Problem Set 13 - Math Tutorial Calculus II

1.(a) Let f be a continuous function of one variable. Show that if f has tow local maxima, then f must also have a local minimum.

(b) The analogue of part (a) does not necessarily hold for continuous functions of more than one variable. Consider the function

$$f(x,y) = 2 - (xy^2 - y - 1)^2 - (y^2 - 1)^2. \quad \leftarrow \text{typo!}$$

Show that f has just two critical points - and that both of them are local maxima.

(c) Draw the graph of a function of two variables which has two local maxima but not local minimum.

2. Show that the largest rectangular box having a fixed surface area must be a cube.

3. True or false?

(a) Every rectangle in $\mathbb{R}^2$ may be denoted by $[a, b] \times [c, d]$.

(b) $\int_0^2 \int_0^x 3\, dydx = \int_0^2 \int_0^y 3\, dxdy$.

(c) $\int_{-1}^1 \int_0^3 x^2 e^{x+y}\, dydx = \left(\int_{-1}^1 x^2 e^x\, dx\right)\left(\int_0^3 e^y\, dy\right)$

(d) The region in $\mathbb{R}^2$ bounded by the graphs of $y = x^3$ and $y = \sqrt{x}$ is a type 3 elementary region in the plane.

(e) The increment $\triangle f$ of a function $f(x, y)$ measure the change in the z-coordinate of the tangent plane of the graph of f.

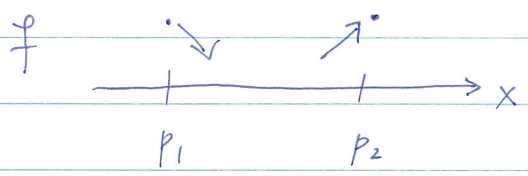
(f) If $\det Hf(\vec{a}) = 0$, then f has a saddle point.

(g) Let f be a differentiable function of two variables. The slope of the line tangent to the curve obtained when intersecting the graph of f with the plane $x + y = 0$ at the point $(a, -a, f(a, -a))$ equals the directional derivative of f at $(a, -a, f(a, -a))$ in the direction $\vec{i} + \vec{j}$.

Problem set 13 - Solutions

#1 (a) $f: \mathbb{R} \to \mathbb{R}$ cont. Suppose f has two local maxima but not a local minimum.

Let p_1, p_2 be the two points at which f attains the two local maxima. Suppose $p_1 < p_2$. Then $f(x)$ is decreasing for $x > p_1$ near p_1 and increasing for $x < p_2$ near p_2.



That means that at some point $p_0 \in [p_1, p_2]$ f must change from decreasing to increasing. Since f is cont. this implies that f attains a loc. min. at p_0.

(b) $f(x,y) = 2 - (xy^2 - y - 1)^2 - (y^2 - 1)^2$

$\Rightarrow f_x(x,y) = -2(xy^2 - y - 1) \cdot y^2$

$f_y(x,y) = -2(xy^2 - y - 1)(2xy - 1) - 2(y^2 - 1) \cdot 2y$

$f_x(x,y) = 0 \Rightarrow$ either $y = 0$ or $xy^2 - y - 1 = 0$.

If $y = 0$, then $f_y(x,y) = 2(-1) \neq 0 \Rightarrow$ does not lead to a critical point.

If $xy^2 - y - 1 = 0$, then

$f_y(x,y) = 0$ if $y^2 - 1 = 0$, i.e. $y = \pm 1$

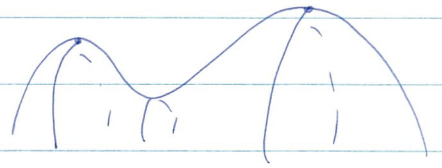
$\Rightarrow$ either $x - 1 - 1 = 0 \Rightarrow x = 2$

or $x + 1 - 1 = 0 \Rightarrow x = 0$

$\Rightarrow (2,1)$ and $(0,-1)$ are critical points.

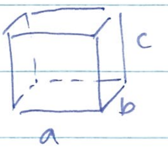
check that they are loc. max!

(c)



two local maxima,
no local minimum

#2



$V = a \cdot b \cdot c$ maximize

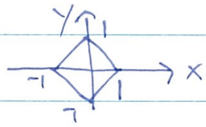
$A = 2ab + 2ac + 2bc$ area = fixed number

$$\Rightarrow c = \left(\frac{A}{2} - ab\right) \cdot \frac{1}{(a+b)}$$

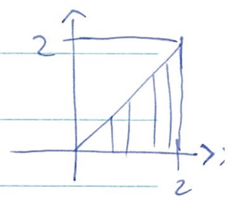
$$\Rightarrow V = a \cdot b \cdot \left(\frac{A}{2} - ab\right) \frac{1}{(a+b)}$$

↑ Find maximum by first finding local max...

#3 (a) False :



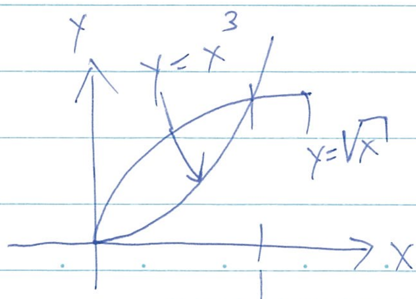
(b) False : $\displaystyle\int_0^2 \int_0^x 3\, dy\, dx = \int_0^2 \int_y^2 3\, dx\, dy$



(c) True : $\displaystyle\int_a^b \int_c^d \underbrace{f(x)}_{\text{constant wrt } y} g(y)\, dy\, dx = \int_a^b f(x) \underbrace{\left(\int_c^d g(y)\, dy\right)}_{\text{constant \#}} dx$

$$= \left(\int_c^d g(y)\, dy\right)\left(\int_a^b f(x)\, dx\right)$$

(d)



is of type I: $D = \{ (x,y) \in \mathbb{R}^2 : x^3 \le y \le \sqrt{x},\ 0 \le x \le 1 \}$

is of type II: $D = \{ (x,y) \in \mathbb{R}^2 : y^2 \le x \le y^{\frac{1}{3}},\ 0 \le y \le 1 \}$

(e) False: · Δf is the actual change of f

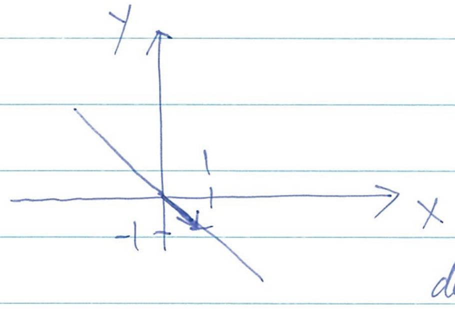
· df measures the change in the z-coordinate of the tangent plane of the graph of f

(f) False: Let $f(x,y) = x^4 + y^4$

$\Rightarrow$ $(\nabla f)(0,0) = 0$ and $(Hf)(\vec{0}) = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$

but $(0,0)$ is not a saddle point (it is a point at which f attains a local min.)

(g) False:



$y = -x$

the correct directional derivative is $\vec{i} - \vec{j}$

in the direction of