Solution

Problem Set 10 - Math Tutorial Calculus II

1. Use the total differential to approximate the value $(6.57)^2(1.23)^3$.

2. To estimate the volume of a cone of radius approximately 2 m and height approximately 6 m, how accurately should the radius and height be measured so that the error in the calculated volume estimate does not exceed 0.2 m^3? Assume that the possible error in measuring the radius and height are the same.

3. Consider the function

$$f(x_1, x_2, \ldots, x_n) = e^{x_1+2x_2+\ldots+nx_n}.$$

(a) Calculate $Df(0, \ldots, 0)$ and $Hf(0, \ldots, 0)$.

(b) Determine the first and second order Taylor polynomials of f at the origin.

Problem set 10 Solution

#1 Let $f(x,y) = x^2 y^3$, approximate $f(6.99, 1.1)$!

Recall that $df(\vec{a},\vec{h}) = (\nabla f)(\vec{a}) \cdot \vec{h}$, and if $\vec{h}$ is close to the zero vector, then $\Delta f \approx df(\vec{a},\vec{h})$.

Here:

$$\Delta f = f(\vec{a}+\vec{h}) - f(\vec{a}).$$

Let $\vec{a} = (7,1)$, $\vec{h} = (-0.01, 0.1)$, then $f(7,1) = 49$

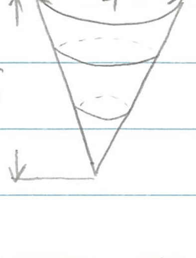
$$(\nabla f)(x,y) = (2xy^3, x^2 3y)$$

$$\Rightarrow (\nabla f)(7,1) = (14, 49\cdot 3) = (14, 147)$$

$$\Rightarrow (df)(\vec{a},\vec{h}) = (14, 147) \cdot (-0.01, 0.1) = -0.14 + 1.47 = 1.33$$

$$\Rightarrow \underbrace{f(6.99, 1.1)}_{= f(\vec{a}+\vec{h})} \approx \underbrace{49}_{= f(\vec{a})} + \underbrace{1.33}_{df(\vec{a},h) \approx \Delta f} = \underline{50.33}$$

#2



$$V(h,r) = \frac{1}{3}\pi r^2 \cdot h$$

height of the cone, radius

$r \approx 2m$
$h \approx 6m$

$$\Delta V \approx dV = \frac{\partial V}{\partial r} \underbrace{\Delta r}_{} + \frac{\partial V}{\partial h} \underbrace{\Delta h}_{} = \left(\frac{2}{3}\pi r \cdot h + \frac{1}{3}\pi r^2\right) \Delta r$$

assumed to be equal

$\Rightarrow dV = \frac{\pi}{3}(24+4)\Delta r = \frac{28}{3}\pi\Delta r$, so $|\Delta r|$ needs to be less than

$$\frac{3\cdot}{28\cdot\pi}\,\frac{2}{10} = \underline{\frac{3}{140}\cdot\frac{1}{\pi}}$$

#3 $\quad f(\vec{x}) = e^{x_1+2x_2+\ldots+nx_n}$

(a) $$(Df)(\vec{0}) = (Df)(\vec{0}) = \left(e^{x_1+2x_2+\ldots+nx_n}\right)\begin{pmatrix}1\\2\\\vdots\\n\end{pmatrix}^T\Bigg|_{\vec{x}=\vec{0}}$$

$$= (1, 2, .., n);$$

$$Hf = \begin{pmatrix} 1 & 2 & 3 & \cdots & n \\ 2 & 2\cdot2 & 2\cdot3 & \cdots & 2n \\ 3 & & & & \vdots \\ \vdots & & & & \vdots \\ n & n\cdot2 & n\cdot3 & \cdots & n\cdot n \end{pmatrix}$$

(b) $$p_1(\vec{x}) = f(\vec{0}) + Df(\vec{0})(\vec{x}-0)$$

$$\Rightarrow p_1(\vec{x}) = 1 + (1,2,..,n)\cdot\vec{x} = 1 + (x_1+2x_2+\ldots+nx_n)$$

$$p_2(\vec{x}) = f(\vec{0}) + Df(\vec{0})(\vec{x}-0) + \frac{1}{2}\vec{x}^T Hf(\vec{0})\vec{x}$$

$$= 1 + (x_1+2x_2+\ldots+nx_n) + \frac{1}{2}\vec{x}^T\cdot\begin{pmatrix} x_1+2x_2+\ldots+nx_n \\ 2(x_1+2x_2+\ldots+nx_n) \\ \vdots \\ n(x_1+2x_2+\ldots+nx_n)\end{pmatrix}$$

$$= 1 + (x_1+2x_2+\ldots+nx_n) + \frac{1}{2}\vec{x}^T\circ\begin{pmatrix}1\\2\\\vdots\\n\end{pmatrix}(x_1+2x_2+\ldots+nx_n)$$

$$\Rightarrow p_2(\vec{x}) = 1 + (x_1 + 2x_2 + \dots + nx_n) + \frac{1}{2}(x_1 + 2x_2 + \dots + nx_n)(x_1 + \dots + nx_n)$$

$$= 1 + (x_1 + 2x_2 + \dots + nx_n) + \frac{1}{2}(x_1 + 2x_2 + \dots + nx_n)^2$$