Problem Set 5 - Math Tutorial Calculus II

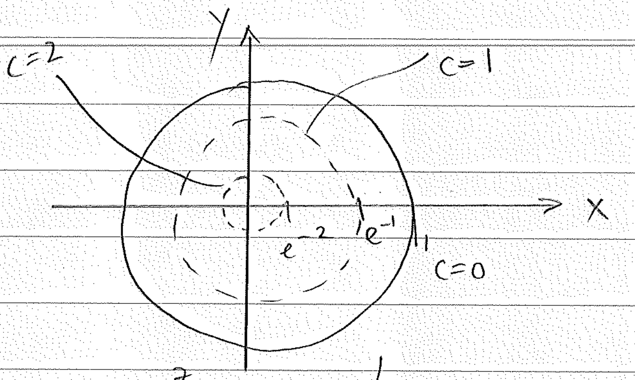
1. Consider the function $f(x,y) = 2 + \ln(x^2 + y^2)$.
   (a) Sketch some level curves and sections of f.
   (b) Use part (a) to give a rough sketch of the graph of $z = f(x,y)$.
2. Consider the function $f(x,y) = \cos\sqrt{x^2 + y^2}$.
   (a) Sketch some level curves and sections of f.
   (b) Use part (a) to give a rough sketch of the graph of $z = f(x,y)$.
3. Consider the function $f(x,y) = \frac{1}{x^2+y^2+4}$.
   (a) Sketch some level curves and sections of f.
   (b) Use part (a) to give a rough sketch of the graph of $z = f(x,y)$.
4. Is $\{(x,y) \in \mathbb{R}^2 : 1 \leq x^2 + y^2 \leq 4\}$ open, closed or neither?
5. Is $\{(x,y) \in \mathbb{R}^2 : -1 < x < 1\} \cup \{(x,y) \in \mathbb{R}^2 : x = 2\}$ open, closed or neither?
6. Evaluate the following limits, or explain why the limit fails to exist.
   (a) $\lim_{(x,y,z)\to(0,0,0)} x^2 + 2xy + yz + z^3 + 2$
   (b) $\lim_{(x,y)\to(0,0)} \frac{(x+y)^2}{x^2+y^2}$
   (c) $\lim_{(x,y)\to(0,0)} \frac{2x^2+y^2}{x^2+y^2}$
   (d) $\lim_{(x,y)\to(0,0)} \frac{x^2+2xy+y^2}{x+y}$
   (e) $\lim_{(x,y)\to(0,0)} \frac{x^4-y^4}{x^2+y^2}$
   (f) $\lim_{(x,y)\to(0,0),x\neq y} \frac{x^2-xy}{\sqrt{x}-\sqrt{y}}$

$f(x,y) = 2 + \ln(x^2+y^2)$

(a) $c=1 \Rightarrow \ln(x^2+y^2) = -1 \Rightarrow x^2+y^2 = e^{-1}$

$c=0 \Rightarrow \ln(x^2+y^2) = -2 \Rightarrow x^2+y^2 = e^{-2}$

$c=2 \Rightarrow \ln(x^2+y^2) = 0 \Rightarrow x^2+y^2 = 1$



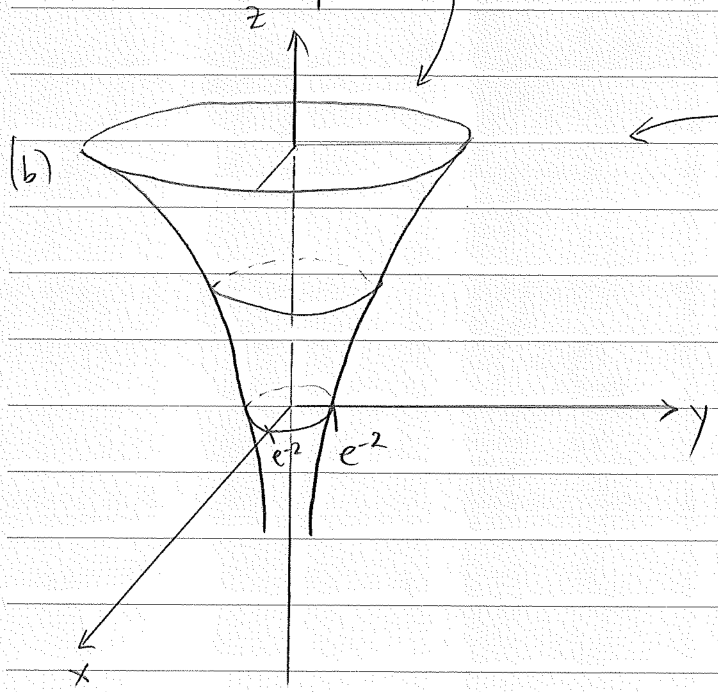
$y=0 \Rightarrow z = 2+\ln x^2 = 2\ln|x| + 2$

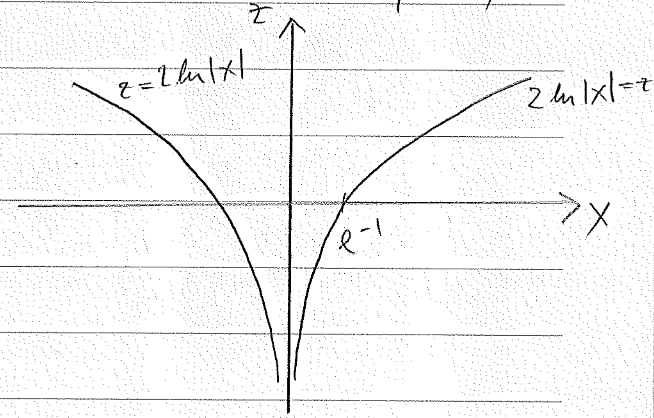
$y=1 \Rightarrow z = \ln(x^2+1)+2$

$x=0 \rightarrow z = 2\ln|y| + 2$

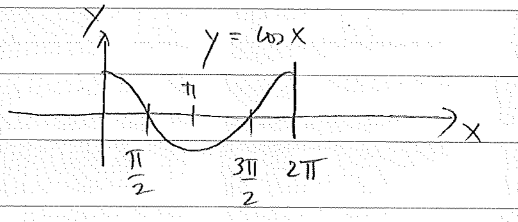
$x=1 \Rightarrow z = \ln(y^2+1)+2$

(b)





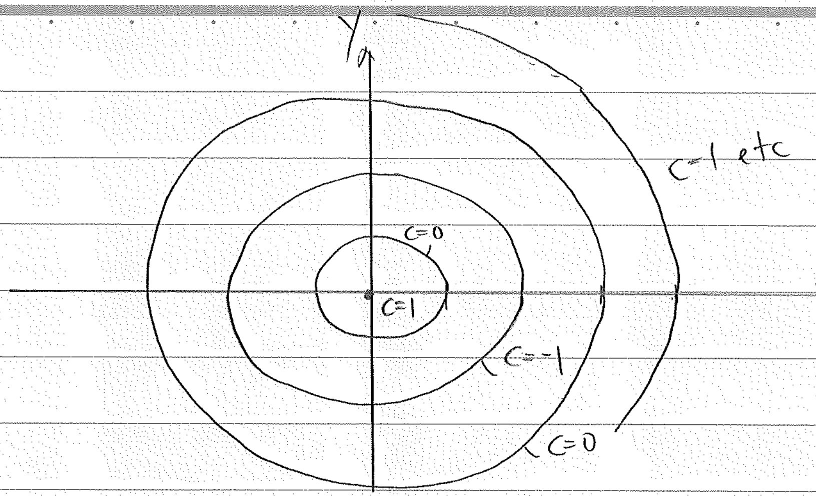
‖ $f(x,y) = \cos\left((x^2+y^2)^{1/2}\right)$



(a) $c=0 \Rightarrow (x^2+y^2)^{1/2} = \frac{\pi}{2} + n\pi \quad n \in \mathbb{Z}$

$c=1 \Rightarrow (x^2+y^2)^{1/2} = 2\pi n \quad n \in \mathbb{Z}$

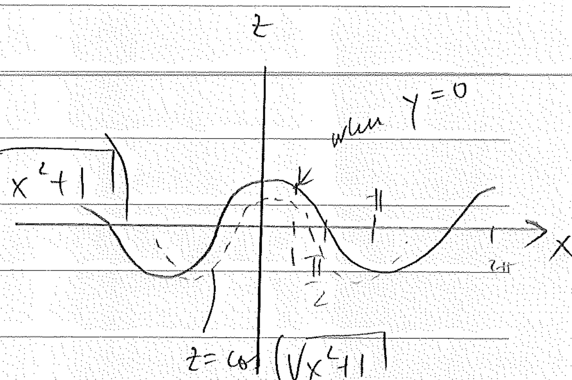
$c=-1 \Rightarrow (x^2+y^2)^{1/2} = \pi + 2\pi n \quad n \in \mathbb{Z}$



Some level sets

Some sections: $y=0 \Rightarrow z=\cos|x|$

$y=1 \Rightarrow z=\cos\left|\sqrt{x^2+1}\right|$



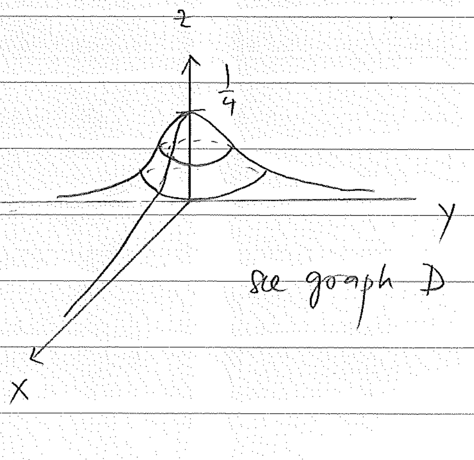
it looks similar to the surface of water after throwing a stone in, see pg. figure B on pg. 172 (this is for sin instead of cos)

$$f(x,y)=\frac{1}{x^2+y^2+4}$$

(a) level curves at height $c>0$ are circles: $c=f(x,y)$

$\Rightarrow x^2+y^2+4=\frac{1}{c}$

(b)



$\Rightarrow x^2+y^2=\frac{1}{c}-4 \Rightarrow$

necessarily $\frac{1}{c}-4\geq 0$

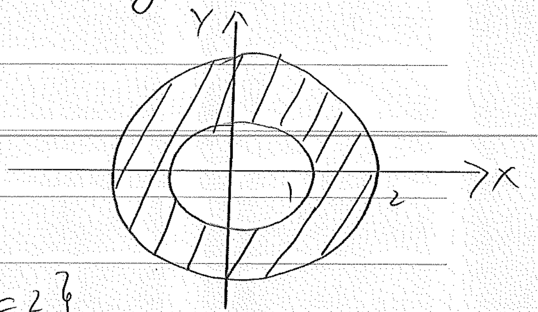
$\Rightarrow \frac{1}{c}\geq 4$

$\Rightarrow c\leq\frac{1}{4}$

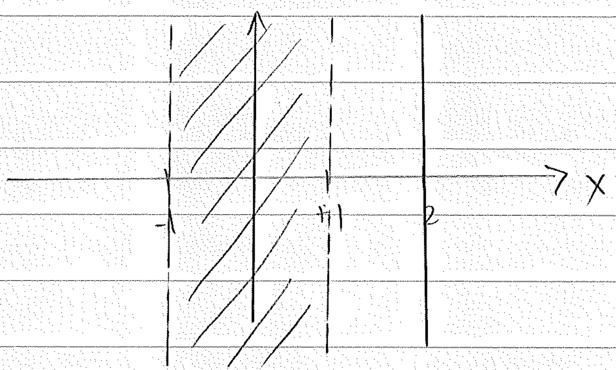
see graph D on pg. 172

#4 $\{(x,y)\in \mathbb{R}^2 : 1\le x^2+y^2\le 4\}$ is closed,

since $x^2+y^2=1$, $x^2+y^2=4$ is the boundary of the set, and all points on the boundary belong to the set



#5 $\{(x,y)\in\mathbb{R}^2 : -1<x<1\} \cup \{(x,y)\in\mathbb{R}^2 : x=2\}$



Is neither open nor closed.

The point: $(2,0)$ is in the boundary and belongs to the set $\Rightarrow$ the set is not open.

The point $(1,0)$ is in the boundary of the set, but does not belong to the set

#6 (a) $\lim_{(x,y,z)\to\vec{0}} x^2+2xy+yz+z^3+2=0$

(b) $\lim_{(x,y)\to(0,0)} \frac{(x+y)^2}{x^2+y^2}$: if $x=-y \Rightarrow \frac{(x+y)^2}{x^2+y^2}=0$

if $x=0, y\to 0$ $\frac{(x+y)^2}{x^2+y^2}=\frac{y^2}{y^2}=1$

$\Rightarrow$ lim DNE

(c) $\lim_{(x,y)\to(0,0)} \frac{2x^2+y^2}{x^2+y^2}$: if $x=0, y\to 0 \Rightarrow \frac{2x^2+y^2}{x^2+y^2}=1$

if $x\to 0, y=0 \Rightarrow \frac{2x^2+y^2}{x^2+y^2}=2$

$\Rightarrow$ lim DNE

(d) $\lim_{(x,y)\to(0,0)} \frac{x^2+2xy+y^2}{x+y} = \lim_{(x,y)\to(0,0)} \frac{(x+y)^2}{x+y} = \lim_{(x,y)\to(0,0)} x+y = 0$

(e) $\lim_{(x,y)\to(0,0)} \frac{x^4-y^4}{x^2+y^2} = \lim_{(x,y)\to(0,0)} \frac{(x^2+y^2)(x^2-y^2)}{(x^2+y^2)} = \lim_{(x,y)\to(0,0)} x^2-y^2=0$

(f) $\lim_{\substack{(x,y)\to(0,0)\\ x\neq y}} \frac{x^2-xy}{\sqrt{x}-\sqrt{y}} = \lim \frac{x(x-y)}{\sqrt{x}-\sqrt{y}} = \lim \frac{x(\sqrt{x}-\sqrt{y})(\sqrt{x}+\sqrt{y})}{\sqrt{x}-\sqrt{y}}$

$= \lim_{\substack{(x,y)\to(0,0)\\ x\neq y.}} x(\sqrt{x}+\sqrt{y}) = 0$