Problem Set 2 - Math Tutorial Calculus II

1. Compute $\vec{a} \circ \vec{b}$, $\|\vec{a}\|$ and $\|\vec{b}\|$ for

   (i) $\vec{a} = (4, -1)$ and $\vec{b} = (\frac{1}{2}, 2)$,

   (ii) $\vec{a} = \vec{i} + 2\vec{j} - \vec{k}$ and $\vec{b} = 2\vec{i} - 3\vec{j} + 2\vec{k}$.

2. Find the angle between $\vec{a} = \vec{i} + \vec{j} - \vec{k}$ and $\vec{b} = -\vec{i} + 2\vec{j} + 2\vec{k}$.

3. Calculate $proj_{\vec{a}}\vec{b}$ and $proj_{\vec{b}}\vec{a}$ for $\vec{a} = \vec{i} + \vec{j}$ and $\vec{b} = 2\vec{i} + 3\vec{j} - \vec{k}$.

4. Use vectors to show that the diagonals of a parallelogram have the same length if and only if the parallelogram is a rectangle.

5. Evaluate the determinants

   (i)

$$\begin{vmatrix} 1 & 5 \\ 2 & 2 \end{vmatrix}.$$

   (ii)

$$\begin{vmatrix} -2 & 0 & \frac{1}{2} \\ 3 & 6 & -1 \\ 4 & -8 & 2 \end{vmatrix}.$$

6. Calculate $(3\vec{i} - 2\vec{j} + \vec{k}) \times (\vec{i} + \vec{j} + \vec{k})$.

7. Find the area of the triangle having vertices $(1, 1)$, $(-1, 2)$ and $(-2, -1)$.

8. For given vectors $\vec{a}$, $\vec{b}$ and $\vec{c}$ in $\mathbb{R}^3$ find expressions for the following vectors:

   (a) A vector orthogonal to $\vec{a}$ and $\vec{b}$.

   (b) A vector of length 2 orthogonal to $\vec{a}$ and $\vec{b}$.

   (c) The vector projection of $\vec{b}$ onto $\vec{a}$.

   (d) A vector with the length of $\vec{b}$ and the direction of $\vec{a}$.

   (e) A vector orthogonal to $\vec{a}$ and $\vec{b} \times \vec{c}$.

   (f) A vector in the plane determined by $\vec{a}$, $\vec{b}$ and perpendicular to $\vec{c}$.

Problem set 2 – Solutions

#1. (i) $\vec{a} = (4,-1)$, $\vec{b} = \left(\frac{1}{2}, 2\right) \Rightarrow \vec{a} \circ \vec{b} = 2-2 = 0 \Rightarrow \vec{a} \perp \vec{b}$

$\|\vec{a}\| = \sqrt{17}$, $\|\vec{b}\| = \frac{\sqrt{17}}{2}$

(ii) $\vec{a} = \vec{i} + 2\vec{j} - \vec{k}$, $\vec{b} = 2\vec{i} - 3\vec{j} + 2\vec{k} \Rightarrow \vec{a} \circ \vec{b} = 2 - 6 - 2 = -6$,

$\|\vec{a}\| = \sqrt{6}$, $\|\vec{b}\| = \sqrt{17}$

#2 $\vec{a} = \vec{i} + \vec{j} - \vec{k}$, $\vec{b} = -\vec{i} + 2\vec{j} + 2\vec{k}$;

$\cos\alpha = \dfrac{\vec{a} \circ \vec{b}}{\|\vec{a}\| \cdot \|\vec{b}\|}$; $\|\vec{a}\| = \sqrt{3}$, $\|\vec{b}\| = \sqrt{9}$, $\vec{a} \circ \vec{b} = -1 + 2 - 2 = -1$

$\Rightarrow \cos\alpha = -\dfrac{1}{3\sqrt{3}} \Rightarrow \underline{\alpha = \arccos\left(-\frac{1}{3\sqrt{3}}\right)}$

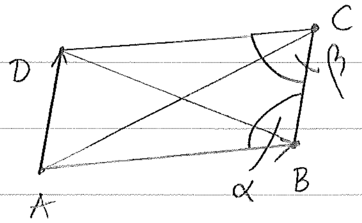
#3 $\vec{a} = \vec{i} + \vec{j}$, $\vec{b} = 2\vec{i} + 3\vec{j} - \vec{k} \Rightarrow \|\vec{a}\|^2 = 2$, $\|\vec{b}\|^2 = 4 + 9 + 1 = 14$

$\vec{a} \cdot \vec{b} = 5$

$\mathrm{proj}_{\vec{b}}\, \vec{a} = \dfrac{\vec{a} \circ \vec{b}}{\|b\|^2}\, \vec{b} = \underline{\frac{5}{14}(2, 3, -1)}$

$\mathrm{proj}_{\vec{a}}\, \vec{b} = \dfrac{\vec{a} \circ \vec{b}}{\|\vec{a}\|^2}\, \vec{a} = \underline{\frac{5}{2}(1, 1, 0)}$

#4



Claim: $\vec{AB} \circ \vec{CD} = 0 \Leftrightarrow \|\vec{AC}\|^2 = \|\vec{BD}\|^2$

$\vec{AC} = \vec{AB} + \vec{BC}$, $\vec{BD} = \vec{BC} + \vec{CD}$

$\Rightarrow \|\vec{AC}\|^2 = \|\vec{AB}\|^2 + 2\, \vec{AB} \circ \vec{BC} + \|\vec{BC}\|^2$

$\|\vec{BD}\|^2 = \|\vec{BC}\|^2 + 2\, \vec{BC} \circ \vec{CD} + \|\vec{CD}\|$

Since ABCD is assumed to be a parallelogram $\|\vec{CD}\| = \|\vec{AB}\|$

$$\Rightarrow \|\overrightarrow{AC}\|^2 = \|\overrightarrow{BD}\|^2 \Leftrightarrow \quad \overrightarrow{AB}\circ\overrightarrow{BC} = \overrightarrow{BC}\circ\overrightarrow{CD}$$

$$\Leftrightarrow \|\overrightarrow{AB}\|\cdot\|\overrightarrow{BC}\|\cos\alpha = \|\overrightarrow{BC}\|\cdot\|\overrightarrow{CD}\|\cdot\cos\beta$$

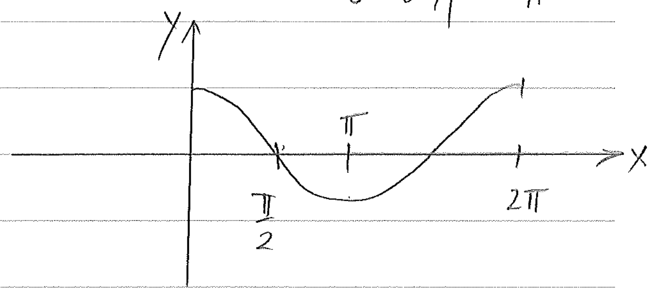
$$\Leftrightarrow \quad \cos\alpha = \cos\beta \qquad (*)$$

Since ABCD is a parallelogram it follows that $\alpha+\beta=\pi$
$0<\alpha,\beta<\pi$

$(*)$ $\cos\alpha = \cos(\pi-\alpha)$

$\Leftrightarrow \quad \alpha = \frac{\pi}{2} = \beta$

$\Rightarrow$ ABCD is a rectangle.



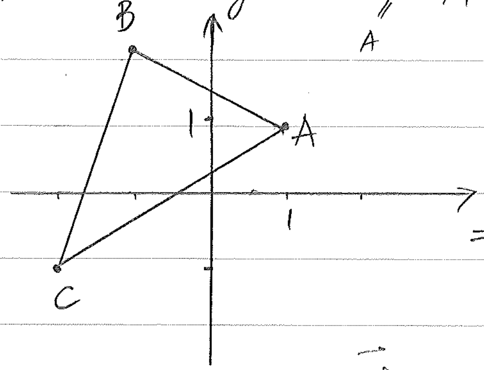
#5 (i) $\begin{vmatrix} 1 & 5 \\ 2 & 2 \end{vmatrix} = 2-10 = \underline{-8}$

(ii) $\begin{vmatrix} -2 & 0 & \frac{1}{2} \\ 3 & 6 & -1 \\ 4 & -8 & 2 \end{vmatrix} = -2(12-8) - 0 + \frac{1}{2}(-24-24) = -8-24$
$= \underline{-32}$

#6 $(3\vec{i}-2\vec{j}+\vec{k})\times(\vec{i}+\vec{j}+\vec{k}) = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 3 & -2 & 1 \\ 1 & 1 & 1 \end{vmatrix}$

$= \vec{i}(-2-1) - \vec{j}(3-1) + \vec{k}(3+2) = \underline{-3\vec{i}-2\vec{j}+5\vec{k}}$

#7 triangle $\underset{A}{(1,1)}, \overset{B}{(-1,2)}, \underset{C}{(-2,-1)}$



$\overrightarrow{CA} = (3,2), \quad \overrightarrow{CB} = (1,3)$

$\Rightarrow$ area of triangle $= \frac{1}{2} \| \overrightarrow{CA} \times \overrightarrow{CB} \|$;

$$\overrightarrow{CA} \times \overrightarrow{CB} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 3 & 2 & 0 \\ 1 & 3 & 0 \end{vmatrix} = \vec{k}\,(9-2) = 7\vec{k}$$

$\Rightarrow$ area of triangle $= \frac{7}{2}$

#8 (a) $\vec{a} \times \vec{b}$ is orthogonal to both $\vec{a}$ and $\vec{b}$

(b) $2 \dfrac{\vec{a} \times \vec{b}}{\|\vec{a} \times \vec{b}\|}$ is of length 2 and orthogonal to $\vec{a}$ and $\vec{b}$

(c) $\mathrm{proj}_{\vec{a}} \vec{b} = \dfrac{\vec{a} \circ \vec{b}}{\|\vec{a}\|^2} \vec{a}$

(d) $\|\vec{b}\| \cdot \dfrac{\vec{a}}{\|\vec{a}\|}$

(e) $\vec{a} \times (\vec{b} \times \vec{c})$

(f) $\vec{v} = s\vec{a} + t\vec{b}$ ← for some scaler s and t such that $\vec{v} \circ \vec{c} = 0$

$\Leftrightarrow \quad s\,\vec{a} \circ \vec{c} = -\,t\,\vec{b} \circ \vec{c}$

$\Rightarrow$ if $\vec{b} \circ \vec{c} = 0 \Rightarrow t$ arbitrary and $s = 0$

if $\vec{b} \circ \vec{c} \neq 0$, then choose $s = 1$ and $t = -\dfrac{\vec{a} \circ \vec{c}}{\vec{b} \circ \vec{c}}$