Problem Set 6 - Math Tutorial Calculus II

1. Calculate the partial derivatives of the following functions:
   (a) $f(x,y) = \sin(xy) + \cos(xy)$
   (b) $f(x,y) = \frac{x^2-y^2}{x^2+y^2}$
   (c) $F(x,y,z) = \sqrt{x^2+y^3+z^4}$

2. Let $f(x,y) = 1 - 9x^2 - 4y^2$. Find the lines tangent to the curves at the given point described in the following.
   (a) The curve obtained by intersecting the graph of f with the plane $y = 0$, at the point $(0,0,f(0,0))$.
   (b) The curve obtained by intersecting the graph of f with the plane $x = 0$, at the point $(0,0,f(0,0))$.
   (c) The curve obtained by intersecting the graph of f with the plane $y = 1$, at the point $(2,1,f(2,1))$.
   (d) The curve obtained by intersecting the graph of f with the plane $x = 2$, at the point $(2,1,f(2,1))$.

3. Let $f(x,y) = 1 - 9x^2 - 4y^2$. Find equations for the following tangent planes.
   (a) The plane tangent to the graph of f which goes through the point $(0,0,f(0,0))$.
   (b) The plane tangent to the graph of f which goes through the point $(2,1,f(2,1))$.

4. Give an equation for the plane tangent to the graph of $z = x^3 - 7xy - e^y$ at $(-1,0,0)$.

Problem set 6 - Solutions

#1 (a) $f(x,y) = \sin(xy) + \cos(xy)$

$$\Rightarrow f_x(x,y) = \cos(xy)\cdot y - \sin(xy)\cdot y$$

$$f_y(x,y) = \cos(xy)\cdot x - \sin(xy)\cdot x$$

(b) $f(x,y) = \dfrac{x^2-y^2}{x^2+y^2}$ $\quad (x,y) \neq (0,0)$

$$\Rightarrow f_x(x,y) = \frac{2x(x^2+y^2) - (x^2-y^2)\cdot 2x}{(x^2+y^2)^2} = \frac{4xy^2}{(x^2+y^2)^2}$$

$$f_y(x,y) = \frac{-2y(x^2+y^2) - (x^2-y^2)\cdot 2y}{(x^2+y^2)^2} = \frac{-4yx^2}{(x^2+y^2)^2}$$

(c) $F(x,y,z) = \sqrt{x^2+y^3+z^4}$ $\quad (x,y,z) \neq (0,0,0)$

$$\Rightarrow F_x(x,y,z) = (x^2+y^3+z^4)^{-\frac{1}{2}}\cdot x$$

$$F_y(x,y,z) = (x^2+y^3+z^4)^{-\frac{1}{2}}\cdot\frac{3}{2}y^2$$

$$F_z(x,y,z) = (x^2+y^3+z^4)^{-\frac{1}{2}}\cdot 2z^3$$

#2 $f(x,y) = 1 - 9x^2 - 4y^2$

(a) $f_x(x,y) = -18x$ is the slope of the line tangent to the curve obtained by intersecting the graph of f with the plane $y =$ constant

$\Rightarrow f_x(0,0) = 0 \Rightarrow \vec{l_1}(t) = (0,0,1) + t\,(1,0,0)$

(b) Similarly to (a), $f_y(0,0) = 0 \Rightarrow \vec{l_2}(t) = (0,0,1) + t(0,1,0)$

(c) Similarly to (a): $f(2,1) = 1 - 36 - 4 = -39$

$f_x(2,1) = -36$

$\Rightarrow \vec{l_3}(t) = (2,1,-39) + t\,(1,0,-36)$

(d) Similarly to (c): $f(2,1) = -39$

$f_y(x,y) = -8y \Rightarrow f_y(2,1) = -8$

$\Rightarrow \vec{l_4}(t) = (2,1,-39) + t\,(0,1,-8)$

#3 (a) From (a) & (b) of #2 it follows that $z = 0$ is the plane tangent to the graph of f at $(0,0,0)$

(b) From (c) & (d) it follows that:

$(1,0,-36) \times (0,1,-8)$ are normal to the plane tangent to the graph at the point $(2,1,-39)$, and

$$(1,0,-36)\times(0,1,-8)=\begin{vmatrix}\vec{i} & \vec{j} & \vec{k}\\ 1 & 0 & -36\\ 0 & 1 & -8\end{vmatrix}$$

$$=\vec{i}(36)-\vec{j}(-8)+\vec{k}=(36,8,1)$$

$\Rightarrow$ equation for the plane tangent to the graph at the point $(2,1,-39)$

$$\underline{36(x-2)+8(y-1)+(z+39)=0}$$

#4 $z=x^3-7xy-e^y$, $(-1,0,-2)$

$f(x,y)=x^3-7xy-e^y \Rightarrow f_x(x,y)=3x^2-7y \Rightarrow f_x(-1,0)=3$

$f_y(x,y)=-7x-e^y \Rightarrow f_y(-1,0)=7-1=6$

Since the tangent plane is: $-f_x(a,b)(x-a)-f_y(a,b)(y-b)+(z-f(a,b))=0$

it follows: $\underline{-3(x+1)-6y+z+2=0}$