Problem Set 12 - Math Tutorial Calculus II

1. Evaluate the double integral $\iint_R (\sqrt{x} - y^2)\ dxdy$ where R is the bounded region enclosed by the curves $y = x^2$ and $y = x^{\frac{1}{4}}$.

2. Evaluate the double integral $\iint_R \cos\left(\frac{\pi}{2}x^2\right)\ dxdy$ where R is the triangle enclosed by the line $y = x$, the vertical line $x = 1$ and the x-axis.

3. Evaluate the following double integrals.

(a) $\int_0^1 \int_{x^2}^{x} xy^2\ dydx$,

(b) $\int_{\pi/2}^{\pi} \int_0^{x^2} \frac{1}{x} \cos\left(\frac{y}{x}\right)\ dydx$,

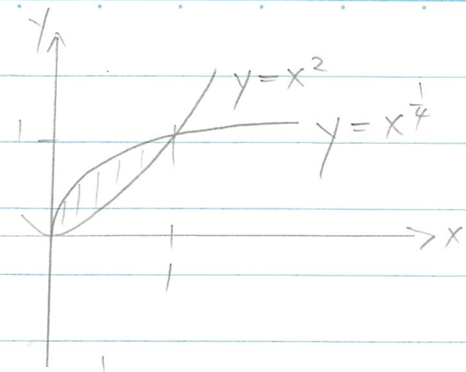
(c) $\int_0^1 \int_{4x}^{4} e^{-y^2}\ dydx$,

(d) $\int_0^1 \int_x^1 (1 - y^2)^{-\frac{1}{2}}\ dydx$,

(e) $\int_0^1 \int_{y^2}^{1} 2\sqrt{x}e^{x^2}\ dxdy$.

Problem set 12 - Solutions

#1 $\displaystyle\iint_R \left(\sqrt{x} - y^2\right) dx\,dy$, R:

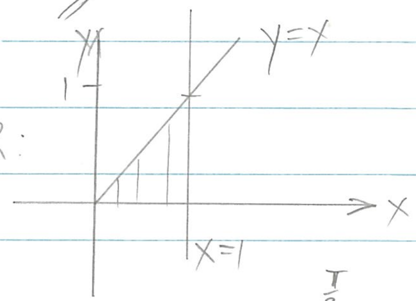


$$= \int_0^1 \int_{x^2}^{x^{\frac{1}{4}}} \left(\sqrt{x} - y^2\right) dy\,dx = \int_0^1 \sqrt{x}\,y - \frac{1}{3}y^3 \Big|_{x^2}^{x^{\frac{1}{4}}} dx$$

$$= \int_0^1 x^{3/4} - x^{5/2} - \frac{1}{3}\left(x^{3/4} - x^6\right) dx = \int_0^1 \frac{2}{3}x^{3/4} - x^{5/2} + \frac{1}{3}x^6\,dx$$

$$= \frac{2}{3}\cdot\frac{4}{7} - \frac{2}{7} + \frac{1}{21} = \frac{8}{21} - \frac{6}{21} + \frac{1}{21} = \underline{\underline{\frac{1}{7}}}$$

#2 $\displaystyle\iint_R \cos\left(\frac{\pi}{2}x^2\right) dx\,dy$, R:

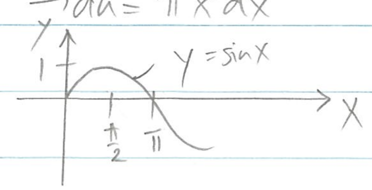


$$= \int_0^1 \int_0^x \cos\left(\frac{\pi}{2}x^2\right) dy\,dx = \int_0^1 \cos\left(\frac{\pi}{2}x^2\right) x\,dx = \frac{1}{\pi}\int_0^{\frac{\pi}{2}} \cos(u)\,du$$

$u = \frac{\pi}{2}x^2$

$\Rightarrow du = \pi x\,dx$

$$= \frac{1}{\pi}\sin(u)\Big|_0^{\frac{\pi}{2}} = \underline{\underline{\frac{1}{\pi}}}$$



#3 (a)

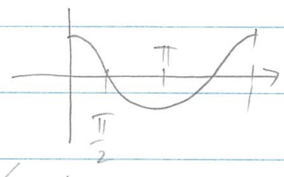
$$\int_0^1\int_{x^2}^{x} xy^2\,dy\,dx = \frac{1}{3}\int_0^1 xy^3\Big|_{x^2}^{x} dx = \frac{1}{3}\int_0^1 x^4 - x^7\,dx$$

$$= \frac{1}{3}\left(\frac{1}{5}-\frac{1}{8}\right) = \frac{1}{3}\ \frac{8-5}{40} = \frac{1}{40}$$

(b)

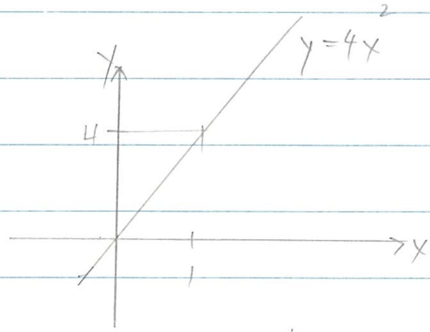
$$\int_{\pi/2}^{\pi}\int_0^{x^2} \frac{1}{x}\cos\left(\frac{y}{x}\right)dy\,dx = \int_{\frac{\pi}{2}}^{\pi}\frac{1}{x}\,x\sin\left(\frac{y}{x}\right)\Big|_0^{x^2} dx$$

$$= \int_{+\frac{\pi}{2}}^{\pi}\sin(x)\,dx = -\cos x\Big|_{\frac{\pi}{2}}^{\pi} = 1$$



(c)

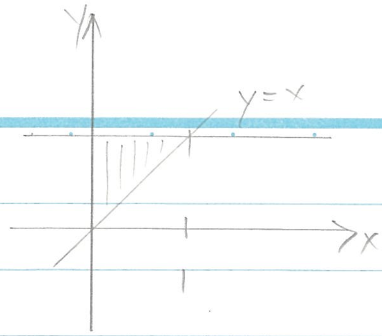
$$\int_0^1\int_{4x}^{4} e^{-y^2}dy\,dx$$



$$= \int_0^4\int_0^{y/4} e^{-y^2}dx\,dy = \int_0^4 e^{-y^2}\cdot x\Big|_0^{y/4} dy = \frac{1}{4}\int_0^4 e^{-y^2}\cdot y\,dy$$

$$= -\frac{1}{8}\int_0^{-16} e^{u}\,du = -\frac{1}{8}e^{u}\Big|_0^{-16} = \frac{1}{8}\left(1-e^{-16}\right)$$

$u=-y^2 \Rightarrow du=-2y\,dy$

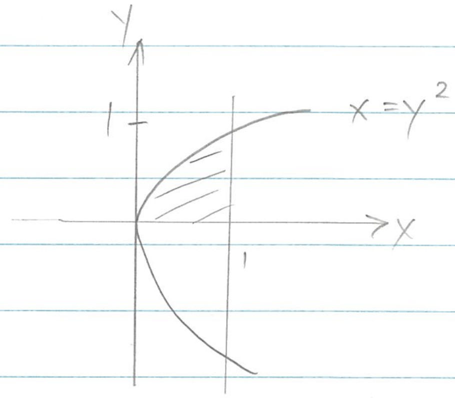


#3 (d) $\displaystyle\int_0^1\int_x^1 (1-y^2)^{-\frac{1}{2}}\,dy\,dx$

$$= \int_0^1\int_0^y (1-y^2)^{-\frac{1}{2}}\,dx\,dy = \int_0^1 (1-y^2)^{-\frac{1}{2}}\,y\,dy$$

$$= -\frac{1}{2}\int_1^0 u^{-\frac{1}{2}}\,du = -\frac{1}{2}\cdot 2\,u^{\frac{1}{2}}\Big|_1^0 = 1$$

$u = 1-y^2$
$\Rightarrow du = -2y\,dy$



(e) $\displaystyle\int_0^1\int_{y^2}^1 2\sqrt{x}\,e^{x^2}\,dx\,dy$

$$= \int_0^1\int_0^{\sqrt{x}} 2\sqrt{x}\,e^{x^2}\,dy\,dx$$

$$= 2\int_0^1 x\,e^{x^2}\,dx \underset{u=x^2}{=} \int_0^1 e^u\,du = e-1$$

$\Rightarrow du = 2x\,dx$