Problem Set 11 - Math Tutorial Calculus II

1. Identify and determine the nature of the critical points of $f(x, y) = xy + \frac{8}{x} + \frac{1}{y}$.
2. Find all critical points of

$$f(x, y) = \frac{2y^3 - 3y^2 - 36y + 2}{1 + 3x^2}. \tag{1}$$

Identify any and all extrema of f.

3. What point on the plane $3x - 4y - z = 24$ is closest to the origin?
4. Find the absolute extrema of $f(x, y) = 2\cos(x) + 3\sin(y)$ on the rectangle $\{(x, y) \mid 0 \leq x \leq 4, 0 \leq y \leq 3\}$.

Problem set 11 - Solutions

#1 $f(x,y) = xy + \frac{8}{x} + \frac{1}{y}$, domain $\{(x,y) \in \mathbb{R}^2 : x \neq 0, y \neq 0\}$

$$\Rightarrow (\nabla f)(x,y) = \left(y - \frac{8}{x^2}, x - \frac{1}{y^2} \right)$$

$$\Rightarrow (\nabla f)(x,y) = 0 \quad \text{if} \quad \underbrace{y = \frac{8}{x^2} \quad \text{and} \quad x = \frac{1}{y^2}}$$

$$\Rightarrow \quad y = \frac{8}{\frac{1}{y^4}} \Rightarrow y = 8y^4$$

$\Rightarrow$ either $y = 0$, $y = \frac{1}{2}$ or $y = -\frac{1}{2}$

Note that $(x,0)$ can not be a critical point of f since it does not belong to the domain of f.
Also $(x, -\frac{1}{2})$ can not be a critical point since $y = \frac{8}{x^2}$ implies that $y \geq 0$

$\Rightarrow f$ has one critical pt $(4, \frac{1}{2})$.

$$H = \begin{pmatrix} \frac{16}{x^3} & 1 \\ 1 & \frac{2}{y^3} \end{pmatrix} \Rightarrow f_{xx}(4, \tfrac{1}{2}) = \frac{16}{64} = \frac{1}{4} > 0$$

$$f_{yy}(4, \tfrac{1}{2}) = 16 > 0$$

$$\Rightarrow f_{xx}(4,\tfrac{1}{2}) f_{yy}(4,\tfrac{1}{2}) - \left(f_{xy}(4,\tfrac{1}{2})\right)^2 = 4 - 1 > 0$$

$\Rightarrow$ f attains a loc min at $(4, \frac{1}{2})$

$f(4, \frac{1}{2})$ is not a global min mia

$$\lim_{\substack{(x,x) \to 0 \\ x<0}} f(x,x) = -\infty < f(4, \tfrac{1}{2}) = 2+2+2 = 6$$

#2

$$f(x,y) = \frac{2y^3 - 3y^2 - 36y + 2}{1+3x^2}$$

$$\Rightarrow f_x(x,y) = \frac{-(2y^3 - 3y^2 - 36y + 2)}{(1+3x^2)^2} \cdot 6x$$

$$f_y(x,y) = \frac{(6y^2 - 6y - 36)}{1+3x^2}$$

$$\Rightarrow \quad y^2 - y - 6 = 0 \quad \text{and} \quad (2y^3 - 3y^2 - 36y + 2) \cdot x = 0$$

$\Downarrow$

$$\left(y - \frac{1}{2}\right)^2 - \frac{1}{4} - \frac{24}{4} = 0 \Rightarrow \quad y = \frac{1}{2} \pm \sqrt{\frac{25}{4}} = \frac{1}{2} \pm \frac{5}{2}$$

$$\Rightarrow \quad y = 3, \quad y = -2$$

$$g(y) = 2y^3 - 3y^2 - 36y + 2 \quad \Rightarrow g(3) = 54 - 27 - 108 + 2 \neq 0$$

$$g(-2) = -16 - 12 + 72 + 2 \neq 0$$

$$\Rightarrow \quad x = 0 \quad \text{for} \quad (\nabla f)(x,y) = 0$$

$\Rightarrow$ 2 critical points: $(0,3)$ and $(0,-2)$

$$f_{xx} = \frac{-(2y^3-3y^2-36y+2)\cdot 6}{(1+3x^2)^2} + \frac{2(2y^3-3y^2-36y+2)}{(1+3x^2)^3}(6x)^2$$

$\Rightarrow f_{xx}(0,y) = -(2y^3-3y^2-36y+2)\cdot 6 \quad \Rightarrow f_{xx}(0,3) > 0$

$f_{xx}(0,-2) < 0$

$f_{xy}(0,y) = 0$

$f_{yy}(0,y) = 12y-6 \quad \Rightarrow \quad f_{yy}(0,3) > 0, \; f_{yy}(0,-2) < 0$

$\Rightarrow$ f has a loc. min at $(0,3)$

f has a loc. max at $(0,-2)$.

#3 $f(x,y,z) = x^2+y^2+z^2$ distance squared to the origin

(x,y,z) is on the plane given by $3x-4y-z=24$

$$\Rightarrow \quad z = 3x-4y-24$$

$\Rightarrow g(x,y) = x^2+y^2+(3x-4y-24)^2$ is the distance squared of a point on the plane $3x-4y-z=24$ w/ (x,y)-coordinates to the origin.

$$\Rightarrow (\nabla g)(x,y) = \left(2x + 2\cdot 3(3x-4y-24),\ 2y + 2\cdot(-4)(3x-4y-24)\right)$$

$\Rightarrow (\nabla g)(x,y) = 0$ if (i) $2x + 6(3x-4y-24) = 0$

(ii) $2y - 8(3x-4y-24) = 0$

(i) $\Rightarrow \quad 20x - 24y - 144 = 0 \Rightarrow 5x - 6y - 36 = 0$

(ii) $\Rightarrow \quad 34y - 24x + 192 = 0$

$\Rightarrow \quad x = \frac{34}{24}y + \frac{192}{24} \Rightarrow x = \frac{17}{12}y + 8$

(i) $\Rightarrow \quad \frac{17\cdot 5}{12}y + 40 - 6y - 36 = 0 \Rightarrow \frac{17\cdot 5 - 72}{12}y = -4$

$\Rightarrow y = -\frac{48}{13} \Rightarrow x = \frac{-17}{12}\cdot\frac{48}{13} + 8 = \frac{-17}{13}\cdot 4 + \frac{104}{13} = \frac{36}{13}$

$\Rightarrow$ 1 critical point $\left(\frac{36}{13}, -\frac{48}{13}\right)$ $\quad H = \begin{pmatrix} 20 & -24 \\ -24 & 34 \end{pmatrix}$

$\Rightarrow |H| > 0$

g has a loc. max at $\left(\frac{36}{13}, -\frac{48}{13}\right) \Leftarrow$

$\Rightarrow \left(\frac{36}{13}, -\frac{48}{13}, 3\cdot\frac{36}{13} + \frac{4\cdot 48}{13} - 24\right)$ is the point on the plane with min. distance to the boundary.

#4

$f(x,y) = 2\cos(x) + 3\sin(y)$ $\quad R = \{(x,y) : 0 \le x \le 4,\ 0 \le y \le 3\}$

$\nabla f(x,y) = \left(-2\sin(x),\ 3\cos(y)\right) = 0 \iff \begin{array}{l} \sin x = 0 \\ \cos y = 0 \end{array}$

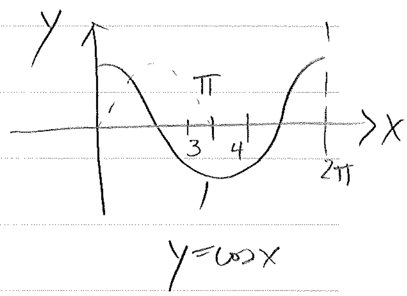
$\Rightarrow x = 0$ or $x = \pi$ and $y = \frac{\pi}{2}$

$\Rightarrow$ only 1 critical point inside of R: $(\pi, \frac{\pi}{2})$

and $f(\pi, \frac{\pi}{2}) = -2 + 3 = 1$

Need to consider (1) $L_1 = \{(x,y) : y = 0,\ 0 \le x \le 4$

$\Rightarrow f(x,0) = 2\cos(x)$



$\Rightarrow f(x,0)$ has $\overset{\text{glob.}}{\downarrow}$ max at $x=0$, $f(0,0) = \underline{2}$
$f(x,0)$ has glob. min at $x=-1$, $f(\pi,0) = \underline{-2}$

(2) $L_2 = \{(x,y) : y = 3,\ 0 \le x \le 4\}$

$\Rightarrow f(x,3) = 2\cos(x) + 3\sin(3) \Rightarrow$ glob. max $f(0,3) = \underline{2 + 3\sin(3)}$
glob. min $f(\pi,3) = \underline{-2 + 3\sin(3)}$

(3) $L_3 = \{(x,y) : 0 \le y \le 3,\ x = 0$

$\Rightarrow f(0,y) = 3\sin y + 2 \Rightarrow$ glob. max : $f(0, \frac{\pi}{2}) = \underline{3} + 2 = 5$
glob. min $f(0,0) = \underline{2}$

(4) $L_4 = \{(x,y) : 0 \le y \le 3,\ x = 4\}$

$\Rightarrow f(4,y) = 2\cos 4 + 3\sin y \Rightarrow$ glob. max : $f(4, \frac{\pi}{2}) = \underline{2\cos 4 + 3}$

glob. min : $f(4,0) = \underline{2\cos 4}$

Comparing the numbers gives: $\underline{f(0, \frac{\pi}{2}) = 5}$ is glob. max

$\underline{f(\pi, 0) = -2}$ is glob. min