Problem Set 3 - Math Tutorial Calculus II

0. Let $\vec{a}$, $\vec{b}$ and $\vec{c}$ be vectors in $\mathbb{R}^3$. Does $\vec{a} \times (\vec{b} \times \vec{c}) = (\vec{a} \times \vec{b}) \times \vec{c}$ hold? What does your result imply for the term $\vec{a} \times \vec{b} \times \vec{c}$?

1. Find an equation for the plane that is perpendicular to the line $x = 3t - 5$, $y = 7 - 2t$, $z = 8 - t$ and that passes through the point $(1, -1, 2)$.

2. Find a value for A so that the planes $8x - 6y + 9Az = 6$ and $Ax + y + 2z = 3$ are parallel.

3. Find the distance between the point $(-11, 10, 20)$ and the line ℓ: $x = 5 - t$, $y = 3$, $z = 7t + 8$.

4. Determine the distance between the two lines $\vec{\ell_1}(t) = t(8, -1, 0) + (-1, 3, 5)$ and $\vec{\ell_2}(t) = t(0, 3, 1) + (0, 3, 4)$.

5. Show that the distance d between the two parallel planes determined by the equations $Ax + By + Cz = D_1$ and $Ax + By + Cz = D_2$ is

$$d = \frac{|D_1 - D_2|}{\sqrt{A^2 + B^2 + C^2}}.$$

6. Find the Cartesian coordinates of the points $(\sqrt{3}, 5\pi/6)$ and $(2, -3\pi/4)$ (which are given in polar coordinates).

7. Find the polar coordinates of the points $(2\sqrt{3}, 2)$ and $(-1, -2)$ (which are given in Cartesian coordinates).

8. Draw the graphs of the following curves in a cartesian coordinates system.

(i) $r^2 = \cos(2\theta)$

(ii) $r = 1 + \cos\theta$

(iii) $r = \cos(3\theta)$

(iv) $r = \cos(2\theta)$.

Problem set 3 - Solutions

#0 Let $\vec{a} = \vec{i} = \vec{b}$, $\vec{c} = \vec{j}$, then

$\Rightarrow$ $\vec{a} \times (\vec{b} \times \vec{c}) = \vec{i} \times (\vec{i} \times \vec{j}) = \vec{i} \times \vec{k} = -\vec{j}$, but
$(\vec{a} \times \vec{b}) \times \vec{c} = (\vec{i} \times \vec{i}) \times \vec{j} = \vec{0} \times \vec{j} = 0$

$\Rightarrow$ ~~$\vec{a} \times (\vec{b} \times \vec{c}) = (\vec{a} \times \vec{b}) \times \vec{c}$ does not hold $\forall \vec{a}, \vec{b}, \vec{c} \in \mathbb{R}^3$~~.

This means that $\vec{a} \times \vec{b} \times \vec{c}$ is not well-defined.

#1 $L = \begin{cases} x = 3t - 5 \\ y = 7 - 2t \\ z = 8 - t \end{cases}$; $P = (1, -1, 2)$

Eqn. for plane Π which perpendicular to L and contains P:

$$\underline{3(x-1) - 2(y+1) - (z-2) = 0}$$

#2 For what A are the planes given by
$8x - 6y + 9Az = 6$
$Ax + y + 2z = 3$
parallel?

Need to find $A \in \mathbb{R}$ with: $(8, -6, 9A) = c(A, 1, 2)$
for some $c \in \mathbb{R}$

$\Rightarrow$ $8 = cA$, $-6 = c$, $9A = 2c$

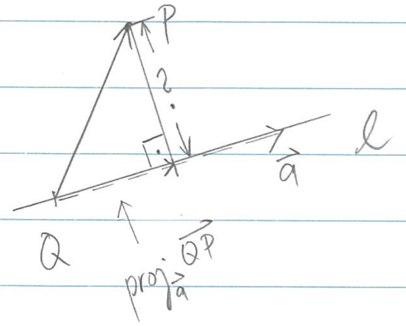
$\Downarrow$ $\Downarrow$

$8 = -6A \Rightarrow A = -\frac{8}{6} \Rightarrow 9A = -\frac{72}{6} = -12$

$2c = -12$ // ✓

$\Rightarrow$ for $A = -\frac{4}{3}$ the 2 planes are parallel.

#3 Distance d between $P(-11,10,20)$ and $l: \begin{cases} x = 5 - t \\ y = 3 \\ z = 7t + 8 \end{cases}$?



$Q = (5,3,8) \Rightarrow \overrightarrow{QP} = (-16, 7, 12)$

$\vec{a} = (-1, 0, 7)$

$$d = \left(\|\overrightarrow{QP}\|^2 - \|\mathrm{proj}_{\vec{a}} \overrightarrow{QP}\|^2 \right)^{1/2}$$

$\Rightarrow \|\overrightarrow{QP}\|^2 = 256 + 49 + 144 = 449$

$$\|\mathrm{proj}_{\vec{a}} \overrightarrow{QP}\| = \frac{|\vec{a} \circ \overrightarrow{QP}|}{\|\vec{a}\|} = \frac{16 + 84}{\sqrt{50}} = \frac{100}{\sqrt{50}} \Rightarrow \|\mathrm{proj}_{\vec{a}} \overrightarrow{QP}\| = \frac{100\cancel{0}\cancel{0}}{5\cancel{0}} = 200$$

$\Rightarrow d = (249)^{1/2}$

#4 $l_1: \begin{cases} x = -1 + t \cdot 8 \\ y = 3 + t \cdot (-1) \\ z = 5 \end{cases}$, $l_2: \begin{cases} x = 0 \\ y = 3 + 3t \\ z = 4 + t \end{cases}$

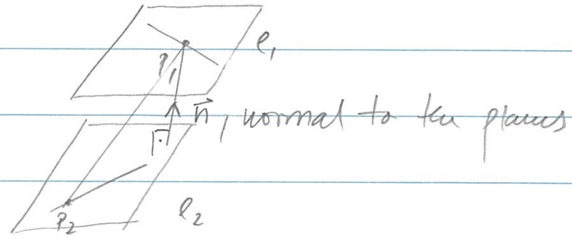
$l_1 \parallel (8,-1,0)$, $l_2 \parallel (0,3,1)$ $\Rightarrow$ since $a_1(8,-1,0) + a_2(0,3,1) = 0$ only for $a_1 = 0 = a_2$ it follows that the two vectors are linearly independent

$\Rightarrow l_1 \nparallel l_2$

Do the lines intersect? $\begin{array}{l} -1 + 8t = 0 \\ 3 - t = 3 + 3s \\ 5 = 4 + s \end{array}$ for some $s, t \in \mathbb{R}$?

$\Rightarrow \begin{array}{l} t = -\frac{1}{8} \\ \left.\begin{array}{l} -t = 3s \\ s = 1 \end{array}\right\} \Rightarrow t = -3 \neq -\frac{1}{8} \end{array}$ $\Rightarrow$ no solutions s, t $\Rightarrow$ the lines do not intersect

$\Rightarrow$ the lines are skew



$$P_1 = (-1,3,5), \quad P_2 = (0,3,4), \quad \vec{n} = (8,-1,0) \times (0,3,1)$$

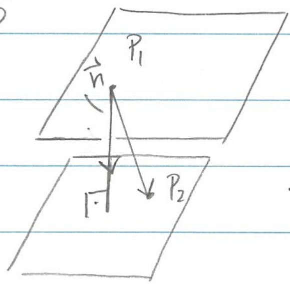
$$\to \text{distance } d(\ell_1,\ell_2) = \left\| \text{proj}_{\vec{n}} \overrightarrow{P_2P_1} \right\|$$

$$\overrightarrow{P_2P_1} = (-1,0,1), \quad \vec{n} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 8 & -1 & 0 \\ 0 & 3 & 1 \end{vmatrix} = \vec{i}(-1) - \vec{j}\cdot 8 + \vec{k}\cdot 24$$

$$= (-1,-8,24)$$

$$\Rightarrow d(\ell_1,\ell_2) = \frac{|\overrightarrow{P_2P_1} \circ \vec{n}|}{\|\vec{n}\|} = \frac{|1+24|}{(1+64+\underbrace{24^2}_{480+80+16=576})^{1/2}} = \boxed{\frac{25}{(641)^{1/2}}}$$

#5



$$\Pi_1 : Ax + By + Cz = D_1, \quad \vec{n} = (A,B,C)$$

$$\Pi_2 : Ax + By + Cz = D_2$$

Suppose $A \neq 0 \Rightarrow$ $P_1 = \left(\frac{D_1}{A}, 0, 0\right)$ is on Π_1

$P_2 = \left(\frac{D_2}{A}, 0, 0\right)$ is on Π_2 $\Big\} \Rightarrow \overrightarrow{P_1P_2} = \left(\frac{D_2-D_1}{A}, 0, \right.$

$$\text{dist}(\Pi_1,\Pi_2) = \left\| \text{proj}_{\vec{n}} \overrightarrow{P_1P_2} \right\| = \frac{|\overrightarrow{P_1P_2} \circ \vec{n}|}{\|\vec{n}\|} = \frac{|D_2 - D_1|}{\sqrt{A^2+B^2+C^2}}$$

If $A = 0$, then either B or C not equal to zero;

If $B \neq 0 \Rightarrow P_i = \left(0, \frac{D_i}{B}, 0\right)$; If $\overset{B=0}{C \neq 0} \Rightarrow P_i = \left(0, 0, \frac{D_i}{C}\right)$

... argument the same as in the case $A \neq 0$.

#6 $(r,\theta) = \left(\sqrt{3}, 5\pi/6\right) \Rightarrow x = \sqrt{3}\cos\left(\frac{5\pi}{6}\right) = -\sqrt{3}\cdot\frac{\sqrt{3}}{2} = -\frac{3}{2}$

$y = \sqrt{3}\sin\left(\frac{5\pi}{6}\right) = \frac{1}{2}$

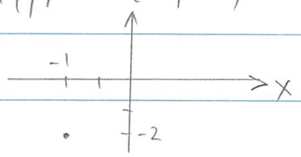
$(r,\theta) = \left(2, -\frac{3\pi}{4}\right) \Rightarrow x = 2\cos\left(-\frac{3\pi}{4}\right) = -1$

$y = 2\sin\left(-\frac{3\pi}{4}\right) = -2\cdot\sqrt{\frac{1}{2}}$

#7 $(x,y) = (2\sqrt{3}, 2) \Rightarrow r = (4\cdot 3 + 4)^{1/2} = 4$

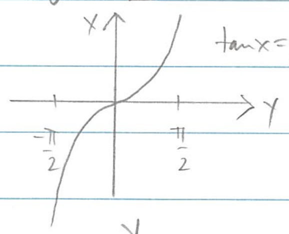
$\tan\theta = \frac{1}{\sqrt{3}} \Rightarrow \theta = \frac{\pi}{6}$

$(x,y) = (-1,-2) \Rightarrow r = \sqrt{5}$



$\Rightarrow \theta \in \left(\pi, \frac{3\pi}{2}\right)$

$\tan\theta = 2 \Rightarrow \theta = \tan^{-1}(2) + \pi$



$\Rightarrow \tan^{-1}(2) \in \left(0, \frac{\pi}{2}\right)$

#8 (i) $r^2 = \cos(2\theta)$

$\Rightarrow \theta \in \left[0, \frac{\pi}{4}\right] \cup \left[\frac{3\pi}{4}, \pi\right]$

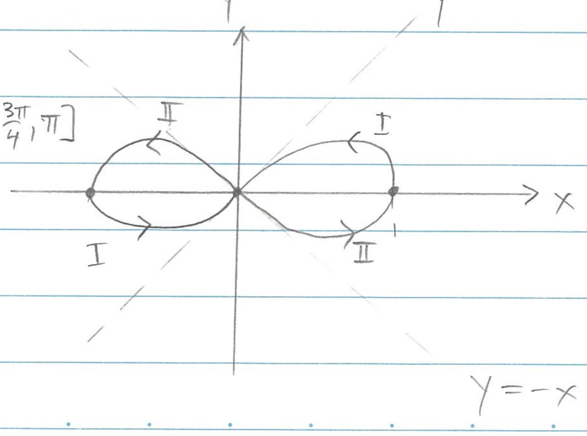
$\theta = 0 \Rightarrow r = \pm 1$

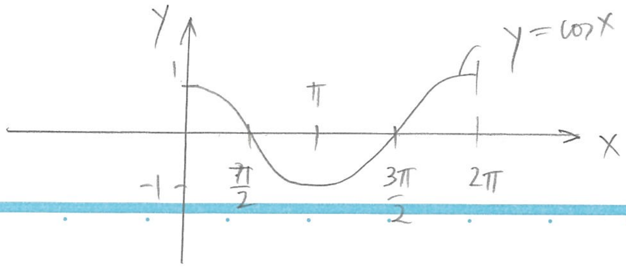
$\theta = \frac{\pi}{4} \Rightarrow r = 0$

$\theta = \frac{3\pi}{4} \Rightarrow r = 0$, $\theta = \pi \Rightarrow r = \pm 1$

(Lemniscate)

$\left(\theta = \frac{\pi}{4}\right)$





(ii) $r = 1 + \cos\theta$

$\theta = 0 \Rightarrow r = 2$

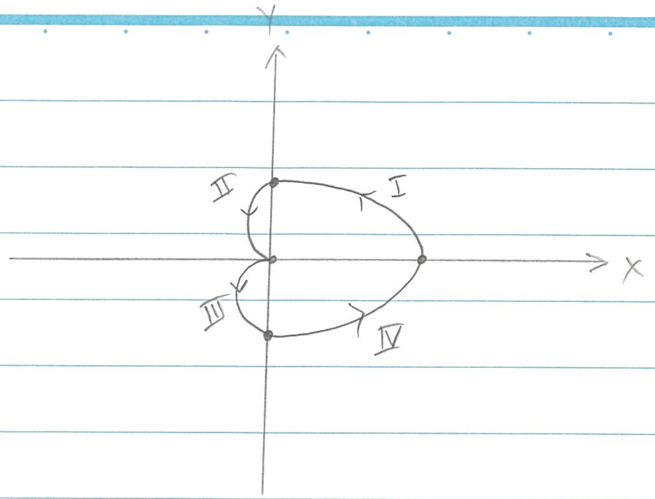
$\theta = \frac{\pi}{2} \Rightarrow r = 1$

$\theta = \pi \Rightarrow r = 0$

$\theta = \frac{3\pi}{2} \Rightarrow r = 1$

$\theta = 2\pi \Rightarrow r = 2$

(cardioid)



(iii) $r = \cos(3\theta)$

$\Rightarrow \theta = 0 \Rightarrow r = 1$

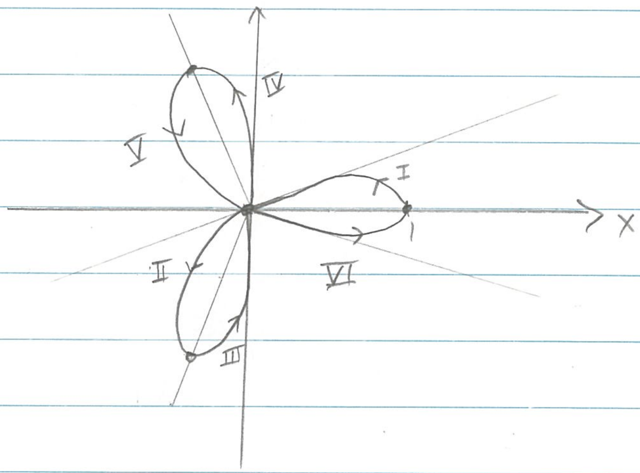
$\theta = \frac{\pi}{6} \Rightarrow r = 0$

$\theta = \frac{\pi}{3} \Rightarrow r = -1$

$\theta = \frac{\pi}{2} \Rightarrow r = 0$

$\theta = \frac{2\pi}{3} \Rightarrow r = 1$

$\cdots$



(iv) $r = \cos(2\theta)$

$\theta = 0 \Rightarrow r = 1$

$\theta = \frac{\pi}{4} \Rightarrow r = 0$

$\theta = \frac{\pi}{2} \Rightarrow r = -1$

$\theta = \frac{3\pi}{4} \Rightarrow r = 0$

$\theta = \pi \Rightarrow r = 1$