# Problem Set 8 - Math Tutorial Calculus II

1. Show that the two surfaces $z = xy$ and $z = \frac{3}{4}x^2 - y^2$ intersect perpendicularly at the point $(2, 1, 2)$.

2. Verify the sum rule for the derivative matrices for $f(x, y) = xy + \cos(x)$ and $g(x, y) = \sin(xy) + y^3$.

3. Verify the product and quotient rules for $f(x, y) = x^2y + y^3$ and $g(x, y) = \frac{x}{y}$.

4. Determine all second-order partial derivatives of $f(x, y) = \frac{1}{\sin^2 x + 2e^y}$.

5. If $f(x, y, z) = x^2 - y^3 + xyz$, and $x = 6t + 7$, $y = \sin(2t)$, $z = t^2$, verify the chain rule by finding $\frac{df}{dt}$ in two different ways.

6. Calculate $D(\vec{f} \circ \vec{g})$ in two ways: (a) by first evaluating $\vec{f} \circ \vec{g}$ and (b) by using chain rule and the derivative matrices $D\vec{f}$ and $D\vec{g}$:

   (i) $\vec{f}(x) = (3x^5, e^{2x})$, $\vec{g}(s, t) = s - 7t$,

   (ii) $\vec{f}(x, y, z) = (x + y + z, x^3 - e^{yz})$, $\vec{g}(s, t, u) = (st, tu, su)$.

7. Find $\frac{dy}{dx}$ when $y$ is defined implicitly by the equation $\sin(xy) - x^2y^7 + e^y = 0$.

8. Find $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$, where $z$ is given implicitly by the equation $x^3z + y\cos(z) + \frac{\sin(y)}{z} = 0$.

Problem set 8

#1 $\quad z = xy \quad , \quad z = \frac{3}{4}x^2 - y^2 \quad , \quad P = (2,1,2)$

Two surfaces intersect perpendicularly at a point if the two tangent planes to the surfaces at that point are perpendicular.

Surface I: $\quad f(x,y) = xy \quad \Rightarrow f_x(x,y) = y, \quad f_y(x,y) = x$

$\Rightarrow f_x(2,1) = 1 \quad , \quad f_y(2,1) = 2$

normal vector to plane tangent to the surface at $(2,1,2)$ is

$\vec{n}_1 = (-1, -2, 1)$

Surface II: $\quad f(x,y) = \frac{3}{4}x^2 - y^2 \Rightarrow f_x(x,y) = \frac{3}{2}x \quad , f_y(x,y) = -2y$

$\Rightarrow f_x(2,1) = 3 \quad , f_y(2,1) = -2$

$\Rightarrow \vec{n}_2 = (-3, +2, 1)$

$\Rightarrow \vec{n}_1 \cdot \vec{n}_2 = (-1,-2,1) \cdot (-3, +2, 1) = +3 - 4 + 1 = 0$

$\Rightarrow$ the two surfaces intersect perpendicularly

#2 $\quad f(x,y) = xy + \cos(x) \quad , g(x,y) = \sin(xy) + y^3$

$Df(x,y) = (y - \sin(x), x) \quad , \quad Dg(x,y) = (y\cos(xy), x\cos(xy) + 3y^2)$

$(Df + Dg)(x,y) = ( y - \sin(x) + y\cos(x,y), \; x + x\cos(xy) + 3y^2 )$ $=$ ✓

$(f+g)(x,y) = xy + \cos(x) + \sin(xy) + y^3$

$\Rightarrow D(f+g)(x,y) = (y - \sin(x) + y\cos(xy), \; x + x\cos(xy) + 3y^2)$

#3 $\quad f(x,y) = x^2y + y^3 \quad , \; g(x,y) = \frac{x}{y}$

$\Rightarrow (Df)(x,y) = (2xy, \; x^2 + 3y^2) \quad , (Dg)(x,y) = \left(\frac{1}{y}, \; -\frac{x}{y^2}\right)$

$(Df(x,y)) \cdot g(x,y) + f(x,y) \cdot Dg(x,y)$

$= (2x^2, \; x^3/y + 3xy) + \left(x^2 + y^2, \; -\frac{x^3}{y} - xy\right)$

$= (3x^2 + y^2, \; +2xy)$ $=$ ✓

$(f \cdot g)(x,y) = x^3 + xy^2$

$\Rightarrow D(f \cdot g)(x,y) = (3x^2 + y^2, \; 2xy)$

$\Rightarrow$ product rule is verified for this example

$$\frac{Df(x,y) \cdot g(x,y) - f(x,y)\, Dg(x,y)}{(g(x,y))^2} = \frac{\left(2x^2, \frac{x^3}{y} + 3xy\right) - \left(x^2 + y^2, -\frac{x^3}{y} - xy\right)}{\frac{x^2}{y^2}}$$

$$= \left(x^2 - y^2, \; 2\frac{x^3}{y} + 4xy\right) \cdot \frac{y^2}{x^2} = \underline{\left(y^2 - \frac{y^4}{x^2}, \; 2xy + \frac{4y^3}{x}\right)}$$

$$\left(\frac{f}{g}\right)(x,y) = xy^2 + \frac{y^4}{x}$$

$$\Rightarrow D\left(\frac{f}{g}\right)(x,y) = \left(y^2 - \frac{y^4}{x^2},\ 2xy + 4\frac{y^3}{x}\right) \quad \checkmark$$

#4 $\quad f(x,y) = \left(\sin^2 x + 2e^y\right)^{-1}$

$$f_x(x,y) = -\left(\sin^2 x + 2e^y\right)^{-2} \cdot 2\sin x \cdot \cos x$$

$$f_y(x,y) = -\left(\sin^2 x + 2e^y\right)^{-2} \cdot 2e^y$$

$$\Rightarrow f_{xx}(x,y) = 2\left(\sin^2 x + 2e^y\right)^{-3} 4\sin^2 x \cdot \cos^2 x - \left(\sin^2 x + 2e^y\right)^{-2} 2\left(\cos^2 x - \sin^2 x\right)$$

$$f_{xy}(x,y) = 2\left(\sin^2 x + 2e^y\right)^{-3} \cdot 2\sin x \cdot \cos x \cdot 2e^y = f_{yx}(x,y)$$

$$f_{yy}(x,y) = 2\left(\sin^2 x + 2e^y\right)^{-3} 4e^{2y} - \left(\sin^2 x + 2e^y\right)^{-2} 2e^y$$

#5 $\quad f(x,y,z) = x^2 - y^3 + xyz, \quad x = 6t+7, \quad y = \sin(2t), \quad z = t^2$

$$\Rightarrow f(x(t), y(t), z(t)) = (6t+7)^2 - \left(\sin(2t)\right)^3 + (6t+7)\sin(2t) \cdot t^2$$

$$\Rightarrow \frac{d}{dt}\left(f(x(t), y(t), z(t))\right) = \underbrace{2(6t+7)\cdot 6}_{(1)} - \underbrace{3\left(\sin(2t)\right)^2 \cdot \cos(2t) \cdot 2}_{(2)} + \underbrace{6\sin(2t)\cdot t^2}_{(3)} + \underbrace{(6t+7)\cos(2t)\cdot 2t^2}_{(4)} + (6t+7)\sin(2t)\cdot 2t$$

$$\frac{d}{dt}\left(f(x,y,z)\circ(x(t),y(t),z(t))\right) = (Df)(x(t),y(t),z(t))\circ D(x(t),y(t),z(t))$$

$$= (Df)(x(t),y(t),z(t))\circ \begin{pmatrix} x'(t)\\ y'(t)\\ z'(t)\end{pmatrix} \quad ; \quad \begin{aligned}&(Df)(x,y,z)\\ &= (2x+yz, -3y^2+xz, xy)\end{aligned}$$

$$= \left(12t+14+\sin(2t)\cdot t^2,\ 3(\sin(2t))^2 + (6t+7)t^2,\ (6t+7)\sin(2t)\right) \circ \begin{pmatrix}6\\ \cos(2t)\cdot 2\\ 2t\end{pmatrix}$$

$$= \underbrace{2(6t+7)\cdot 6}_{(1)} + \underbrace{6\sin(2t)\cdot t^2}_{(3)} \underbrace{- 3(\sin(2t))^2\cos(2t)\cdot 2}_{(2)} + \underbrace{(6t+7)t^2\cdot\cos(2t)\cdot 2}_{(4)}$$

$$+ \underbrace{(6t+7)\sin(2t)\cdot 2t}_{(5)}$$

#6 (i) $\vec{f}(x) = (3x^5, e^{2x})$, $\vec{g}(s,t) = s-7t$ $\Rightarrow$ $\vec{f}\circ\vec{g}: \mathbb{R}^2\to\mathbb{R}^2$

$$(\vec{f}\circ\vec{g})(s,t) = \left(3(s-7t)^5, e^{2(s-7t)}\right)$$

$$\Rightarrow D(\vec{f}\circ\vec{g})(s,t) = \begin{pmatrix}15(s-7t)^4 & 15(s-7t)^4\cdot(-7)\\ 2e^{2s-7t} & -14e^{2(s-7t)}\end{pmatrix}$$

$$(D\vec{f})(g(s,t))\circ \vec{D}g(s,t) = \begin{pmatrix}15x^4\\ 2e^{2x}\end{pmatrix}_{x=g(s,t)} \circ (1, -7)$$

$$= \begin{pmatrix} 15(s-7t)^4 \\ 2e^{2(s-7t)} \end{pmatrix} \circ (1, -7) = \begin{pmatrix} 15(s-7t)^4 & 15(s-7t)^4(-7) \\ 2e^{2(s-7t)} & -14e^{2(s-7t)} \end{pmatrix}$$

(ii) $\vec{f}(x,y,z) = (x+y+z,\ x^3 - e^{yz})$ , $\vec{g}(s,t,u) = (st,\ tu,\ su)$

$$\rightarrow (\vec{f} \circ \vec{g})(s,t,u) = (st + tu + su,\ s^3t^3 - e^{tu^2s})$$

$$\Rightarrow (D(\vec{f} \circ \vec{g}))(s,t,u) = \begin{pmatrix} t+u & s+u & t+s \\ 3s^2t^3 - e^{tu^2s}tu^2 & 3s^3t^2 - e^{tu^2s}u^2s & -e^{tu^2s} \cdot 2tus \end{pmatrix}$$

$$(D\vec{f})\Big|_{(x,y,z)=g(s,t,u)} \circ \vec{D}g(s,t,u)$$

$$= \begin{pmatrix} 1 & 1 & 1 \\ 3x^2 & -e^{yz} \cdot z & -e^{yz} \cdot y \end{pmatrix}_{x,y,z=g(s,t,u)} \circ \begin{pmatrix} t & s & 0 \\ 0 & u & t \\ u & 0 & s \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 1 & 1 \\ 3s^2t^2 & -e^{tu^2s} \cdot su & -e^{tu^2s} \cdot tu \end{pmatrix} \cdot \begin{pmatrix} t & s & 0 \\ 0 & u & t \\ u & 0 & s \end{pmatrix}$$

$$= \begin{pmatrix} t+u & s+u & t+s \\ 3s^2t^3 - e^{tu^2s}tu^2 & 3s^3t^2 - e^{tu^2s}su^2 & -2e^{tu^2s}tus \end{pmatrix}$$

#7 $\sin(xy) - x^2 y^7 + e^y = 0$

$$\xrightarrow{\frac{d}{dx}} \cos(xy)\cdot y + \cos(xy)\cdot x\frac{dy}{dx} - 2xy^7 - x^2\cdot 7y^6\frac{dy}{dx} + e^y\frac{dy}{dx} = 0$$

$$\Rightarrow \frac{dy}{dx}\left(\cos(xy)\cdot x - 7x^2y^6 + e^y\right) = -\cos(xy)\cdot y - 2xy^7$$

$$\Rightarrow \frac{dy}{dx} = \frac{-\cos(xy)\cdot y - 2xy^7}{\cos(xy)\cdot x - 7x^2y^6 + e^y}$$

#8 $x^3 z + y\cos(z) + \frac{\sin(y)}{z} = 0$

$$\Rightarrow 3x^2\cdot z + x^3\frac{\partial z}{\partial x} + -y\sin(z)\frac{\partial z}{\partial x} - \frac{\sin y}{z^2}\frac{\partial z}{\partial x} = 0$$

$$\Rightarrow \frac{\partial z}{\partial x}\left(x^3 - y\sin z - \frac{\sin y}{z^2}\right) = -3x^2z$$

$$\Rightarrow \frac{\partial z}{\partial x} = \frac{-3x^2z}{x^3 - y\sin z - \frac{\sin y}{z^2}}$$

$$x^3\cdot\frac{\partial z}{\partial y} + \cos(z) + -y\sin(z)\frac{dz}{dy} + \frac{\cos(y)}{z} - \frac{\sin(y)}{z^2}\frac{\partial z}{\partial y} = 0$$

$$\Rightarrow \frac{\partial z}{\partial y}\left(x^3 - y\sin(z) - \frac{\sin(y)}{z^2}\right) = -\cos(z) - \frac{\cos(y)}{z}$$

$$\Rightarrow \frac{\partial z}{\partial y} = \frac{-\cos(z) - \frac{\cos(y)}{z}}{x^3 - y\sin(z) - \frac{\sin(y)}{z^2}}$$