left(\frac{1}{2}(1+T)\right)^2 = \frac{1}{4}(1+T)(1+T) = \frac{1}{4}(1+T+T+T^2)$

$= \frac{1}{4}(2+2T) = \frac{1}{2}(1+T) = P.$

$T^2=1$ Thus $P^2 = P$ which means P is a projection.

Exercise 2 *Determine if the following map is linear:* $\mathrm{F} : \mathbb{R}^3 \to \mathbb{R}^4$ *with* $\mathrm{F}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x \\ y \\ z \\ x+y+z \end{pmatrix}$.

$$F\begin{pmatrix} x_1+x_2 \\ y_1+y_2 \\ z_1+z_2 \end{pmatrix} = \begin{pmatrix} x_1+x_2 \\ y_1+y_2 \\ z_1+z_2 \\ x_1+x_2+y_1+y_2+z_1+z_2 \end{pmatrix} = \begin{pmatrix} x_1 \\ y_1 \\ z_1 \\ x_1+y_1+z_1 \end{pmatrix} + \begin{pmatrix} x_2 \\ y_2 \\ z_2 \\ x_2+y_2+z_2 \end{pmatrix} =$$

$$= F\begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix} + F\begin{pmatrix} x_2 \\ y_2 \\ z_2 \end{pmatrix}, \quad \text{for any } \begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix}, \begin{pmatrix} x_2 \\ y_2 \\ z_2 \end{pmatrix} \in \mathbb{R}^3$$

$$F\begin{pmatrix} \lambda x \\ \lambda y \\ \lambda z \end{pmatrix} = \begin{pmatrix} \lambda x \\ \lambda y \\ \lambda z \\ \lambda(x+y+z) \end{pmatrix} = \lambda\begin{pmatrix} x \\ y \\ z \\ x+y+z \end{pmatrix} = \lambda F\begin{pmatrix} x \\ y \\ z \end{pmatrix}, \quad \text{for any } \begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \mathbb{R}^3, \lambda \in \mathbb{R}$$

$\Rightarrow$ F is linear.

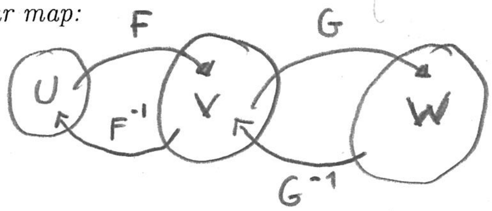
What is the dimension of its kernel and the dimension of its range ?

$\mathrm{Ker}(F) = \{0\}$ because $F\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \Rightarrow \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$

and therefore $\dim(\ker(F)) = \underline{\underline{0}}$ and

$\dim(\mathrm{Ran}(F)) = 3 - 0 = \underline{\underline{3}}.$

Exercise 3 *Let U, V, W be distinct vector spaces. Let $\mathrm{F} : U \to V$ and $\mathrm{G} : V \to W$ be invertible linear maps. Circle the letter(s) which corresponds to a well defined linear map:*



(a) $\mathrm{G}^{-1} \circ \mathrm{F} : U \to V$,

(b) $\mathrm{F}^{-1} \circ \mathrm{G}^{-1} : W \to U$,

(c) $\mathrm{F}^{-1} : U \to V$,

(d) $\mathrm{G} \circ \mathrm{F} : U \to W$.

Exercise 4 *Let F, G be invertible linear maps from a vector space into itself. Is $\mathrm{F} \circ \mathrm{G}$ invertible, and if so what is its inverse ? If $\mathrm{F} \circ \mathrm{G}$ is not invertible, what can you say about the inverse of $\mathrm{G} \circ \mathrm{F}$?*

<u>Yes</u>, $F \circ G$ is invertible, with $(F \circ G)^{-1} = G^{-1} \circ F^{-1}$.