Nagoya University, G30 program Spring 2015

Linear algebra II Instructor : Serge Richard

Quiz I Name : --

Exercise 1 *Complete the following sentence:*
A map $F : V \to W$ between two sets V and W is injective if...

for any $X, Y \in V$ with $X \neq Y$, one has $F(X) \neq F(Y)$.

Exercise 2 *Let $L : V \to V$ be a linear map on a real vector space V. Let $\mathcal{L}$ be the matrix associated with L with respect to a basis $\{V_1, \dots, V_n\}$ of V, and let $\mathcal{L}'$ be the matrix associated with L with respect to another basis $\{V_1', \dots, V_n'\}$ of V. If $\mathcal{L}$ is an invertible matrix, is $\mathcal{L}'$ always/sometimes/never an invertible matrix ? Justify very briefly your answer.*

$\mathcal{L}$ and $\mathcal{L}'$ are similar i.e. $\mathcal{L}' = B \mathcal{L} B^{-1}$ for some invertible matrix B. Thus, $\mathcal{L}'$ is invertible with inverse $(\mathcal{L}')^{-1} = (B \mathcal{L} B^{-1})^{-1} = B \mathcal{L}^{-1} B^{-1}$.

Exercise 3 *Let $\mathrm{L}_{\mathcal{A}} : \mathbb{R}^3 \to \mathbb{R}^3$ be the linear map defined on any $X \in \mathbb{R}^3$ by $\mathrm{L}_{\mathcal{A}}(X) = \mathcal{A}X$, with $\mathcal{A} := \begin{pmatrix} 0 & 1 & 2 \\ 0 & 2 & 4 \\ 1 & 1 & 2 \end{pmatrix}$. Without doing any computation, answer the following questions:*

(i) Is $\mathrm{L}_{\mathcal{A}}$ a surjective linear map, and why ? No, because the second and third column of $\mathcal{A}$ are linearly dependent.
$\Rightarrow \operatorname{rank} \mathcal{A} = 2$.

(ii) What is the dimension of the kernel of $\mathrm{L}_{\mathcal{A}}$, and why ?

$$\text{Since } 3 = \dim(\ker \mathrm{L}_{\mathcal{A}}) + \dim(\operatorname{Ran} \mathrm{L}_{\mathcal{A}}) = \dim(\ker \mathrm{L}_{\mathcal{A}}) + \operatorname{rank} \mathcal{A}$$
$$\Rightarrow \dim(\ker \mathrm{L}_{\mathcal{A}}) = 3 - 2 = \underline{1}.$$

Exercise 4 *Let F, G be arbitrary linear maps from a vector space into itself. Is $\mathrm{F} \circ \mathrm{G}$ equal to $\mathrm{G} \circ \mathrm{F}$?* in general

No, as for example $F = L_{\mathcal{A}}$, $G = L_B$ and $F \circ G = L_{\mathcal{A}} L_B = L_{\mathcal{A}B} \neq L_{B\mathcal{A}} = L_B L_{\mathcal{A}} = G \circ F$.

If F is invertible, is $\mathrm{F}^2 = \mathrm{F} \circ \mathrm{F}$ invertible, and if so what is its inverse ?

Yes, F^2 is invertible, with $(F^2)^{-1} = (F \circ F)^{-1} = F^{-1} \circ F^{-1} = (F^{-1})^2$.