Problem Set 9 - Math Tutorial Calculus II

1. The surface $z = 3x^2 + \frac{1}{6}x^3 - \frac{1}{8}x^4 - 4y^2$ is intersected by the plane $2x - y = 1$. The resulting intersection is a curve of the surface. Find a set of parametric equations for the line tangent to this curve at the point $(1, 1, -\frac{23}{24})$.

2. Show that the plane tangent to a sphere at a point P on the sphere is always perpendicular to the vector $\overrightarrow{OP}$ from the center O of the sphere to the point P.

3. Consider the surface $z = f(x, y)$, and a given vector $(a, b) \in \mathbb{R}^2$. What is a tangent vector at $(x_0, y_0, f(x_0, y_0))$ that has (a, b) as its (x, y)-component?

4. Let $f(x, y) = x^2 + y^2$.

   (a) Describe the level curve $f(x, y) = 2$.

   (b) Without calculation, find the directional derivative of f at $(1, 1)$ in the direction $\frac{1}{\sqrt{2}}(-1, 1)$.

   (c) By computation, find the directional derivative at $(1, 1)$ in the direction of $\frac{1}{\sqrt{2}}(-1, 1)$

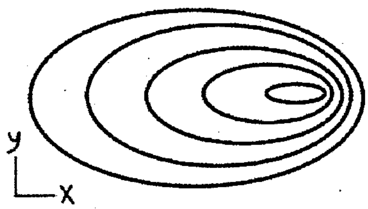
5. In the picture below, equispaced level curves of a function $f(x, y)$ are given.

   (a) Where is ∇f largest in magnitude?

   (b) Where is ∇f smallest in magnitude?

   (c) For what (x, y) is $f_x(x, y)$ equal to 0?

   (d) Where is the directional derivative $D_{(1/\sqrt{2}, 1/\sqrt{2})} f$ equal to 0?



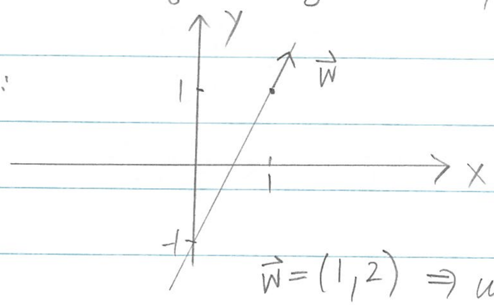
Problem set 9 - Solutions

#1 Intersect $z = 3x^2 + \frac{1}{6}x^3 - \frac{1}{8}x^4 - 4y^2$ with plane $2x - y = 1$, tangent line to the resulting curve at $(1, 1, -\frac{23}{24})$?

Need to find the directional derivative of

$$f(x,y) = 3x^2 + \frac{1}{6}x^3 - \frac{1}{8}x^4 - 4y^2 \quad \text{at } (x,y) = (1,1)$$

in direction:



$2x - y = 1$

$\Leftrightarrow \; y = 2x - 1$

slope $= 2$

$\vec{w} = (1,2) \Rightarrow$ unit vector with same direction $\vec{v} = \frac{1}{\sqrt{5}}(1,2)$

$$(\nabla f)(x,y) = \left(6x + \frac{1}{2}x^2 - \frac{1}{2}x^3, -8y\right)$$

$$\Rightarrow (\nabla f)(1,1) = (6, -8) \quad \Rightarrow \quad (D_{\vec{v}} f)(1,1) = \frac{1}{\sqrt{5}}(6 - 16) = \frac{-10}{\sqrt{5}}$$

$$= -\sqrt{\frac{100}{5}} = -\sqrt{20}$$

$\Rightarrow$ slope of that tangent line (considered in the $2x-y=1$ plane) $= -\sqrt{20}$

parametric equation for the line tangent

$$\vec{l}(t) = \begin{pmatrix} 1 \\ 1 \\ -\frac{23}{24} \end{pmatrix} + t \begin{pmatrix} 1/\sqrt{5} \\ 2/\sqrt{5} \\ -10/\sqrt{5} \end{pmatrix}$$

#2 We may assume that the center of the sphere lies at the origin, so that the sphere is described by the equation

$$x^2 + y^2 + z^2 = r^2 \quad \text{for some } r \in \mathbb{R}.$$

Let $P = (P_1, P_2, P_3)$ be a point on the sphere

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Since $f(x,y,z) = x^2+y^2+z^2$ has as a level set the given sphere it follows that $(\nabla f)(P)$ is normal to the plane tangent to the sphere at P. That is,

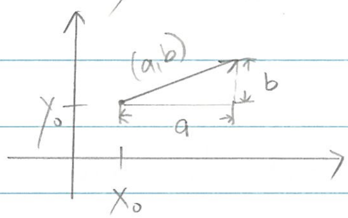
$$(\nabla f)(\vec{x}) = (2x, 2y, 2z)$$

$\Rightarrow$ $(\nabla f)(P) = 2(P_1, P_2, P_3)$ is normal to the tangent plane, meaning (P_1, P_2, P_3) is perpendicular to the tangent plane, but $(P_1, P_2, P_3) = \overrightarrow{OP}$

$\square$

#3 $\quad z = f(x,y)$, $(a,b) \in \mathbb{R}^2$, $(x_0, y_0) \in \mathbb{R}^2$

Need directional derivative



$$\vec{v} = \frac{(a,b)}{\sqrt{a^2+b^2}}$$

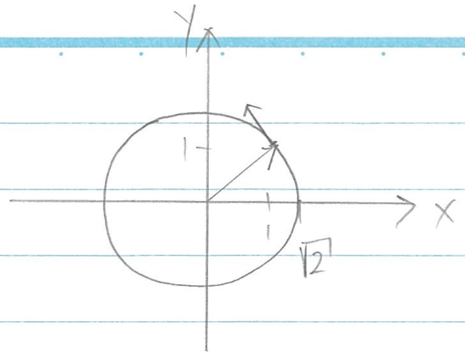
$$(D_{\vec{v}} f)(x_0,y_0) = \frac{1}{\sqrt{a^2+b^2}} \left(a f_x(x_0,y_0) + b f_y(x_0,y_0) \right)$$

$\Rightarrow$ tangent vector:

$$\begin{pmatrix} a/\sqrt{a^2+b^2} \\ b/\sqrt{a^2+b^2} \\ \frac{1}{\sqrt{a^2+b^2}} \left(a f_x(x_0,y_0) + b f_y(x_0,y_0) \right) \end{pmatrix}$$

#4 $\quad f(x,y) = x^2+y^2$

(a) The level curve of $f(x,y)$ of height 2 is a circle of radius $\sqrt{2}$ with center $(0,0)$



(b) The vector $\frac{1}{\sqrt{2}}(-1,1)$ is tangent to the circle at $(1,1)$. That is $\frac{1}{\sqrt{2}}(-1,1) \perp$ to $(1,1)$ (which is normal)

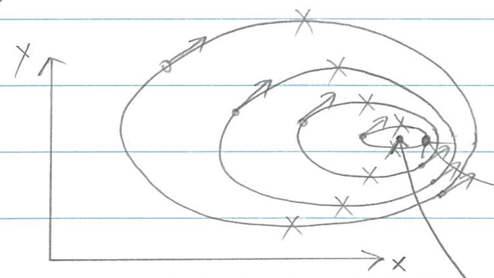
Therefore $D_{\frac{1}{\sqrt{2}}(-1,1)} f(1,1) = 0$.

Because f does not change along this circle.

(c) $(\nabla f)(x,y) = (2x, 2y) \Rightarrow (\nabla f)(x,y) = (2,2)$

$$\Rightarrow \left(D_{\frac{1}{\sqrt{2}}(-1,1)} f\right)(1,1) = \frac{1}{\sqrt{2}}(-1,1) \cdot (2,2) = 0.$$

#5



(a) The directional derivative is largest there where the curves are the closest together

max $|\nabla f|$

(b) $|\nabla f(x)|$ is small if f does not change in any direction.

(c) $f_x(x,y) = 0$ means
f does not change in $(1,0)$ direction at (x,y) : see x-marks

(d) $D_{(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})} f(x,y) = 0$ means
$f(x,y)$ does not change in $\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$ direction : see ↗-points.